Efficient consumption set under recursive utility and unknown beliefs.*

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Abstract

In a context of complete financial markets where asset prices follow Ito's processes, we characterize the set of consumption processes which are optimal for a given stochastic differential utility (e.g. Duffie and Epstein (1992)) when beliefs are unknown. Necessary and sufficient conditions for the efficiency of a consumption process, consists of the existence of a solution to a quadratic backward stochastic differential equation and a martingale condition. We study the efficiency condition in the case of a class of homothetic stochastic differential utilities and derive some results for those particular cases. In a Markovian context, this efficiency condition becomes a partial differential equation.

Keywords: Recursive utility, quadratic backward stochastic differential equations, beliefs, martingale condition

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1 Introduction

In this paper we consider the following "invertibility" problem: in a continuous time setting, we observe the optimal intertemporal contingent consumption plan of a single agent who also invests in a financial market. This agent that has recursive utility of the type modeled in Duffie and Epstein (1992) (that they call Stochastic Differential Utility or SDU) and that offers some modeling flexibility in the separation between the concepts of risk aversion, intertemporal substitution and preference for early or late resolution of uncertainty.[1] However, we do not know neither the preferences of the agent nor the "beliefs" of the agent (by beliefs we mean, out of all probability spaces that can explain the dynamics of securities prices, which one the agent uses). Our problem is then to: 1) check if an observed consumption is consistent with preference maximization for some unknown beliefs (testability) and, 2) when it is possible, recover the set of fundamentals (preferences and beliefs) that are consistent with the observed consumption (identifiability).

At the multiple agents and general equilibrium level, this invertibility problem has historically attracted the interest of many economists (see Chiappori *et al.* (1999) for some recent advances in this field and for references). Our approach may be seen as a single agent version of the invertibility literature in a partial equilibrium environment where the agent's intertemporal consumption is the outcome of trading risky financial assets in continuous time. However, instead of assuming knowledge of the equilibrium price manifold, (Chiappori *et al.* (1999)), we suppose that we know the individual intertemporal contingent consumption as a function of intertemporal contingent Arrow Debreu prices.[2] Therefore, our approach is similar to Mas-Colell (1977) who identifies the preferences of an individual from demand behavior as commodities prices and income vary in an atemporal and riskless environment. We attempt to address the Mas-Colell (1977) question for a consumer who is trading in an intertemporal and uncertain financial market. Initial wealth is known and given.[3] We observe intertemporal consumption and the parameters of its stochastic dynamics (the trend and the volatility) for both *realized* and *unrealized* states.

Furthermore, since we are in a risky environment, we introduce a new component in the invertibility problem by incorporating non-observability of the consumer's beliefs about future asset returns. We take the cue from Kraus and Sick (1980). We ask, for instance, whether low consumption rate is the result of pessimism about the general business conditions or it reflects increases in risk aversion, changes in intertemporal substitution possibilities or even increases in the rate of

[1] We recall that expected utility implies that the investor is indifferent to the timing of resolution of uncertainty (see Duffie and Epstein (1992) and their references). In the SDU case, the model offers the flexibility to model both preference for early resolution of uncertainty (a form of *anxiety*) and preference for late resolution of uncertainty (a form of *optimism*). For instance in the homothetic subclass of SDU (a continuous time version of the Kreps-Porteus recursive utility), we have a scalar parameter whose value determines if the decision maker exhibits preference for late or early resolution of uncertainty.

[2] Note that the relationship between demand and prices is expressed in term of a system of stochastic differential equations where the uncertainty is driven the Brownian shocks.

[3] Mas-Colell (1977) uses the responses to changes in wealth as a source of information to infer the utility of the agent.

preference for early resolution of uncertainty.

Finally we point out that, for a representative agent holding the market portfolio of a complete markets, pure exchange economy, our results extend to the stochastic differential utility case the strand of the finance literature that seeks to verify if a stochastic process for the market portfolio is consistent with equilibrium (See Bick (1987, 1990) and He and Leland (1993)).

Our first result (Theorem 2) concerns a class of non parametric utilities and is methodological. In order to obtain it, we use some recent results in the theory of Backward Stochastic Differential Equations (or BSDE) in El Karoui, Peng and Quenez (1997, 1999) and Schroder and Skiadas (1999). We characterize the set of preferences (within the class of recursive preferences) and beliefs that would be consistent with the observed consumption. This characterization consists of a martingale condition (a restriction on the stochastic process that represents consumption) jointly with an existence requirement on a quadratic BSDE.[4] When the utility is time additive, this quadratic BSDE becomes a standard linear BSDE whose existence is automatically guaranteed in our framework. Consequently, we extend Cuoco and Zapatero (2000) to the SDU case and find that a given consumption process is not compatible with any parameterization of preferences, even if we do not know the beliefs and we allow them to adjust so as to make that consumption optimal for the given parameterization of preferences.

In the parametric case, we obtain a more constructive result. For a class of homothetic SDU considered by Schroder and Skiadas (1999), that we call the logarithm SDU class, and that reduces to the standard logarithm expected utility in the time additive case, testability obtains. We provide a preference-free necessary and sufficient condition for a given consumption process[5] to be optimal and, consequently, we characterize the set of efficient consumption plans (those consumption plans that are optimal for some hypothetical consumer who maximizes a SDU in the logarithm SDU class and whose beliefs about future returns are unknown). However, identifiability does not obtain because the fundamentals (preferences and beliefs) that are consistent with the given consumption processes are never unique, when they exist. We interpret this result as an observational equivalence between logarithm SDU and logarithm time additive utility when beliefs are unknown. In fact, our result shows that, given any consumption process which is optimal for logarithm time additive utility under some beliefs, there are always beliefs that will make it optimal for any logarithm SDU. In other terms, our result states that without information on consumer beliefs about future asset returns, the set of efficient consumption plans is identical for logarithm SDU and logarithm time additive utility. In particular, we prove that observing contingent consumption alone does not allow to assert that the consumer exhibits preference for early resolution of uncertainty, preference

[4] Quadratic BSDE's existence is not covered by the standard theory on BSDE's (Pardoux and Peng (1990)) because the intertemporal aggregator (defined in Section 2) is not a Lipschitz function of its arguments. Kobylanski (2000) has a systematic study of the quadratic BSDE's. In the specific parametric case under consideration, we use some mathematical results from Schroder and Skiadas (1999) to prove that, in fact, the quadratic BSDE exists and hence our martingale condition requirement is sufficient for testability to hold.

[5] Those restrictions may be seen as a stochastic equivalent to the Slutsky equations in our continuous time environment.

for late resolution of uncertainty or indifference for the timing of resolution of uncertainty.

We conduct a similar analysis for another class of homothetic SDU considered by Schroder and Skiadas (1999), the power SDU that reduces to constant relative risk aversion expected utility in the time additive case. Although a preference dependant characterization of efficiency is obtained, we discuss how this condition can provide some efficiency verification tools. Moreover, when the efficiency condition is satisfied, the associated beliefs are directly obtained from preference parameters and consumption dynamics parameters.

Finally, we also extend the result of Cuoco and Zapatero (2000) related to the recoverability of preferences and beliefs in a Markovian setting. We state a verification result that allows to determine if a consumption process is optimal for a given parameterization of preferences and, if that is the case, what would be the beliefs of the individual. This result extends the recoverability literature in finance (See Décamps and Lazrak (2000) for some recent results and references) to the SDU case.

The paper is organized as follows. In section 2 we describe in detail the setting. In section 3 we formally characterize the problem. In section 4 we derive the main result and we apply it to two particular cases, logarithmic utilities and power utilities. In section 5 we consider the problem of recoverability in a Markovian setting. We close the paper with some conclusions.

2 The model

We start with a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with the augmented filtration $\mathcal{F}_{(.)} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of a standard $\mathbb{R}^n$-valued Brownian motion $(W_t = (W_{1,t}, W_{2,t}, ..., W_{n,t}), 0 \leq t \leq T)$. The terminal time $T < \infty$ is fixed constant and we assume that $\mathcal{F} = \mathcal{F}_T$. All stochastic processes introduced in the paper are assumed progressively measurable with respect to the filtration $\mathcal{F}_{(.)}$. The conditional expectation $E(. \mid \mathcal{F}_t)$ will be abbreviated to E_t throughout.

We shall denote by $\mathcal{P}$ the set of predictable $\sigma-$field and for each integer d, we define $\mathcal{H}^2(\mathbb{R}^d) = \{\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}^d / \varphi \in \mathcal{P} \text{ and } ||\varphi||^2 = E \int_0^T |\varphi_t|^2 dt < \infty\}$.

Consumption Set. The set $\mathcal{C}$ of *consumption processes* is formed by any strictly positive process $c : \Omega \times [0,T] \longrightarrow \mathbb{R}^+$ such that $c \in \mathcal{H}^2(\mathbb{R})$, with c_t representing a time-t consumption rate in terms of a single numéraire good. To simplify the exposition, we avoid terminal consumption although our analysis extends easily to that case.

Density generators of beliefs. We define Υ, the set of possible "beliefs" as the set of progressively measurable processes $\gamma : \Omega \times [0,T] \longrightarrow \mathbb{R}^n$ the integrability condition

$$E\left[\exp\left\{-\frac{1}{2}\int_0^t |\gamma_s|^2 ds - \int_0^t \gamma_s{}^* \cdot dW_s\right\}\right] = 1, \quad \forall t \in [0,T].$$

where * represents "transpose." In fact, each $\gamma \in \Upsilon$ characterizes a possible belief represented by a probability measure P^γ on $(\Omega, \mathcal{F})$ equivalent to P, with a Radon Nikodym derivative of the form

$$\frac{dP^\gamma}{dP} = \xi_T^\gamma,$$

where $(\xi_t^\gamma, 0 \le t \le T)$ is the martingale,

$$\xi_t^\gamma = \exp\left(-\frac{1}{2}\int_0^t |\gamma_s|^2 ds - \int_0^t \gamma_s{}^* \cdot dW_s\right), \quad 0 \le t \le T.$$

Note that, by Girsanov Theorem, the process $W_t^\gamma = W_t + \int_0^t \gamma_s ds$ is a Brownian motion under the measure P^γ. Finally, note that since the beliefs of consumers shall be unknown in our context, the measure P is part of the description of the environment only because it defines the null sets. In the inverse problem of section 3 we will state the optimality conditions under the probability P, but any other equivalent probability could be used.

Preferences. An *intertemporal aggregator* is a deterministic function f mapping $\mathbb{R}^+ \times \mathbb{R}$ onto $\mathbb{R}$ that satisfies that there exists some constants k_1, k_2 such that, $\forall c \in \mathbb{R}^+$, $| f(c, 0) | \le k_1 + k_2 c^p$, for some constant $0 < p < 1$.[6]) We now introduce some assumptions that we will use in different parts of the paper.

(**A1**) *Lipschitz.* There exists a constant $K \ge 0$ such that

$$| f(c, y) - f(c, y') | \le K \mid y - y' \mid, \quad \forall c \in \mathbb{R}^+, \forall (y, y') \in \mathbb{R} \times \mathbb{R}.$$

(**A2**) *Concavity and monotonicity.* f is concave with respect to (c, y) and increasing with respect to c.

(**A3**) *Differentiability.* f is three times continuously differentiable with respect to (c, y), and f_c and f_y are bounded.

(**A4**) *Inada condition.* For each y, $\lim_{c \downarrow 0} f_c(c, y) = +\infty$.

(**A5**) *Logarithmic.* The intertemporal aggregator takes the following form

$$f(c, y) = (1 + \alpha y)\left(\log(c) - \frac{\beta}{\alpha}\log(1 + \alpha y)\right),$$

with parameter restrictions: $\beta \ge Max(0, \alpha)$.

(**A6**) *Power.* The intertemporal aggregator takes the following form

$$f(c, y) = (1 + a)\left(\frac{c^\nu}{\nu} \mid y \mid^{\frac{a}{1+a}} - \beta y\right),$$

with parameter restrictions: $\beta \ge 0$, $a \in (-1, 1)$, $\nu < Min(1, 1/(1 + a))$ and $\nu \ne 0$.

We denote by $\mathcal{I}$ the set of intertemporal aggregators defined as $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_p \cup \mathcal{I}_l$. Here $\mathcal{I}_1$ is the set of intertemporal aggregators that satisfies the non parametric Assumptions $A1 - A4$, and $\mathcal{I}_l$ (resp. $\mathcal{I}_p$) is the set of logarithmic (resp. power) intertemporal aggregators that satisfy Assumption $A5$ (resp. $A6$).

Assumptions $A1$, $A5$ and $A6$ are mutually exclusive. Assumption $A1$ is required to define the non parametric SDU of Duffie and Epstein (1992). The concavity part of assumption $A2$ is required

[6] Since $c \in \mathcal{H}^2(\mathbb{R})$, The Condition $| f(c, 0) | \le k_1 + k_2 c^p$ implies that $(f(c_t, 0), 0 \le t \le T) \in \mathcal{H}^{2/p}(\mathbb{R}) \subset \mathcal{H}^2(\mathbb{R})$. The fact that $(f(c_t, 0), 0 \le t \le T) \in \mathcal{H}^2(\mathbb{R})$ is required to prove the existence of the SDU defined further (e.g. Pardoux and Peng (1992) and El Karoui, Peng and Quenez (1997)).

for the existence of an optimal utility and allows us to define (under assumption $A3$) the inverse of f_c, that is the function $I(.,.)$ defined by, $I(c, f(c, y)) = y$; whereas the increasing part of assumption $A2$ implies that the utility function is an increasing functional of consumption. Assumption $A3$ is a technical regularity that allows us to expand Itô's rules and to formulate optimality conditions with the help of the partial derivatives of the intertemporal aggregator. The purpose of assumption $A4$ is to simplify the optimality conditions.

The specifications of the intertemporal aggregators given in Assumptions $A5$ and $A6$ are proposed by Schroder and Skiadas (1999) to define a parametric homothetic class of SDU. Assumption $A5$ defines the logarithm SDU. For this specification, the parameter α has no impact on the ordinal ranking of deterministic consumption plans. However, a negative α indicates preference for early resolution of uncertainty, and a positive α indicates preference for late resolution of uncertainty (See Schroder and Skiadas (1999)). Assumption $A6$ defines the power SDU and, again the parameter a has no impact on the ordinal ranking of deterministic consumption plans. When $\nu > 0$ (resp. $\nu < 0$), a negative a indicates preference for early (resp. late) resolution of uncertainty, and a positive a indicates preference for late (resp. early) resolution of uncertainty (See Schroder and Skiadas (1999)).

Given an intertemporal consumption process $c \in \mathcal{C}$, consumer preferences are represented by a SDU (Duffie and Epstein (1992) under Assumptions $A1 - A4$ and Schroder and Skiadas (1999) under Assumption $A5$ or Assumption $A6$) Y defined by

$$Y_t^c = E_t^\gamma \left[\int_t^T f(c_s, Y_s^c) ds \right], \tag{1}$$

where E_t^γ represents $\mathcal{F}_t-$conditional expectation under the subjective beliefs P^γ associated to the density generator $\gamma \in \Upsilon$. Alternatively, the utility represented by (1) may as well be characterized by the BSDE (see Pardoux and Peng (1992) and El Karoui, Peng and Quenez (1997)),

$$-dY_t^c = f(c_t, Y_t^c)dt - Z_t^{c*} \cdot dW_t^\gamma, \qquad Y_T^c = 0, \tag{2}$$

where the intensity process $Z^c \in \mathcal{H}^2(\mathbb{R}^n)$ is part of the solution of the BSDE.

Existence and uniqueness of recursive utility is studied by Duffie and Epstein (1992) when the intertemporal aggregator satisfies assumption A1. Schroder and Skiadas (1999) proves the existence of the associated homothetic SDU when the intertemporal aggregator satisfies either assumption $A4$ or assumption $A5$.

Financial markets We assume that financial markets are complete and the parameters that define the dynamics of the securities are summarized in the following *state price density* that we take as a primitive

$$H_t = \exp\left\{-\frac{1}{2}\int_0^t r_s ds\right\} \exp\left\{-\frac{1}{2}\int_0^t |\eta_s|^2 ds - \int_0^t \eta_s^* \cdot dW_s\right\}, \quad 0 \leq t \leq T,$$

where $\int_0^T (|r_t| + |\eta_s|^2)dt < \infty$ a.s.. The process r is the *short-rate* process and the process η is the *market price of risk* process. The process $(H_t, 0 \leq t \leq T)$ represents the intertemporal contingent Arrow Debreu prices.

We are now ready to formalize the consumption optimization problem of the consumer when the intertemporal aggregator is given by $f \in \mathcal{I}_1 \cup \mathcal{I}_p \cup \mathcal{I}_l$ and when beliefs are given by the density generator $\gamma \in \Upsilon$, as

$$\mathcal{P}^{f,\gamma} \quad \begin{cases} \sup_{c \in \mathcal{C}} Y_0^c \\ s.t.\ E\left[\int_0^T H_t c_t dt\right] \leq w_0, \end{cases}$$

where w_0 is a nonnegative scalar representing initial wealth[7] and Y_0^c is the initial value of the solution Y of the BSDE (2).

3 An inverse problem

3.1 Characterization of optimality

By Girsanov Theorem, it is clear that, given any $\gamma^* \in \Upsilon$ the solution (Y^c, Z^c) of the BSDE (1) solves also the following BSDE

$$-dY_t^c = (f(c_t, Y_t^c) - \gamma_t^*.Z_t^c)dt - Z_t^{c*} \cdot dW_t, \qquad Y_T^c = 0,$$

and admits the representation

$$Y_t^c = E_t\left[\int_t^T (f(c_s, Y_s^c) - \gamma_s^* \cdot Z_s^c)ds\right]. \tag{3}$$

Therefore, we can ignore the measure P^γ and consider (Y^c, Z^c) as a generalized SDU (see Lazrak and Quenez (1999) and see also a related model in Chen and Epstein (1999)) under the benchmark measure P with a stochastic intertemporal aggregator of the form

$$g(t, c, y, z) = f(c, z) - \gamma_t^* \cdot z, \quad \forall (t, c, y, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n.$$

This remark allows us to apply directly the optimality characterization result of El Karoui, Peng and Quenez (1999) (see also Duffie and Skiadas (1994) and Schroder and Skiadas (1999)).

Theorem 1 *Suppose that assumptions A2, A3 and A4 are satisfied and that either of assumptions A1, A5 or A6 holds. Then, for each $\gamma \in \Upsilon$, the consumption process $(c_t, 0 \leq t \leq T) \in \mathcal{C}$ is optimal under the beliefs given by γ if and only if*

$$c_t = I(e^{A_t}, Y_t), \quad dP \otimes dt\ a.s., \tag{4}$$

where the pair $(A; (Y, Z))$ solves a Forward-backward system that has a forward component,

$$\begin{cases} dA_t = -(r_t + f_y(I(e^{A_t}, Y_t), Y_t) + \frac{1}{2}(|\eta_t|^2 - |\gamma_t|^2))dt - (\eta_t - \gamma_t)^* dW_t, \\ A_0 = \log(\lambda), \end{cases} \tag{5}$$

[7] Notice that the initial wealth $w_0 > 0$ is fixed for the rest of the paper.

and a backward component

$$-dY_t = (f(I(e^{A_t}, Y_t), Y_t) - \gamma_t^* \cdot Z_t)dt - Z_t^* dW_t, \qquad Y_T = 0, \tag{6}$$

for some constant $\lambda > 0$ *which is fixed in a such way that* $E\left[\int_0^T H_t I(e^{A_t}, Y_t)dt\right] = w_0$.

Proof: In the context of assumption $A1$, this Theorem is a specialization of Theorem 6.1 of El Karoui, Peng and Quenez (1999) to the case of linear wealth and an intertemporal aggregator which is linear with respect to z. For the optimality first order conditions of the homothetic SDU implied by either assumption $A5$ or assumption $A6$, we refer to Schroder and Skiadas (1999). $\diamond$

3.2 The problem

The main problem considered in this paper is that of an observer (a "financial economist") who tries to verify whether a given consumption process is optimal for some combination of tastes and beliefs. Formally, this question translates into the problem of characterizing for each consumption process $(c_t, 0 \leq t \leq T) \in \mathcal{C}$, the set

$$\mathcal{I}^c = \left\{ (f, \gamma) \in \mathcal{I} \times \Upsilon \ / \ c \ solves \ \mathcal{P}^{f,\gamma} \right\}.$$

The optimality first order conditions (4)-(6) imply that each optimal consumption process should be indistinguishable from an Itô process of the form

$$\frac{dc_t}{c_t} = \mu_t dt + \rho_t^* dW_t, \quad c_0 > 0, \tag{7}$$

for some processes $(\mu_t, 0 \leq t \leq T) \in \mathcal{H}^2(\mathbb{R})$ and $(\rho_t, 0 \leq t \leq T) \in \mathcal{H}^2(\mathbb{R}^n)$. Therefore, we shall restrict our attention to the set of consumption processes of the form (7) and for each consumption process, we shall express the characterization of the set $\mathcal{I}^c$ in terms of μ and ρ. Finally, we point out that as a byproduct of this characterization, we shall be able to identify the beliefs that support each optimal consumption given an intertemporal aggregator.

4 Viable consumption plans, quadratic BSDE and martingale restrictions

4.1 Characterization of viable consumption plans

Theorem 2 *A consumption process* $c \in \mathcal{C}$ *that satisfies the dynamics* (7) *is optimal for a fixed intertemporal aggregator* $f \in \mathcal{I}$ *if and only if the process* ξ *defined below is a martingale,*

$$(\xi_t := \frac{H_t e^{-\int_0^t f_y(c_s, Y_s)ds}}{f_c(c_t, Y_t)}, \quad 0 \leq t \leq T), \tag{8}$$

where Y *is the first element of a pair* (Y, Z) *that solves the quadratic BSDE,*

$$-dY_t = (f(c_t, Y_t) - (\eta_t + c_t \frac{f_{cc}(c_t, Y_t)}{f_c(c_t, Y_t)}\rho_t + \frac{f_{cy}(c_t, Y_t)}{f_c(c_t, Y_t)} Z_t)^* \cdot Z_t)dt - Z_t^* dW_t, \qquad Y_T = 0. \tag{9}$$

Moreover, if the above conditions are satisfied then c solves $\mathcal{P}^{f,\gamma}$ for the beliefs density generator γ given by

$$\gamma_t = \eta_t + c_t \frac{f_{cc}(c_t, Y_t)}{f_c(c_t, Y_t)} \rho_t + \frac{f_{cy}(c_t, Y_t)}{f_c(c_t, Y_t)} Z_t, \tag{10}$$

and the optimal level of utility is given by the solution to (9).

Proof: *Necessity.* If c is optimal for $f \in \mathcal{I}$ and $\gamma \in \Upsilon$, from Theorem 1 the first order conditions (4)-(6) have to be satisfied. Integrating (5), taking exponential and using (4) yields

$$\frac{H_t e^{-\int_0^t f_y(c_s, Y_s) ds}}{f_c(c_t, Y_t)} = \frac{1}{\lambda} \exp\left\{ -\frac{1}{2} \int_0^t |\gamma_s|^2 ds - \int_0^t \gamma_s{}^* \cdot dW_s \right\}. \tag{11}$$

Then, applying Itô's Lemma to the right and left hand side of this last equality results (10) that we substitute into the BSDE (6) to obtain that in fact, (Y, Z) solves the quadratic BSDE (9). Finally, since $\gamma \in \Upsilon$, the process $(\xi_t, \, 0 \le t \le T)$ defined in (8) and identified in (11) is a martingale.

Sufficiency. Applying Itô's Lemma to (8) we get

$$d\xi_t = -\xi_t \gamma_t^* \cdot dW_t,$$

where γ is given by (10). Now letting $A_t := \log(H_t e^{-\int_0^t f_y(c_s, Y_s) ds} / \xi_t)$ we obtain, by construction, (4) and (5). Additionally, (6) is easily obtained after substituting (10) into (9). $\diamond$

Theorem 2 extends Cuoco and Zapatero (2000) (that only consider additively separable expected utility) to the SDU case. As in Cuoco and Zapatero (2000), Theorem 2 shows that a given consumption plan is not necessarily optimal for some intertemporal aggregator, even if we allow beliefs to adjust. Beliefs are part of the optimality condition, but even if we use them as an additional degree of freedom, a given consumption plan might not be consistent with that optimality condition *in general* (we will see at least one case in which this degree of freedom guarantees that any consumption process of a very large set will be compatible with the optimality condition). The problem we consider here is that of "testability" rather than "identifiability" or "recoverability," as defined in the introduction. Therefore, the result in Cuoco and Zapatero (2000) is robust to the generalization to the SDU case. Furthermore, an analysis of the optimality conditions of Theorem 2 shows that for a given consumption process to be optimal for a pair of preferences and beliefs in a SDU setting, two conditions have to be satisfied.

The first condition is the existence of a solution to the quadratic BSDE (9) that has no equivalent in the additively separable case. In fact, when the utility is additive, the intertemporal aggregator takes the form $f(c, y) = u(c) - \beta y$, for some constant β and some increasing and concave u, and the quadratic BSDE (9) becomes the linear BSDE

$$-dY_t = (u(c_t) - \beta Y_t - (\eta_t + c_t \frac{u''(c_t)}{u'(c_t)} \rho_t)^* \cdot Z_t) dt - Z_t^* dW_t, \qquad Y_T = 0,$$

which has the following explicit and unique solution

$$Y_t = E_t \left[\int_t^T u(c_s) e^{-\beta(s-t)} \exp\left\{ -\frac{1}{2} \int_0^t |\eta_s + c_s \frac{u''(c_s)}{u'(c_s)} \rho_s|^2 ds - \int_0^t (\eta_s + c_s \frac{u''(c_s)}{u'(c_s)} \rho_s)^* dW_s \right\} \right].$$

Also, when beliefs are objective, the Brownian motion W_t^γ coincides with the Brownian motion W and therefore, the BSDE (9) takes the standard form (2) and existence and uniqueness are guaranteed under our assumptions on $\mathcal{I}$. Informally, this additional restriction would seem to make the separation between beliefs and tastes more likely in the SDU case. Note also, that only the volatility (ρ) of the consumption process is involved in this restriction.

The second condition is a result of requiring the process defined by (8) to be a martingale (in fact this process may be seen as a component of the quadratic BSDE (9)) and amounts to a joint restriction on the drift (μ) and the volatility (ρ) of the consumption process (in the additive case this joint restriction is the optimality restriction given in Cuoco and Zapatero (2000)).

Corollary 1 *Let $c \in \mathcal{C}$ be a consumption process of the form* (7) *and let $f \in \mathcal{I}$ be a fixed intertemporal aggregator. Suppose that the quadratic BSDE* (9) *has a solution* (Y, Z) *that satisfies the integrability condition*

$$E\left[\exp\left\{-\frac{1}{2}\int_0^t |\eta_t + c_t\frac{f_{cc}(c_t,Y_t)}{f_c(c_t,Y_t)}\rho_t + \frac{f_{cy}(c_t,Y_t)}{f_c(c_t,Y_t)}Z_t|^2 ds - \int_0^t (\eta_t + c_t\frac{f_{cc}(c_t,Y_t)}{f_c(c_t,Y_t)}\rho_t + \frac{f_{cy}(c_t,Y_t)}{f_c(c_t,Y_t)}Z_t)^* dW_s\right\}\right] = 1, \tag{12}$$

for all $t \in [0, T]$. Then, there exists some beliefs generator $\gamma \in \Upsilon$, such that c is optimal for $\mathcal{P}^\gamma$ if and only if the following equality,

$$\begin{aligned} 0 \;=\; & r_t - \frac{c_t f_{cc}(c_t,Y_t)}{f_c(c_t,Y_t)}(\rho_t^*\eta_t - \mu_t) + f_y(c_t,Y_t) + \rho_t^2\left(\frac{1}{2}\frac{c_t^2 f_{ccc}(c_t,Y_t)}{f_c(c_t,Y_t)} - \left(\frac{c_t f_{cc}(c_t,Y_t)}{f_c(c_t,Y_t)}\right)^2\right) \\ & + \frac{1}{2}\frac{f_{cyy}(c_t,Y_t)}{f_c(c_t,Y_t)}|Z_t|^2 - \frac{f_{cy}(c_t,Y_t)}{f_c(c_t,Y_t)}\left(f(c_t,Y_t) + \frac{c_t f_{cc}(c_t,Y_t)}{f_c(c_t,Y_t)}\rho_t^* Z_t\right) + \frac{c_t f_{ccy}(c_t,Y_t)}{f_c(c_t,Y_t)}\rho_t^* Z_t \end{aligned} \tag{13}$$

holds $dP \otimes dt$ a.s.

Moreover, if (13) *holds, the beliefs density generator is given by* (10)

Proof: Necessity is immediate by Itô's Lemma. Conversely, if (13) is satisfied, the integrability condition (12) says that the process defined in (8) is a martingale and the result follows from Theorem 2. ⋄

Therefore, equation (13) combined with (10) characterize the set $\mathcal{I}^c$, under the assumption that the BSDE (9) admits a solution that satisfies (12). This restriction on the drift (μ) and volatility (ρ) of the consumption process depends on the intertemporal aggregator. This relationship between μ and ρ is difficult to interpret since (13) involves the variables (Y, Z) which depend on (c, ρ) in an abstract way via the quadratic BSDE (9). Therefore, it is useful to analyze our inverse problem in the context of some parametric intertemporal aggregator and that is the objective of the next two subsections. The existence of these parametric utilities as well as the existence of an optimal consumption plan is shown in Schroder and Skiadas (1999) for the case of objective beliefs under some appropriate technical conditions. We also have to adapt these technical conditions to our context and, more specifically, we need to restrict our attention to the set of consumption processes:

$$\mathcal{D} = \left\{ c \in \mathcal{C} \quad : \quad E\left[\int_0^T c_t{}^l dt\right] < \infty, \quad \forall l \in \mathbb{R} \right\}.$$

Finally, we need to make the following relatively weak assumption on consumption processes c of the form given by (7):

(**A**7) There exists some scalar $k \leq 1$ such that the processes ξ^{δ^k} and $\xi^{-\delta^k}$, defined as the unique local martingales that satisfy

$$d\xi_t^{\delta^k} = -\xi_t^{\delta^k} \delta_t^{k*} dW_t, \quad \xi_0^{\delta^k} = 1, \tag{14}$$

and

$$d\xi_t^{-\delta^k} = \xi_t^{\delta^k} \delta_t^{k*} dW_t, \quad \xi_0^{\delta^k} = 1,$$

with $\delta_t^k := \eta_t - k\rho_t$, are square integrable martingales.

The processes ξ are the Radon-Nikodym derivative of the reference probability measure with respect to an alternative probability measure (or beliefs). More about it in the Appendix. Assumption $A7$ will hold for example if both η and ρ are bounded.

4.2 The logarithm SDU and an observational equivalence result

The following lemma shows that, under some technical requirements, the quadratic BSDE (9) has a unique solution (in a sense made more precise in the Appendix) for a logarithmic intertemporal aggregator.

Lemma 1 *Suppose that the intertemporal aggregator satisfies assumption* $A5$ *and the consumption process* $c \in \mathcal{D}$ *is of the form* (7) *and satisfies* $A7$ *with* $k = 1$. *Then, the quadratic BSDE* (9), *which takes the specialized form*

$$-dY_t = \left((1+\alpha Y_t)(\log(c_t) - \frac{\beta}{\alpha}\log(1+\alpha Y_t)) - (\eta_t - \rho_t + \frac{\alpha}{1+\alpha Y_t} Z_t)^* \cdot Z_t \right)dt - Z_t^* dW_t, \qquad Y_T = 0, \tag{15}$$

has a unique solution that satisfies $1 + \alpha Y_t > 0$, $dP \otimes dt$ *a.s., and the integrability condition*

$$E\left[\exp\left\{-\frac{1}{2}\int_0^t |\eta_s - \rho_s + \frac{\alpha}{1+\alpha Y_s} Z_s|^2 ds - \int_0^t (\eta_s - \rho_s + \frac{\alpha}{1+\alpha Y_s} Z_s)^* \cdot dW_s\right\}\right] = 1, \tag{16}$$

for all $t \in [0, T]$.

Proof: See Appendix. ⋄

Lemma 1 shows that under proper technical restrictions (that are preference free) on c, the first optimality condition of Theorem 2 (existence of a solution to a quadratic BSDE) is always satisfied in the logarithmic case. Optimality is then characterized by (13) alone, as the following proposition shows.

Proposition 1 *Suppose that $c \in \mathcal{D}$ is a consumption process of the form* (7) *that satisfies assumption A7 with $k = 1$ and $f \in \mathcal{I}_l$ is a logarithmic intertemporal aggregator that satisfies assumption A5. There exists a beliefs generator $\gamma \in \Upsilon$, such that c is optimal for $\mathcal{P}^\gamma$ if and only if*

$$\mu_t - \rho_t^* \eta_t = r_t - \beta, \quad dP \otimes dt \ a.s. \tag{17}$$

Moreover, if (17) *is satisfied, the beliefs density generator is given by*

$$\gamma_t = \eta_t - \rho_t + \frac{\alpha}{1 + \alpha Y_t} Z_t,$$

where (Y, Z) is the solution of (15).

Proof: This is an immediate consequence of Corollary 1, Lemma 1 and (17). ⋄

The characterization (17) provides an easy way to check if a consumption plan may be optimal for some logarithm SDU and some unknown beliefs: one should just check if the process $(r_t - \mu_t + \rho_t^* \eta_t, \quad 0 \leq t \leq T)$ is deterministic, time invariant and non negative.

For instance, if we assume lognormal dynamics for asset prices and a constant interest rate, η and r will be deterministic and time invariant, and therefore, any lognormal consumption process (μ and ρ are constants) should satisfy (17) as long as μ is not too large.

In summary, testability, as defined in the introduction, obtains in this model. However, identifiability does not obtain in the sense that, if the above test is positive, we can find a supporting beliefs generator for each intertemporal aggregator in the class $\mathcal{I}_l$ with a discount factor β that satisfies the Equality (17). We conclude that, in the context of the logarithmic intertemporal aggregator, it is not possible to disentangle beliefs from tastes (represented by the parameter α that determines both risk aversion and information seeking/aversion) by only observing the optimal consumption process. In particular, we have shown that observing contingent consumption alone does not allow to assert that the consumer exhibits preference for early resolution of uncertainty, preference for late resolution of uncertainty or indifference for the timing of resolution of uncertainty. We interpret this as an observational equivalence result.

4.3 The power intertemporal aggregator

We now turn to the case of a power intertemporal aggregator that satisfies assumption $A6$. As in the logarithmic case, the following lemma shows that in the quadratic BSDE (9) has a unique solution (in a sense made more precise in the Appendix) when the consumption process satisfies some technical requirements.

Lemma 2 *Suppose that the intertemporal aggregator satisfies assumption A6 and the consumption process $c \in \mathcal{D}$ is of the form* (7) *and satisfies assumption A7 with $k = \alpha$. Then, the quadratic BSDE* (9)*, which takes the specialized form*

$$-dY_t = (\ (1+a)(\frac{c_t^\nu}{\nu} Y_t^{a/(1+a)} - \beta Y_t) - (\eta_t - (1-\nu)\rho_t + \frac{a}{1+a} Y_t^{-1} Z_t)^* \cdot Z_t\ )dt - Z_t^* dW_t, \qquad Y_T = 0, \tag{18}$$

has a unique solution that satisfies $Y_t > 0$, $dP \otimes dt$ a.s.

Proof: See the Appendix. $\diamond$

Lemma 2 shows that, as in the logarithmic case, under proper technical restrictions on c, (9) always has a solution and we can characterize optimality via (13). The following proposition specializes (13) to the power case when the integrability condition

$$E\left[\exp\left(-\frac{1}{2}\int_0^t |\eta_s - (1-\nu)\rho_s + \frac{a}{1+a}Y_s^{-1}Z_s|^2 ds - \int_0^t (\eta_s - (1-\nu)\rho_s + \frac{a}{1+a}Y_s^{-1}Z_s)^* \cdot dW_s\right)\right] = 1, \tag{19}$$

holds.

Proposition 2 *Suppose that $c \in \mathcal{D}$ is a consumption process of the form* (7) *that satisfies assumption A7 with $k = \alpha$ and $f \in \mathcal{I}_p$ is a power intertemporal aggregator that satisfies assumption A6. Suppose also that (Y, Z), the solution of the BSDE* (18) *satisfies the integrability condition* (19). *Then there exists a beliefs generator $\gamma \in \Upsilon$, such that c is optimal for $\mathcal{P}^\gamma$ if and only if*

$$(1-\nu)(\mu_t - \rho_t^*\eta_t) = r_t - \beta + \frac{1}{2}\nu(1-\nu) \mid \rho_t \mid^2 - \frac{1}{2}\frac{a}{(1+a)^2}Y_t^{-2}Z_t^2, \quad dP \otimes dt \ a.s. \tag{20}$$

Moreover, the beliefs density generator is given by

$$\gamma_t = \eta_t - (1-\nu)\rho_t + sgn(a)\sqrt{\frac{2}{\mid a \mid}}\sqrt{(r_t - \beta) - (1-\nu)(\mu_t - \rho_t^*\eta_t - \frac{1}{2}\nu \mid \rho_t \mid^2)},$$

where $sgn(x) = x/ \mid x \mid$ if $x \in \mathbb{R}^*$ and $sgn(0) = 0$.

Proof: This is an immediate consequence of Corollary 1, Lemma 2 and (13). $\diamond$

Note that, unlike in the logarithmic case, the optimality condition (20) for power utilities depends on the utility parameters a and ν. Therefore, we need to know the preference parameters a and ν in order to check efficiency. Nevertheless, (20) maybe useful even if we do not know the utility parameters in order to exclude some consumption policies. For instance one necessary condition of efficiency that is derived from is (20) is that the process $((1-\nu)(\mu_t - \rho_t^*\eta_t) - r_t + \beta - \frac{1}{2}\nu(1-\nu) \mid \rho_t \mid^2, \quad 0 \le t \le T)$ should be either nonnegative or nonpositive.

Example 1 *Assume lognormal dynamics for asset prices and constant interest rate -η and r will be deterministic and time invariant- and let us consider a consumption process of the form $c_t = \exp[\varepsilon_t]$ where ε_t is a mean reverting Ornstein Uhlenbeck process of the form $d\varepsilon_t = (\varepsilon_t - \theta)dt + dW_{1,t}$ with θ a given constant. Then, by Itô's Lemma, it is clear that $\rho_t^* = (1, 0, ..., 0)$ and, on the other hand, $\mu_t = (\varepsilon_t - \theta + 1/2)$ is a Gaussian process and, as such, cannot have an invariant sign. Therefore, the consumption process $c_t = \exp[\varepsilon_t]$ will never be optimal for a power SDU.*

5 Optimal PDE in a Markovian environment

In this section we intend to elucidate the implication of the former results in a Markovian context. With this purpose, consider an intertemporal aggregator $f \in \mathcal{I}$ and suppose that the consumption process follows a Markovian diffusion of the form

$$\frac{dc_t}{c_t} = \mu(t, c_t)dt + \rho(t, c_t)^* dW_t, \quad c_0 > 0, \tag{21}$$

where the functions $\mu : [0, T] \times (0, \infty) \longrightarrow \mathsf{R}$, and $\rho : [0, T] \times (0, \infty) \longrightarrow \mathsf{R}^n$ are such that a unique (strong) positive solution of (21) exists (e.g. Karatzas and Shreve (1997)). Let us also assume that $r_t = r(t, c_t)$, and $\eta_t = \eta(t, c_t)$ for some measurable functions $r : [0, T] \times (0, \infty) \longrightarrow \mathsf{R}$ and $\eta : [0, T] \times (0, \infty) \longrightarrow \mathsf{R}^n$.

Now, observe that in this Markovian context the quadratic BSDE (9) amounts to the partial differential equation (PDE)

$$\begin{cases} \phi_t(t, c) + \mathcal{L}\phi(t, c) = p(t,\ c,\ \phi(t, c),\ c\rho(t, c)\phi_c(t, c)\ ), & \forall (t, c) \in [0, T) \times (0, \infty), \\ \phi(T, c) = 0, \end{cases} \tag{22}$$

where the operator $\mathcal{L}$ is defined for $u \in C([0, T] \times (0, \infty)) \cap C^{1,2}([0, T) \times (0, \infty))$ as

$$\mathcal{L}u(t, c) = c\mu(t, c)u_c(t, c) + \frac{1}{2} \mid \rho(t, c) \mid^2 c^2 u_{cc}(t, c),$$

and where the function p is given by

$$p(t, c, y, z) = -f(c, y) + \left(\eta(t, c) + \frac{c f_{cc}(c, y)}{f_c(c, y)} \rho(t, c) + \frac{f_{cy}(c, y)}{f_c(c, y)} z \right)^* z, \tag{23}$$

for all $(t, c, y, z) \in [0, T] \times (0, \infty) \times \mathsf{R} \times \mathsf{R}^n$.

To be more explicit, one can verify easily by Itô's Lemma that if a function $\phi \in C([0, T] \times (0, \infty)) \cap C^{1,2}([0, T) \times (0, \infty))$ satisfies the PDE (22), then,

$$Y_t := \phi(t, c_t),\ and Z_t := c_t\ \rho(t, c_t)\ \phi_c(t, c_t),$$

solves the quadratic BSDE (9).

The following proposition characterizes the set of tastes and beliefs $(f, \gamma) \in \mathcal{I} \times \Upsilon$ that guarantee optimality of the consumption process defined in (21), under the assumption that the PDE (22) has a solution.

Proposition 3 *Consider a consumption process c that satisfies* (21)*, and $f \in \mathcal{I}$ and suppose that the function ϕ solves the PDE* (22)*. There exist beliefs $\gamma \in \Upsilon$ such that c is optimal for $\mathcal{P}^\gamma$ if and only if the function $k : [0, T] \times (0, \infty) \to \mathsf{R}$ defined as*

$$k(t, c) := \log\left(f_c(c, \phi(t, c))\right) \tag{24}$$

satisfies,

i) the PDE

$$\begin{cases} k_t(t,c) + \mathcal{M}k(t,c) - \frac{1}{2} \mid \rho(t,c) \mid^2 c^2 k_c^2(t,c) = -r(t,c) - f_y(c,\phi(t,c)), & \forall (t,c) \in [0,T) \times (0,\infty), \\ k(T,c) = \log\left(f_c(c,0)\right), \end{cases} \tag{25}$$

where the operator $\mathcal{M}$ *is defined for* $u \in C([0,T] \times (0,\infty)) \cap C^{1,2}([0,T) \times (0,\infty))$ *by*

$$\mathcal{M}u(t,c) = c(\mu(t,c) - \eta^*(t,c)\rho(t,c))u_c(t,c) + \frac{1}{2} \mid \rho(t,c) \mid^2 c^2 u_{cc}(t,c),$$

ii) and the integrability condition

$$E\left[\exp\left\{-\frac{1}{2}\int_0^t |\eta(t,c_t) + c_t k_c(t,c_t)\rho(t,c_t)|^2 ds - \int_0^t (\eta(t,c_t) + c_t k_c(t,c_t)\rho(t,c_t))^* dW_s\right\}\right] = 1, \tag{26}$$

for all $t \in [0,T]$.

Moreover, if the above conditions hold, the beliefs generator is given by $\gamma_t \equiv \gamma(t,c_t)$ *where the measurable function* $\gamma : [0,T] \times (0,\infty) \longrightarrow \mathbb{R}^n$ *is defined as*

$$\gamma(t,c) = \eta(t,c) + c k_c(t,c)\rho(t,c), \quad \forall (t,c) \in [0,T) \times (0,\infty).$$

Proof: Suppose that the function k defined in (24) satisfies (25) and (26). Making $Y_t := \phi(t,c_t)$, and $Z_t := c_t\, \rho(t,c_t)\, \phi_c(t,c_t)$, it is easy to see by applying Itô's Lemma that (Y,Z) solve the quadratic BSDE (9). Now, again by Itô's Lemma, the process

$$\xi_t := \frac{H_t e^{-\int_0^t f_y(c_s,Y_s)ds}}{f_c(c_t,Y_t)} \equiv H_t e^{-\int_0^t f_y(c_s,Y_s)ds - k(t,c_t)},$$

satisfies the stochastic differential equation,

$$\begin{aligned} \frac{d\xi_t}{\xi_t} &= -(k_t(t,c_t) + \mathcal{M}k(t,c_t) - \frac{1}{2} \mid \rho(t,c_t) \mid^2 c_t^2 k_c^2(t,c_t) + r(t,c_t) + f_y(c_t,Y_t))dt \\ &\quad -(\eta(t,c_t) + c_t k_c(t,c_t)\rho(t,c_t))^* \cdot dW_t \\ &= -\left[\eta(t,c_t) + c_t k_c(t,c_t)\rho(t,c_t)\right]^* \cdot dW_t, \end{aligned}$$

where the second equality follows from (25). Therefore, by the integrability condition (26), the process $(\xi_t, 0 \le t \le T)$ is a martingale and the result follows from Theorem 2.

The proof of necessity is similar. $\diamond$

When the utility is time-additive, the intertemporal aggregator takes the form $f(c,y) = u(c) - \beta y$, for some constant β and some increasing and concave u. In that case the function k defined in (24) takes the form $k(t,c) = \log(u'(c))$ which is time independent and independent of the function ϕ. Therefore, the PDE (22) is not restrictive for the inverse problem.

More importantly, in the additive utility case the optimality restriction (25) becomes

$$c(\mu(t,c) - \eta^*(t,c)\rho(t,c))k_c(c) + \frac{1}{2} \mid \rho(t,c) \mid^2 c^2 k_{cc}(c) - \frac{1}{2} \mid \rho(t,c) \mid^2 c^2 k_c^2(c) = -r(t,c) - \beta. \tag{27}$$

This is "preference free" since it may be seen as a differential equation with k as an unknown function, that has to be satisfied for each $t \in [0,T]$. For instance, when the functions μ, η, ρ and r are time independent, Cuoco and Zapatero (2000) show that this equation reduces to a Ricatti ordinary differential equation that, under mild conditions exhibits existence and uniqueness of the solutions. Cuoco and Zapatero (2000) use this approach to solve the recoverability problem (see Bick (1987, 1990) He and Leland (1996) and Decamps and Lazrak (2000)) in a continuous time, complete market, representative agent and Markovian aggregate consumption pure exchange economy.

Now, it appears that the restrictions that we obtain in the SDU (when the intertemporal aggregator is nonlinear with respect to y) are not preference free since (25) involves the knowledge of the function f_y as well as the function ϕ (which depends on the intertemporal aggregator f via (23)). Note also that this result holds when the beliefs are objective. The restriction in (25) becomes a tool for verification of the compatibility of a given intertemporal aggregator f and a given consumption process of the form (21), rather than a way to recover preferences and beliefs from a given consumption process. Of course, we cannot rule out the existence of a transformation that would allow recoverability of preferences and beliefs (maybe with additional restrictions to our Markovian setting). The existence of such a rule is not obvious, however.

6 Conclusions

We consider an inverse problem with unknown beliefs for an agent that has recursive utilities. We use some recent results in the theory of BSDE (in El Karoui, Peng and Quenez (1997, 1999) and Schroder and Skiadas (1999)) and show that a given consumption process might not be optimal for any parameterization of preferences, even if we allow beliefs to adjust. For logarithmic SDU, we show that a consumption process that satisfies some technical requirements might can be optimal for an infinite number of pairs of a specific parameterization of the logarithmic SDU and beliefs. We also derive a recoverability result in a Markovian setting. Some technical questions remain open. Namely, the main result of the paper involves the existence of a solution to a quadratic BSDE. The existence (and uniqueness) of a solution of this quadratic BSDE is established in the case of logarithmic or power SDU. However, in the case of non parametric intertemporal aggregator, it is not clear whether such a solution is guaranteed by some technical conditions or if it is an additional restriction (that does not seem to have a counterpart in the additive separable case).

7 Appendix

We now prove Lemma 1 and Lemma 2. The proofs rely heavily on ideas of Appendix A of Schroder and Skiadas (1999). First, we denote by $\mathcal{L}^2(\mathbb{R}^n)$ the set of $n-$dimensional progressively measurable processes X such that $\int_0^T |X_t|^2 dt < \infty$, a.s. Furthermore, the following subsets of $\mathcal{H}^2(\mathbb{R})$ (defined in Section 2) will be used:

$$\begin{aligned}
\mathcal{D}_0 &= \left\{ X \in \mathcal{H}^2(\mathbb{R}) \quad : \quad E ess \sup_t |X_t|^l < \infty, \quad \forall l \in \mathbb{R} \right\}, \\
\mathcal{D}_1 &= \left\{ X \in \mathcal{H}^2(\mathbb{R}) \quad : \quad E \int_0^T |X_t|^l dt < \infty, \quad \forall l > 0 \right\}, \\
\mathcal{D}_0^{\exp} &= \left\{ X \in \mathcal{H}^2(\mathbb{R}) \quad : \quad E(\exp(ess \sup_t l|X_t|^l) < \infty, \quad \forall l \in \mathbb{R} \right\}, \\
\mathcal{D}_1^{\exp} &= \left\{ X \in \mathcal{H}^2(\mathbb{R}) \quad : \quad E(\exp(l \int_0^T |X_t| dt) < \infty, \quad \forall l \in \mathbb{R} \right\}.
\end{aligned}$$

For each set $S \subset \mathcal{H}^2(\mathbb{R})$, we define $S^{++} = S \cap \mathcal{H}^{2++}(\mathbb{R})$, where $\mathcal{H}^{2++}(\mathbb{R})$ is the strictly positive cone of $\mathcal{H}^2(\mathbb{R})$, that is $\mathcal{H}^{2++}(\mathbb{R}) = \left\{ X \in \mathcal{H}^2(\mathbb{R}) \quad : \quad X_t > 0, \quad dP \otimes dt \ a.s. \right\}$.

Proposition 4 *(Schroder and Skiadas (1999)) Suppose that* $U \in \mathcal{D}_1^{\exp}$ *and* $\beta > 0$*. Then, there exists a unique pair* $(Y, Z) \in \mathcal{D}_0^{\exp} \times \mathcal{H}^2(\mathbb{R}^n)$ *that solves the quadratic BSDE*

$$-dY_t = \left(U_t - \beta Y_t + \frac{1}{2} \mid Z_t \mid^2 \right) dt - Z_t^* dW_t, \qquad Y_T = 0.$$

Proposition 5 *(Schroder and Skiadas (1999)) Suppose that* $U \in \mathcal{D}_1^{++}$ *and* $m > -1$*. Then, there exists a unique pair* $(Y, Z) \in \mathcal{D}_0^{++} \times \mathcal{L}^2(\mathbb{R}^n)$ *that solves the quadratic BSDE*

$$-dY_t = \left(U_t + \frac{m}{2} \frac{\mid Z_t \mid^2}{Y_t} \right) dt - Z_t^* dW_t, \qquad Y_T = 0.$$

It will be convenient, for each consumption process $c \in \mathcal{D}$ of the form (7) that satisfies assumption $A7$, to express the BSDE in terms of the probability measures P_k, defined by the Radon Nikodym derivative

$$\frac{dP_k}{dP} = \xi_T^{\delta^k},$$

where the process ξ^{δ^k} is defined in (14) and $\delta_t^k := \eta_t - k\rho_t$. Note that by the Girsanov Theorem, the process $W_t^{\delta^k} = W_t + \int_0^t \delta_s^k ds$ is a Brownian motion under the probability P_k.

Proof of Lemma 1: Given any $c \in \mathcal{D}$ that satisfies assumption $A7$, we define the process $U_t = -\alpha \log(c_t)$. Now, for each $l \in \mathbb{R}$, we have

$$\begin{aligned}
\left(E_{P_1} \left(\exp(l \int_0^T |U_t| dt) \right) \right)^2 &= \left(E \left(\xi_T^{\delta^1} \exp(l \int_0^T |U_t| dt) \right) \right)^2 \\
&\leq E \left[\left(\xi_T^{\delta^1} \right)^2 \right] E \left(\exp \left(\frac{1}{T} \int_0^T |\log(c_t^{2\alpha l T})| dt \right) \right) \\
&\leq E \left[\left(\xi_T^{\delta^1} \right)^2 \right] \frac{1}{T} E \left(\int_0^T \exp |\log(c_t^{2\alpha l T})| dt \right) \\
&\leq E \left[\left(\xi_T^{\delta^1} \right)^2 \right] \frac{1}{T} E \left(\int_0^T \left[c_t^{2\alpha l T} + c_t^{-2\alpha l T} \right] dt \right) < \infty,
\end{aligned}$$

where we have used the Cauchy Schwarz inequality, the Jensen inequality, the inequality $\exp(|x|) \leq \exp(x) + \exp(-x)$ and the fact that $c \in \mathcal{D}$. It follows that $U \in \mathcal{D}_1^{\exp,1}$ and therefore, it follows from Proposition 4 that the BSDE[8]

$$-dY_{1,t} = (U_t - \beta Y_{1,t} + \frac{1}{2} Z_{1,t}^* \cdot Z_{1,t})dt - Z_{1,t}^* dW_t^{\delta^1}, \qquad Y_{1,T} = 0, \tag{28}$$

has a unique solution $(Y_1, Z_1) \in \mathcal{D}_0^{\exp,1} \times \mathcal{H}^{2,1}(\mathbb{R}^n)$. Now, by Itô's Lemma it is easy to show that the process $Y_t := \frac{\exp(-Y_{1,t})-1}{\alpha}$ solves the BSDE (15) with the intensity $Z_t := -\frac{1+\alpha Y}{\alpha} Z_{1,t}$.

Now, in order to establish the integrability condition (16), we define the process $M_{1,t} = 1 + \int_0^t M_{1,s} Z_{1,s}^* dW_s^{\delta^1}$. The process M_1 is a local martingale under the probability P_1, and therefore there exists an increasing sequence of stopping times $\{\tau(n)\}$ that converges to T, such that the stopped process $\{M_{1,t\wedge\tau(n)},\quad 0 \leq t \leq T\}$ is a martingale, for every n. Following Schroder and Skiadas (1999, Lemma A1), we integrate (28) between any $0 \leq t \leq u \leq T$ and take the exponential to obtain

$$\frac{M_{1,u}}{M_{1,t}} \exp(Y_{1,t}) = \exp\left(Y_{1,u} + \int_t^u (U_s - \beta Y_{1,s}) ds \right). \tag{29}$$

Now, by the optional sampling theorem, $M_{1,t} = E_t^{P_1}(M_{1,t\wedge\tau(n)})$, and therefore, when $u = \tau(n)$, the identity (29) leads to

$$\exp(Y_{1,t}) = E_t^{P_1}\left(\exp\left(Y_{1,\tau(n)} + \int_t^{\tau(n)} (U_s - \beta Y_{1,s}) ds \right)\right), \tag{30}$$

on the event $\{\tau(n) \geq t\}$. Letting $n \to \infty$ in (30), the dominated convergence theorem (made possible by the fact that $Y_1 \in \mathcal{D}_0^{\exp,1}$ and $U \in \mathcal{D}_1^{\exp,1}$) yields

$$\exp(Y_{1,t}) = E_t^{P_1}\left(\exp\left(\int_t^T (U_s - \beta Y_{1,s}) ds \right)\right),$$

which, in conjunction with (29) when $u = T$, implies that $M_{1,t} = E_t^{P_1}(M_{1,T})$ and, hence, M_1 is after all a true martingale under the probability P_1. Therefore, from Bayes rule (Karatzas and Shreve (1988), Lemma 5.3, page 193), the process $M_t := \xi_t^{\delta^1} M_{1,t}$ is a martingale under the probability P and, since by Itô's Lemma,

$$dM_t = -M_t(\delta_t^1 - Z_{1,t})^* dW_t \equiv -M_t(\eta_t - \rho_t + \frac{\alpha}{1+\alpha Y_t} Z_t)^* dW_t,$$

the integrability condition (16) holds. ⋄

Proof of Lemma 2: Letting $U_t = \frac{c_t^\nu}{\nu} e^{-\beta t}$, it follows from Proposition 5, that if $a < 1$ (this is satisfied under $A6$) the BSDE

$$-dY_{a,\nu,t} = (\, U_t - \frac{1}{2} a Y_{a,\nu,t}^{-1} Z_{a,\nu,t}{}^* Z_{a,\nu,t}\,)dt - Z_{a,\nu,t}^* dW_t^{\delta^{1-\nu}}, \qquad Y_{a,\nu,T} = 0,$$

[8] Here, the sets $\mathcal{D}_1^{\exp,1}$ and $\mathcal{D}_0^{\exp,1}$ as well as the set $\mathcal{H}^{2,1}(\mathbb{R}^n)$ are defined with respect to the probability P_1 instead of the probability P.

has a unique solution in $(Y_{a,\nu}, Z_{a,\nu}) \in \mathcal{D}_0^{++} \times \mathcal{L}^2(\mathsf{R}^n)$.[9] Now, by Itô's Lemma, it is easy to show that the process $Y_t := Y_{a,\nu,t}^{1+a} e^{(1+a)\beta t}$ solves the BSDE (18) with the intensity $Z_t := (1+a)e^{(1+a)\beta t} Y_{a,\nu,t}^{a} Z_{a,\nu,t}$. $\diamond$

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