

Evaluation of American Strangles

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September 18, 2002

Abstract

This paper presents a generalisation of McKean's free boundary value problem for American options by considering an American strangle position, where exercising one side of the payoff early knocks-out the remaining side. The Fourier transform technique is used to derive a coupled integral equation system for the strangle's free boundaries. A numerical algorithm is provided to solve this system, and these free boundaries are then used to determine the price of the American strangle position. Numerical comparisons between the strangle price and the price of a portfolio formed using a long American call and a long American put option are presented.

JEL classification **C61, D11**

Keywords: American options, Volterra integral equation, free-boundary problem.

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1 Introduction

American options are highly common derivative securities in today's financial markets. American calls and puts are frequently written on a range of underlying assets, including stocks, futures, and foreign exchange rates. Since the ground-breaking results of Merton (1973) and Black and Scholes (1973) regarding the analytic pricing of European call and put options, a great deal of research has been conducted into applying the Black-Scholes framework to American options. While McKean (1965) and Kim (1990) successfully extended the Black-Scholes European option pricing methodology to American calls and puts, the method has never been generalised to allow a broader range of payoff-functions. This paper extends the results of McKean and Kim to a special type of American strangle position, where the early exercise of one side of the position will knock-out the remaining side. Through this example, a general American option pricing framework for convex payoffs is provided.

The fundamental difference between American and European options is that an American option can be exercised at any time up to and excluding the expiry date. Although this difference is conceptually simple, it adds a large degree of mathematical complexity to the option pricing problem. Using mathematical results from Kolodner (1956), McKean (1965) first derived the integral equation for both the price and early exercise boundary of an American call option as the solution to a free boundary problem. This was a natural extension of the Black-Scholes method for European options, and was explored further by Van Moerbeke (1974, 1976). A range of alternative methods based on the Black-Scholes partial differential equation (PDE) were proposed, including the quadratic approximation of Barone-Adesi and Whaley (1987), and the compound option approach of Geske and Johnson (1984). Ho, Stapleton and Subrahmanyam (1997) extended the Geske-Johnson technique to include stochastic interest rates. Other methods that have been considered include the finite difference method (Brennan and Schwartz, 1977) and the binomial approximation (Parkinson, 1977).

Karatzas (1988) was one of the first to re-visit the topic of pricing American options as a solution to a free boundary problem. Kim (1990) reproduced McKean's results, extending them to

the American put case, and deriving the exact behaviour of the early exercise boundary near expiry. By taking the limit of the Geske-Johnson method as the number of early exercise dates is increased to infinity, he found a new set of integral equations for the free boundaries and prices of American calls and puts. These new integral equations were proven to be mathematically equivalent to those found by McKean, and it was possible to express McKean's integral equations in Kim's form using integration by parts. While McKean's integral equations follow naturally from applying the Fourier transform technique to the Black-Scholes partial differential equation, they involve the derivative of the free boundary, which is difficult to handle numerically. Furthermore there is no obvious economic interpretation available for McKean's integral equations. Kim's integral equations are free of the derivative of the free boundary, and their structure decomposes the value of an American option into the sum of its European value and its early exercise premium. Thus it is possible to apply McKean's approach (which has the advantage of being more straight forward to apply to complex payoffs, as we shall see below) to extend the Black-Scholes analysis to American options, while then transforming to Kim's representation to express the integral equations in a more numerically tractable and economically intuitive form. Several important papers on American options were published around the time of Kim's findings, exploring American option prices in the context of free boundary problems. Kim's results were confirmed from a different approach by Elliot, Myneni and Viswanathan (1990). Jacka (1991) proved both existence and uniqueness for the American put option price and early exercise boundary. Carr, Jarrow and Myneni (1992) expressed the American option price as the sum of its early exercise and intrinsic value. Jamshidian (1989a, 1989b, 1992) has also conducted extensive research into the free boundary method for pricing American calls and puts.

Though Kim's integral equations for American call and put prices could be interpreted as the sum of their European values and their early exercise premiums, the early exercise boundary could not be found explicitly. Numerical solutions for the early exercise boundary have been considered by many authors. Underwood and Wang (2000) implemented Kim's equations using a fixed-point iterative scheme, which was prone to slow convergence. Huang, Subrahmanyam and Yu (1996) used a recursive method. They estimated the free boundary at a

small number of time points, combining the Geske-Johnson compound option technique with analytic valuation formulae. The complete early exercise boundary was then developed using Richardson extrapolation. This approximation was used in the analytic integral equations to find the American option price.

Ju (1998) approximated the free boundary using a piece-wise exponential function, finding that the prices obtained using these estimates were highly accurate. AitSahalia and Lai (2001) used linear splines with very few knots to approximate the free boundary. Solving a transformed free boundary integral equation at several time points, they estimated the entire free boundary using linear interpolation, and were able to obtain very accurate American option prices using these early exercise boundary estimates. As a consequence of their own numerical findings, these authors postulated that the price of an American option was not overly sensitive to the early exercise boundary approximation.

Despite the large amount of research conducted into the American option pricing problem, there is still no clear framework with which one can derive the integral equations and free boundaries for a generic payoff function, either monotonic, convex or concave. It should be noted that Elliot, Myneni and Viswanathan (henceforth EMV) (1990) considered the American straddle pricing problem, deriving the coupled integral equation system for the straddle's free boundaries. While they used probability theory in their analysis, in this paper we revisit McKean's incomplete Fourier transform method for American call options and derive the coupled integral equation system via this approach. We extend on the findings of EMV by applying McKean's method to a special kind of American strangle. If exercised early, the entire payoff is optimally realised, making this fundamentally different to an American strangle formed using individual calls and puts. Thus the strangle under consideration is the analogue of the straddle considered by EMV.

The American strangle is an example of a more general American option position with a convex payoff function and indeed the methodology developed in this paper may be applied to evaluate such positions. The application of McKean's method is based on the exposition in Kucera and Ziogas (2002), who amplify the analysis of the incomplete Fourier transform

techniques used by McKean. We also transform the results into Kim’s integral equation form, and then proceed to implement these equations numerically to find firstly the strangle’s early exercise boundaries, and finally the strangle’s price.

Such a strangle has several implicit advantages over a “traditional” American strangle constructed using a long position in both an American call and an American put. The new strangle is self-closing, since exercising one side of the position will knock-out the other. These implicit knock-out barriers will make this strangle cheaper than a “traditional” one, and may have market applications within pure-volatility strategies. It is important to note that the proposed strangle loses the flexibility to be decomposed into its component options. An analytic delta exists for the strangle, and the strangle can be hedged in the same manner as any American call or put.

The remainder of this paper is structured as follows. Section 2 outlines McKean’s free boundary problem that arises from this paper’s American strangle option pricing problem. Section 3 applies the incomplete Fourier transform to solve the PDE in terms of a transform variable. The transform is inverted in Section 4, to provide McKean’s integral equations for the American strangle price, and a corresponding integral equation system for the strangle’s two early exercise boundaries. Section 5 outlines the numerical solution method for both the free boundaries and strangle price, including the transform from McKean’s equations to Kim’s equations. A selection of numerical results are provided in Section 6, with concluding remarks presented in Section 7.

2 Problem Statement

Let $A_{a_1, a_2}(S, t)$ be the price of an American strangle position written on an underlying asset with price S at time t , with time to expiry $(T - t)$. This position is formed using a long put with strike K_1 , and a long call with strike K_2 . Note that $K_1 < K_2$. Let the early exercise boundary on the put side be denoted by $a_1(t)$, and the early exercise boundary on the call side

be denoted by $a_2(t)$. Figure 1 demonstrates the payoff and continuation region for $A_{a_1, a_2}(S, t)$.

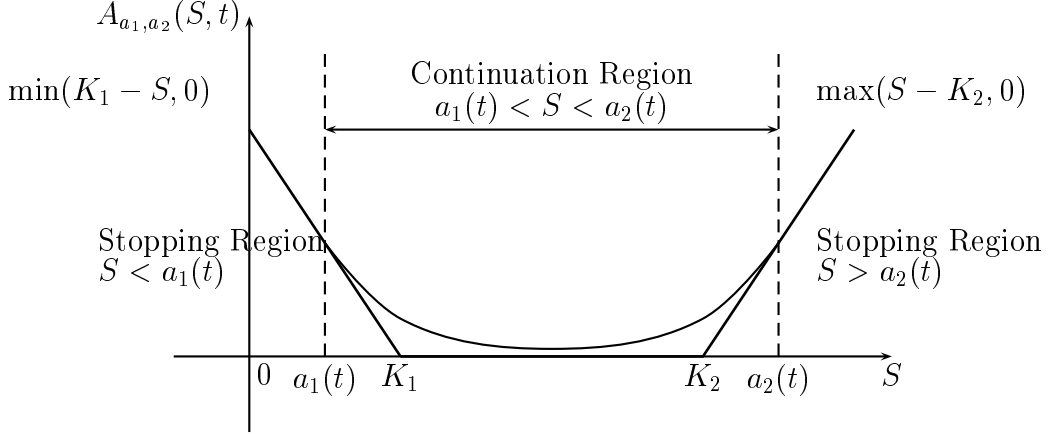


Figure 1: Continuation region in S-space

Under the assumption that the price of the underlying asset is driven by geometric Brownian motion

$$dS = \mu S dt + \sigma S dW$$

with drift μ , volatility σ and Wiener process increments dW , it is known that A satisfies the Black-Scholes PDE:

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 A}{\partial S^2} + (r - q)S \frac{\partial A}{\partial S} - rA = 0, \quad 0 \leq t \leq T \quad (1)$$

in the region $a_1(t) < S < a_2(t)$, where

r = risk-free rate, and

q = dividend rate of S (continuously compounded),

subject to the following final time and boundary conditions:-

$$A_{a_1, a_2}(S, T) = \max(K_1 - S, 0) + \max(S - K_2, 0), \quad 0 < S < \infty \quad (2)$$

$$A_{a_1, a_2}(a_1(t), t) = K_1 - a_1(t), \quad t \geq 0 \quad (3)$$

$$A_{a_1, a_2}(a_2(t), t) = a_2(t) - K, \quad t \geq 0 \quad (4)$$

$$\lim_{S \rightarrow a_1(t)} \frac{\partial A}{\partial S} = -1, \quad t \geq 0 \quad (5)$$

$$\lim_{S \rightarrow a_2(t)} \frac{\partial A}{\partial S} = 1, \quad t \geq 0. \quad (6)$$

Condition (2) is the payoff function for the strangle at expiry, while conditions (3)-(6) are collectively known as the “smooth-pasting” conditions. These ensure that the price, $A_{a_1, a_2}(S, \tau)$, and its first derivative with respect to S are both continuous. This is necessary to maintain the Black-Scholes assumption of an arbitrage-free market.

Firstly, we shall transform the PDE (1) to a forward-in-time equation with constant coefficients. Let

$$A_{a_1, a_2}(S, t) = V_{c_1, c_2}(x, \tau), \quad (7)$$

where

$$S = e^x, \quad t = T - \tau. \quad (8)$$

The transformed PDE for V is then

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + k \frac{\partial V}{\partial x} - rV, \quad 0 \leq \tau \leq T, \quad (9)$$

in the region $\ln c_1(\tau) < x < \ln c_2(\tau)$ where

$$k = r - q - \frac{1}{2}\sigma^2,$$

and the transformed free boundaries are given by

$$c_1(\tau) = a_1(t) \text{ and } c_2(\tau) = a_2(t).$$

The transformed initial and boundary conditions are

$$V_{c_1, c_2}(x, 0) = \max(K_1 - e^x, 0) + \max(e^x - K_2, 0), \quad -\infty < x < \infty \quad (10)$$

$$V_{c_1, c_2}(\ln c_1(\tau), \tau) = K_1 - c_1(\tau), \quad \tau \geq 0 \quad (11)$$

$$V_{c_1, c_2}(\ln c_2(\tau), \tau) = c_2(\tau) - K_2, \quad \tau \geq 0 \quad (12)$$

$$\lim_{x \rightarrow \ln c_1(\tau)} \frac{\partial V}{\partial x} = -c_1(\tau) \quad (13)$$

$$\lim_{x \rightarrow \ln c_2(\tau)} \frac{\partial V}{\partial x} = c_2(\tau). \quad (14)$$

In what follows, we will use the notation $c_1 \equiv c_1(\tau)$ and $c_2 \equiv c_2(\tau)$ for simplicity.

In order to be able to apply transform methods to solve this PDE for $V_{c_1, c_2}(x, \tau)$, the x domain shall be extended to $-\infty < x < \infty$ by expressing the PDE as

$$H(\ln c_2 - x)H(x - \ln c_1) \left(\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - k \frac{\partial V}{\partial x} + rV \right) = 0,$$

$$-\infty < x < \infty, \quad 0 \leq \tau \leq T,$$

where $H(x)$ is the Heaviside step function, defined as

$$H(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0. \end{cases} \quad (15)$$

The reason for the appearance of the factor of $\frac{1}{2}$ at the point of discontinuity is explained below. The initial and boundary conditions remain unchanged.

3 Applying the Fourier Transform

We propose to solve the problem defined by equations (9)-(14) by using the Fourier transform technique to reduce the PDE to an ODE, whose solution is readily obtainable. Note that in using the Fourier transform approach, we employ all the standard techniques that apply when solving the Black-Scholes PDE for the price of European options. This includes the assumption that for the purposes of the transform method, the function V and its first two derivatives with respect to x can be treated as zero when x tends to $\pm\infty$. The assumption is subsequently justified by virtue of the facts that the solution obtained satisfies the partial

differential equation and associated boundary conditions, and that the solution is unique.

Since the x domain is now $-\infty < x < \infty$, the Fourier transform of the PDE can be found.

The Fourier transform of V , $\mathcal{F}\{V_{c_1, c_2}(x, \tau)\}$, is defined as

$$\mathcal{F}\{V_{c_1, c_2}(x, \tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V_{c_1, c_2}(x, \tau) dx.$$

Thus, the transformed PDE appears as:

$$\begin{aligned} \mathcal{F}\{H(\ln c_2 - x)H(x - \ln c_1)\frac{\partial V}{\partial \tau}\} &= \frac{1}{2}\sigma^2 \mathcal{F}\{H(\ln c_2 - x)H(x - \ln c_1)\frac{\partial^2 V}{\partial x^2}\} \\ &+ k \mathcal{F}\{H(\ln c_2 - x)H(x - \ln c_1)\frac{\partial V}{\partial x}\} \\ &- r \mathcal{F}\{H(\ln c_2 - x)H(x - \ln c_1)V\}. \end{aligned}$$

By definition

$$\begin{aligned} &\mathcal{F}\{H(\ln c_2 - x)H(x - \ln c_1)V_{c_1, c_2}(x, \tau)\} \\ &= \int_{-\infty}^{\infty} e^{i\eta x} H(\ln c_2 - x)H(x - \ln c_1)V_{c_1, c_2}(x, \tau) dx \\ &= \int_{\ln c_1}^{\ln c_2} e^{i\eta x} V_{c_1, c_2}(x, \tau) dx \\ &\equiv \mathcal{F}^c\{V_{c_1, c_2}(x, \tau)\}, \end{aligned} \tag{16}$$

and for convenience, let us introduce the notation

$$\hat{V}_{c_1, c_2}(\eta, \tau) \equiv \mathcal{F}^c\{V_{c_1, c_2}(x, \tau)\}.$$

We note that, \mathcal{F}^c is an incomplete Fourier transform, since it is equivalent to a standard Fourier transform applied to $V_{c_1, c_2}(x, \tau)$ in the x -domain of $\ln c_1 < x < \ln c_2$. In Appendix A we show how the incomplete Fourier transform may be derived as a consequence of the standard Fourier transform and there derive the corresponding inversion theorem.

To apply the incomplete Fourier transform to the PDE (9), we need to consider three specific properties of \mathcal{F}^c .

Proposition 1:

Given the definition of \mathcal{F}^c in equation (16), the following identities exist for \mathcal{F}^c :

$$\mathcal{F}^c \left\{ \frac{\partial V}{\partial x} \right\} = (c_2 - K_2)e^{i\eta \ln c_2} - (K_1 - c_1)e^{i\eta \ln c_1} - i\eta \hat{V}; \quad (17)$$

$$\mathcal{F}^c \left\{ \frac{\partial^2 V}{\partial x^2} \right\} = e^{i\eta \ln c_2} (c_2 - i\eta(c_2 - K_2)) - e^{i\eta \ln c_1} (-c_1 - i\eta(K_1 - c_1)) - \eta^2 \hat{V}; \quad (18)$$

$$\mathcal{F}^c \left\{ \frac{\partial V}{\partial \tau} \right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{c_2'}{c_2} e^{i\eta \ln c_2} (c_2 - K_2) + \frac{c_1'}{c_1} e^{i\eta \ln c_1} (K_1 - c_1). \quad (19)$$

Proof: Refer to Appendix B.1. Note that in deriving the above results, we make use of the so-called “smooth pasting” conditions given in equations (11)-(14).

□

The PDE can now be transformed, as required.

Proposition 2:

The incomplete Fourier transform of the PDE (9) with respect to x satisfies the ordinary differential equation

$$\frac{d\hat{V}}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + k i\eta + r \right) \hat{V} = F(\eta, \tau), \quad (20)$$

where

$$F(\eta, \tau) = e^{i\eta \ln c_2} \left[\frac{\sigma^2 c_2}{2} + \left(\frac{c_2'}{c_2} - \frac{\sigma^2 i\eta}{2} + k \right) (c_2 - K_2) \right] - e^{i\eta \ln c_1} \left[-\frac{\sigma^2 c_1}{2} + \left(\frac{c_1'}{c_1} - \frac{\sigma^2 i\eta}{2} + k \right) (K_1 - c_1) \right], \quad (21)$$

with initial condition

$$\mathcal{F}\{V_{c_1, c_2}(x, 0)\} \equiv \hat{V}_{c_1, c_2}(\eta, 0).$$

being calculated from equation (10).

Proof: Refer to Appendix B.2.

□

Instead of solving a PDE for the function $V_{c_1, c_2}(x, \tau)$, we are now faced with the simpler task of solving the ODE (20) for the function $\hat{V}_{c_1, c_2}(\eta, \tau)$. This can then be inverted via the Fourier inversion theorem (see Appendix A) to recover the desired function $V_{c_1, c_2}(x, \tau)$. Before concluding this section, we obtain the solution to (20).

Proposition 3:

The solution for $\hat{V}_{c_1, c_2}(\eta, \tau)$ is given by

$$\hat{V}_{c_1, c_2}(\eta, \tau) = \hat{V}_{c_1, c_2}(\eta, 0)e^{-\left(\frac{1}{2}\sigma^2\eta^2 + k\eta + r\right)\tau} + \int_0^\tau e^{-\left(\frac{\sigma^2\eta^2}{2} + k\eta + r\right)(\tau-s)} F(\eta, s) ds. \quad (22)$$

Proof: Recalling that c_1 and c_2 are functions of τ , the ODE (20) is of the form

$$\frac{d\hat{V}}{d\tau} + b_1(\eta)\hat{V} = b_2(\eta, \tau).$$

Using the integrating factor $e^{b_1(\eta)\tau}$, the ODE becomes

$$\frac{d}{d\tau} \left(\hat{V} e^{b_1(\eta)\tau} \right) = b_2(\eta, \tau) e^{b_1(\eta)\tau},$$

whose solution may be expressed as

$$\hat{V}_{c_1, c_2}(\eta, \tau) e^{b_1(\eta)\tau} - \hat{V}_{c_1, c_2}(\eta, 0) = \int_0^\tau b_2(\eta, s) e^{b_1(\eta)s} ds.$$

Referring back to the original ODE, the solution for $\hat{V}(\eta, \tau)$ is found to be equation (22).

□

4 Inverting the Fourier Transform

Having now found $\hat{V}_{c_1, c_2}(\eta, \tau)$, it is necessary to recover $V_{c_1, c_2}(x, \tau)$, the American strangle price in the x - τ plane. Taking the inverse (complete) Fourier transform of (22) gives

$$\begin{aligned} V_{c_1, c_2}(x, \tau) &= \mathcal{F}^{-1}\{\hat{V}(\eta, 0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}\} \\ &\quad + \mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{\sigma^2\eta^2}{2} + ki\eta + r)(\tau-s)} F(\eta, s) ds\right\} \\ &\equiv V_{c_1, c_2}^{(1)}(x, \tau) + V_{c_1, c_2}^{(2)}(x, \tau); \quad \ln c_1(\tau) < x < \ln c_2(\tau). \end{aligned}$$

We must now determine $V_{c_1, c_2}^{(1)}(x, \tau)$ and $V_{c_1, c_2}^{(2)}(x, \tau)$.

Proposition 4:

The function $V_{c_1, c_2}^{(1)}(x, \tau)$ is given by

$$\begin{aligned} V_{c_1, c_2}^{(1)}(x, \tau) &= [K_1 e^{-r\tau} N(-\hat{d}_2(x, \tau; K_1)) - e^x e^{-q\tau} N(-\hat{d}_1(x, \tau; K_1))] \\ &\quad + [e^x e^{-q\tau} N(\hat{d}_1(x, \tau; K_2)) - K_2 e^{-r\tau} N(\hat{d}_2(x, \tau; K_2))] \\ &\quad - [K_1 e^{-r\tau} N(-\hat{d}_2(x, \tau; c_1(0))) - e^x e^{-q\tau} N(-\hat{d}_1(x, \tau; c_1(0)))] \\ &\quad - [e^x e^{-q\tau} N(\hat{d}_1(x, \tau; c_2(0))) - K_2 e^{-r\tau} N(\hat{d}_2(x, \tau; c_2(0)))], \quad (23) \end{aligned}$$

where

$$\begin{aligned} \hat{d}_1(x, \tau; \beta) &= \frac{x - \ln \beta + \left(r - q + \frac{\sigma^2}{2}\right) \tau}{\sigma \sqrt{\tau}}, \\ \hat{d}_2(x, \tau; \beta) &= \hat{d}_1(x, \tau; \beta) - \sigma \sqrt{\tau}, \end{aligned}$$

and

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\alpha^2}{2}} d\alpha.$$

Proof: Refer to Appendix C.1.

□

Proposition 5:

The function $V_{c_1, c_2}^{(2)}(x, \tau)$ is given by

$$V_{c_1, c_2}^{(2)}(x, \tau) = \int_0^\tau \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} [e^{-h_2(x,s)}Q_2(x, s) + e^{-h_1(x,s)}Q_1(x, s)]ds, \quad (24)$$

where

$$h_j(x, s) = \frac{(x - \ln c_j(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)}, \quad (25)$$

and

$$Q_j(x, s) = \frac{\sigma^2 c_j(s)}{2} + \left(\frac{c_j'(s)}{c_j(s)} + \frac{1}{2} \left[k - \frac{(x - \ln c_j(s))}{(\tau - s)} \right] \right) (c_j(s) - K_j) \quad (26)$$

for $j = 1, 2$ and $\ln c_1(\tau) < x < \ln c_2(\tau)$

Proof: Refer to Appendix C.2.

□

Hence, the value of the American strangle is given in the x - τ plane by

$$V_{c_1, c_2}(x, \tau) = V_{c_1, c_2}^{(1)}(x, \tau) + V_{c_1, c_2}^{(2)}(x, \tau), \quad (27)$$

$$0 \leq \tau \leq T; \quad \ln c_1(\tau) < x < \ln c_2(\tau).$$

Equation (27) expresses the value of the American strangle position in terms of the early exercise boundaries $c_1(\tau)$ and $c_2(\tau)$. At this point these remain unknown, but we are able to obtain an integral equation system that determines them by requiring the expression for $V_{c_1, c_2}(x, \tau)$ to satisfy the early exercise boundary conditions (11) and (12). Recalling our definition for the Heaviside function, the following integral equation system is obtained:

$$\frac{c_2(\tau) - K_2}{2} = V_{c_1, c_2}(\ln c_2(\tau), \tau), \quad (28)$$

$$\frac{K_1 - c_1(\tau)}{2} = V_{c_1, c_2}(\ln c_1(\tau), \tau), \quad (29)$$

where $V_{c_1, c_2}(x, \tau)$ is given by equation (27) in conjunction with (23)-(26). The factor of $\frac{1}{2}$ appears in the left hand side of (28) and (29) due to properties of the Fourier transform. Recall that the complete Fourier transform was applied to the discontinuous functions of the form $H(\ln c_1 - x)H(x - \ln c_2)f(x, \tau)$. As proved in Dettman (1965, p.360), the inverted Fourier transform of a discontinuous function will converge to the midpoint of the discontinuity. Thus in equations (28)-(29), when V is evaluated at either $\ln c_1(\tau)$ or $\ln c_2(\tau)$, the factor of $\frac{1}{2}$ must be introduced into the left hand side. This is accounted for by our Heaviside function definition in equation (15).

The system of integral equations (28) and (29) must be solved simultaneously using numerical methods to find $c_1(\tau)$ and $c_2(\tau)$ since analytical solutions seem impossible. Once these are found, it is a simple matter to evaluate $V_{c_1, c_2}(x, \tau)$ via numerical integration.

5 Numerical Implementation

To make the task of numerical implementation less complicated, we will transform equations (23)-(27) into the form presented by Kim (1990). This essentially has the effect of removing the $c'_1(\tau)$ and $c'_2(\tau)$ terms from the integral by using integration by parts. The first step is to re-write the pricing equation in terms of the original underlying asset S .

Proposition 6:

The solution to the boundary value problem (1)-(6) in terms of S and τ is given by

$$A_{c_1, c_2}(S, \tau) = A_{c_1, c_2}^{(1)}(S, \tau) + A_{c_1, c_2}^{(2)}(S, \tau), \quad (30)$$

where

$$\begin{aligned} A_{c_1, c_2}^{(1)}(S, \tau) &= [K_1 e^{-r\tau} N(-d_2(A, \tau; K_1)) - S e^{-q\tau} N(-d_1(S, \tau; K_1))] \\ &+ [S e^{-q\tau} N(d_1(S, \tau; K_2)) - K_2 e^{-r\tau} N(d_2(x, \tau; K_2))] \end{aligned}$$

$$\begin{aligned}
& -[K_1 e^{-r\tau} N(-d_2(S, \tau; c_1(0))) - S e^{-q\tau} N(-d_1(S, \tau; c_1(0)))] \\
& -[S e^{-q\tau} N(d_1(S, \tau; c_2(0))) - K_2 e^{-r\tau} N(d_2(S, \tau; c_2(0)))] , \quad (31)
\end{aligned}$$

with

$$\begin{aligned}
d_1(S, \tau; \beta) &= \frac{\ln(S/\beta) + \left(r - q + \frac{\sigma^2}{2}\right) \tau}{\sigma \sqrt{\tau}}, \\
d_2(S, \tau; \beta) &= d_1(S, \tau; \beta) - \sigma \sqrt{\tau},
\end{aligned}$$

and

$$A_{c_1, c_2}^{(2)}(S, \tau) = \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma \sqrt{2\pi(\tau-\xi)}} [e^{-\hat{h}_2(S, \xi)} \hat{Q}_2(S, \xi) + e^{-\hat{h}_1(S, \xi)} \hat{Q}_1(S, \xi)] d\xi, \quad (32)$$

with

$$\hat{h}_j(S, \xi) = \frac{(\ln(S/c_j(\xi)) + k(\tau - \xi))^2}{2\sigma^2(\tau - \xi)}, \quad (33)$$

and

$$\hat{Q}_j(S, \xi) = \frac{\sigma^2 c_j(\xi)}{2} + \left(\frac{c_j'(\xi)}{c_j(\xi)} + \frac{1}{2} \left[k - \frac{\ln(S/c_j(\xi))}{(\tau - \xi)} \right] \right) (c_j(\xi) - K_j) \quad (34)$$

for $j = 1, 2$ and $c_1(\tau) < S < c_2(\tau)$.

Proof: Recall that $x = \ln S$, and substitute this into equations (23)-(27).

□

Proposition 7:

Equations (30)-(34) can be expressed as follows:

$$A_{c_1, c_2}(S, \tau) = AP(S, \tau) + AC(S, \tau), \quad (35)$$

where

$$\begin{aligned}
AP(S, \tau) &= K_1 e^{-r\tau} N(-d_2(S, \tau; K_1)) - S e^{-q\tau} N(-d_1(S, \tau; K_1)) \\
&+ \int_0^\tau [K_1 r e^{-r(\tau-\xi)} N(-d_2(S, \tau - \xi; c_1(\xi))) \\
&\quad - S q e^{-q(\tau-\xi)} N(-d_1(S, \tau - \xi; c_1(\xi)))] d\xi, \tag{36}
\end{aligned}$$

and

$$\begin{aligned}
AC(S, \tau) &= S e^{-q\tau} N(d_1(S, \tau; K_2)) - K_2 e^{-r\tau} N(d_2(S, \tau; K_2)) \\
&+ \int_0^\tau [S q e^{-q(\tau-\xi)} N(d_1(S, \tau - \xi; c_2(\xi))) \\
&\quad - K_2 r e^{-r(\tau-\xi)} N(d_2(S, \tau - \xi; c_2(\xi)))] d\xi. \tag{37}
\end{aligned}$$

Proof: The above is derived using integration by parts, as outlined in the appendix of Kim (1990). Note that equations (36)-(37) do not involve c'_1 or c'_2 .

□

The integral equation system for the free boundaries $c_1(\tau)$ and $c_2(\tau)$ is now

$$c_2(\tau) - K_2 = A_{c_1, c_2}(c_2(\tau), \tau), \tag{38}$$

$$K_1 - c_1(\tau) = A_{c_1, c_2}(c_1(\tau), \tau). \tag{39}$$

It is of value to note that equation (35) is simply the sum of the integral pricing equations for an American put and an American call option. The added complexity in pricing an American strangle therefore arises from the early exercise boundaries. Each free boundary is dependent upon the other free boundary in the system (38)-(39), and therefore these boundaries are not equal to those found when valuing an American call and put option separately. Thus it is important to understand the nature of the early exercise boundaries for American option portfolios in order to obtain the correct early exercise boundary values.

To solve this system, we propose using a numerical scheme similar to that usually used for Volterra integral equations. Firstly, discretise the time variable τ into n equally spaced intervals of length h . Thus $\tau_i = ih$ for $i = 0, 1, 2, \dots, n$, and $h = T/n$. Following the methods of Kim (1990), it can be readily shown that the initial values are given by (see Appendix E)

$$c_1(0) = \min \left(K_1, K_1 \frac{r}{q} \right), \quad (40)$$

$$c_2(0) = \max \left(K_2, K_2 \frac{r}{q} \right). \quad (41)$$

Thus by starting at $i = 1$, there are only two unknown values in the system (38)-(39) for each i , namely $c_1(\tau_i)$ and $c_2(\tau_i)$. We use Simpson's rule to evaluate the integral terms.

For each i beginning with $i = 1$, the bisection method is applied to the following nonlinear equation to find $c_1(\tau_i)$:

$$c_1(ih) = K_1 - \hat{A}P(c_1(ih), ih; c_1) - \hat{A}C(c_1(ih), ih; c_2), \quad (42)$$

where

$$\begin{aligned} \hat{A}P(c_1(ih), ih; c_1) &= K_1 e^{-rih} N(-d_2(c_1(ih), ih; K_1)) \\ &\quad - c_1(ih) e^{-qih} N(-d_1(c_1(ih), ih; K_1)) \\ &\quad + h \sum_{j=0}^i w_j [K_1 r e^{-rh(i-j)} N(-d_2(c_1(ih), h(i-j); c_1(jh))) \\ &\quad - c_1(ih) q e^{-qh(i-j)} N(-d_1(c_1(ih), h(i-j); c_1(jh)))], \end{aligned} \quad (43)$$

and

$$\begin{aligned} \hat{A}C(c_1(ih), ih; c_2) &= c_1(ih) e^{-qih} N(d_1(c_1(ih), ih; K_2)) \\ &\quad - K_2 e^{-rih} N(-d_2(c_1(ih), ih; K_2)) \\ &\quad + h \sum_{j=0}^i w_j [c_1(ih) q e^{-qh(i-j)} N(d_1(c_1(ih), h(i-j); c_2(jh))) \\ &\quad - K_2 r e^{-rh(i-j)} N(d_2(c_1(ih), h(i-j); c_2(jh)))]. \end{aligned} \quad (44)$$

The summation weights w_j are those dictated by the numerical integration scheme, in this case Simpson's rule, and the extended Simpson's rule (used on the end furthest from expiry whenever i is odd). The bisection method was chosen over more complex techniques, such as Newton's method, because the monotonic nature of the free boundaries enables us to efficiently reduce our search region for the unknown root as i increases. The bisection method also saves us from having to evaluate the derivatives of the free boundary integral equations.

Similarly, we apply the bisection method to the following to find $c_2(\tau_i)$:

$$c_2(ih) = K_2 + \hat{A}P(c_2(ih), ih; c_1) + \hat{A}C(c_2(ih), ih; c_2), \quad (45)$$

where

$$\begin{aligned} \hat{A}P(c_2(ih), ih; c_1) &= K_1 e^{-rih} N(-d_2(c_2(ih), ih; K_1)) \\ &\quad - c_2(ih) e^{-qih} N(-d_1(c_2(ih), ih; K_1)) \\ &\quad + h \sum_{j=0}^i w_j [K_1 r e^{-rh(i-j)} N(-d_2(c_2(ih), h(i-j); c_1(jh))) \\ &\quad - c_2(ih) q e^{-qh(i-j)} N(-d_1(c_2(ih), h(i-j); c_1(jh)))], \end{aligned} \quad (46)$$

and

$$\begin{aligned} \hat{A}C(c_2(ih), ih; c_2) &= c_2(ih) e^{-qih} N(d_1(c_2(ih), ih; K_2)) \\ &\quad - K_2 e^{-rih} N(-d_2(c_2(ih), ih; K_2)) \\ &\quad + h \sum_{j=0}^i w_j [c_2(ih) q e^{-qh(i-j)} N(d_1(c_2(ih), h(i-j); c_2(jh))) \\ &\quad - K_2 r e^{-rh(i-j)} N(d_2(c_2(ih), h(i-j); c_2(jh)))]. \end{aligned} \quad (47)$$

It is important to note that at $j = i$, both d_1 and d_2 are singular. This can be handled by using the following:

$$\lim_{\xi \rightarrow \tau} N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) = 0, \quad (48)$$

$$\lim_{\xi \rightarrow \tau} N(d_1(c_2(\tau), \tau - \xi; c_2(\xi))) = 0.5, \quad (49)$$

$$\lim_{\xi \rightarrow \tau} N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi))) = 0.5, \quad (50)$$

$$\lim_{\xi \rightarrow \tau} N(-d_1(c_2(\tau), \tau - \xi; c_1(\xi))) = 0. \quad (51)$$

These limits are the same for d_2 . It is also important to see that while, for example, (42)-(44) depends upon $c_2(\tau_i)$ it does not explicitly require $c_2(\tau_i)$, due to the need to use the limits (48)-(51) at $j = i$. Hence, the simultaneous integral equation system (38)-(39) can be solved by finding $c_1(\tau_i)$ using all known values of $c_1(\tau_j)$ and $c_2(\tau_j)$, $j = 0, 1, 2, \dots, i - 1$. That is, the interdependence at the i th time point is removed due to the need to consider the limits of d_1 and d_2 when evaluating (42)-(44) and (45)-(47).

The above numerical scheme is firstly carried out using using a time-step size of h , and then repeated using $h/2$. In each case, since it is necessary to alternate between two different numerical integration schemes (for odd and even values of i) it turns out that the free boundaries have non-monotonic gradients. This is rectified by combining the two estimates using Richardson's extrapolation. Pricing the American strangle is then achieved via numerical integration using Simpson's rule, combined with the estimates of $c_1(\tau_i)$ and $c_2(\tau_i)$.

6 Results

Firstly we examined the accuracy of the numerical scheme being implemented. The method was applied to a 1-year American strangle position, using $n = 100, 200, 400$ and 800 . This has been compared against a standard Crank-Nicolson scheme using 4 time steps per day. The r and q parameters were chosen to be non-zero and unequal. The results are summarised in table 1 for 5 spot prices, chosen to represent at-the-money, in-the-money and out-of-the-money prices for the strangle within the continuation region.

From table 1, it can be seen that for the strangle at-the-money on the call side, the Crank-Nicolson scheme has converged to 4 decimal places, while it has converged to 5 decimal places

S	CN 60,000	CN 120,000	Kim 100	Kim 200	Kim 400	Kim 800
0.75	0.275648	0.275648	0.275647	0.275647	0.275647	0.275647
1.00	0.100322	0.100319	0.100332	0.100332	0.100332	0.100332
1.25	0.038560	0.038560	0.038563	0.038562	0.038561	0.038561
1.50	0.092316	0.092314	0.092344	0.092341	0.092341	0.092340
1.75	0.255619	0.255619	0.255631	0.255632	0.255633	0.255633

Table 1: American Strangle price found numerically. The Crank-Nicolson finite difference scheme involved 4 time steps per day, and involved 60,000 and 120,000 space-nodes, as indicated in the table. The numerical scheme for solving Kim's integral equations used $n = 100, 200, 400$ and 800 respectively, as indicated in the table. The parameter values were $r = 5\%$, $q = 10\%$, $T - t = 1$; $K_1 = 1$; $K_2 = 1.5$; $\sigma = 20\%$.

at the other spot values. Thus we take the Crank-Nicolson results as being the true solution to an accuracy of around 4 decimal places.

For all the values of n used, the American strangle prices found using Kim's integral equation system match those found using Crank-Nicolson to 4 decimal places. We conclude from these results that the numerical method employed in solving the integral equations has an accuracy of 4 decimal places. It can also be seen that the numerical scheme for Kim's integral equations has converged to 5 decimal places for n as low as 100. We therefore select $n = 200$ for the purposes of generating all further results. The algorithm was implemented using LAHEYTMFORTRAN 95 on a PC with a Pentium III processor, 128MB of RAM, and running the Windows 98 operating system. With $n = 200$, the code takes approximately 3 minutes and 15 seconds to solve the integral equation system for the American strangle's free boundaries.

To demonstrate the early exercise boundaries and price properties of the American strangle, we implemented the method using $n = 200$ time nodes. To improve the accuracy of the method where the free boundaries change rapidly, a finer grid was used between the first three nodes (specifically, 40 nodes between $i = 0$ and $i = 2$). The method was also applied in the same manner to the American call and put contracts which define the components of the strangle's payoff function. By comparing the results for the strangle against those of the independent call and put, we can demonstrate how the American strangle's free boundaries and price are affected by the interdependence between $c_1(\tau)$ and $c_2(\tau)$.

Firstly, consider an American strangle with one year until maturity. Let the put-side strike be 1 and the call-side strike be 1.1, with the volatility of the underlying at 20%. In figures 2-7, we present the call- and put-side boundaries for the American strangle, with $r > q$ (figures 2-3), $r < q$ (figures 4-5), and finally $r = q$ (figures 6-7). In all cases, we include the free boundary for the corresponding American call or put. The same results are presented in figures 8-13, but the call-side strike has been reduced to 1.001, moving the strangle position closer to a straddle.

There are several distinct features that can be ascertained from these free boundary plots. The first is that the relative values of r and q directly affect whether or not the American strangle free boundaries will show significant divergence from the corresponding American call and put boundaries. In particular, when $r > q$, only the put-side boundaries diverge, and when $r < q$ only the call-side boundaries diverge. When $r = q$, there is divergence in both boundaries, but it is smaller than in the other two cases.

Since the early exercise of the in-the-money side of the strangle will knock-out the other side, it is expected that the strangle will have to be deeper in-the-money to warrant early exercise than one formed using independent American calls and puts. In all cases, the call-side free boundary for the strangle is always greater than or equal to that of the corresponding American call free boundary, while the put-side is always less than or equal to that of the corresponding American put free boundary. This is in keeping with the economic intuition behind the American strangle position.

In all three cases of r and q values, moving the call-side's strike closer to the put-side's strike increases any divergence between the American strangle free boundaries and those of the corresponding American call and put. This is again as one would expect, since the closer the strangle is to being a straddle, the more intrinsic value the out-of-the-money strangle component will contribute to the early exercise decision. It can also be seen that as the time to maturity increases, the divergence between the strangle free boundaries and the corresponding call and put boundaries increases. When the strangle has a very short time to maturity, say 2 weeks or less, then the divergence between the two free boundaries becomes minimal.

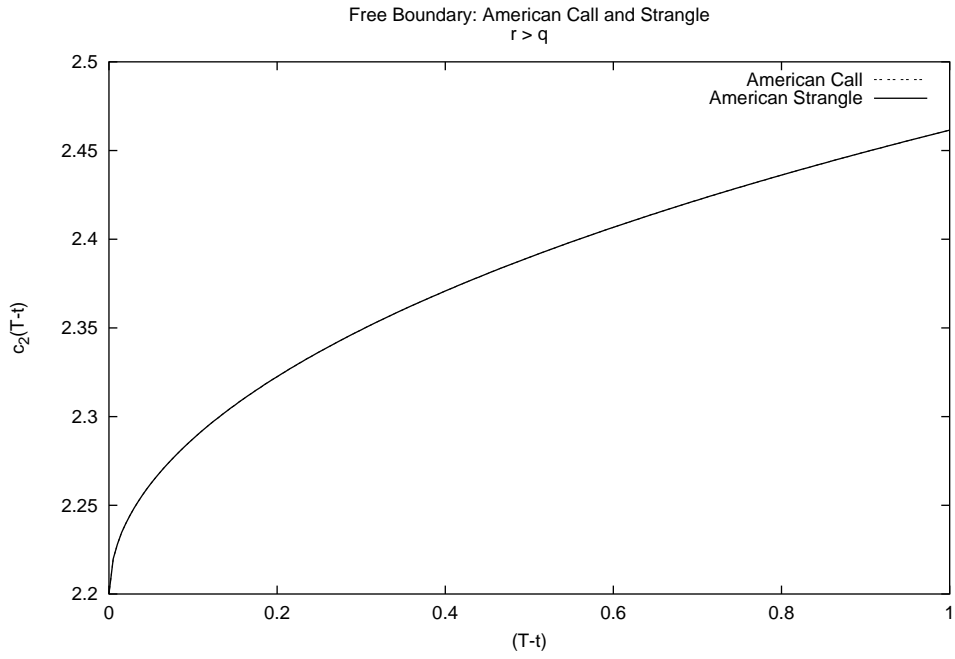


Figure 2: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 10\%, q = 5\%$

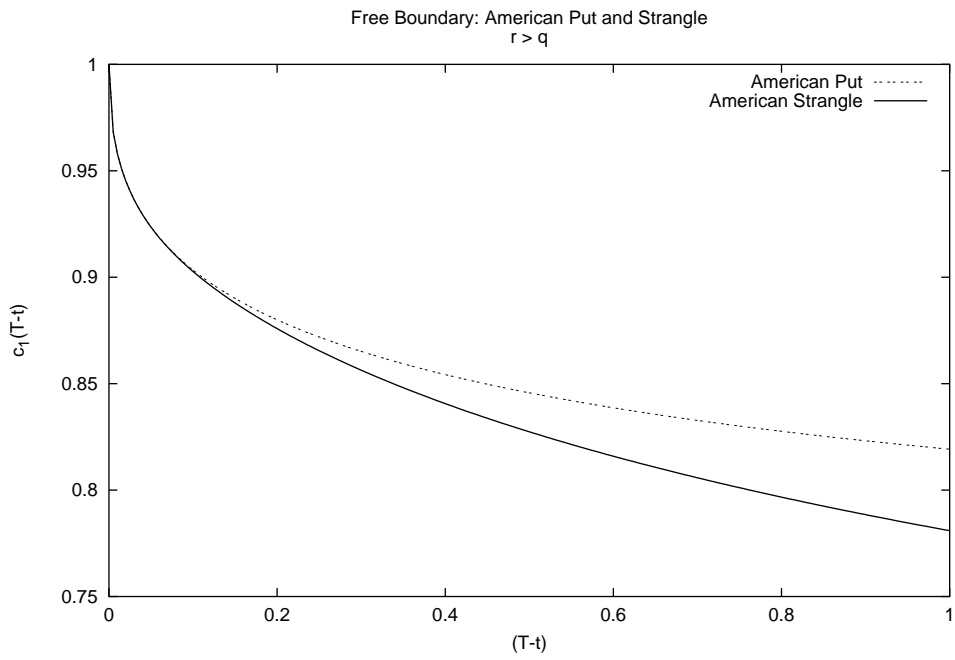


Figure 3: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 10\%, q = 5\%$

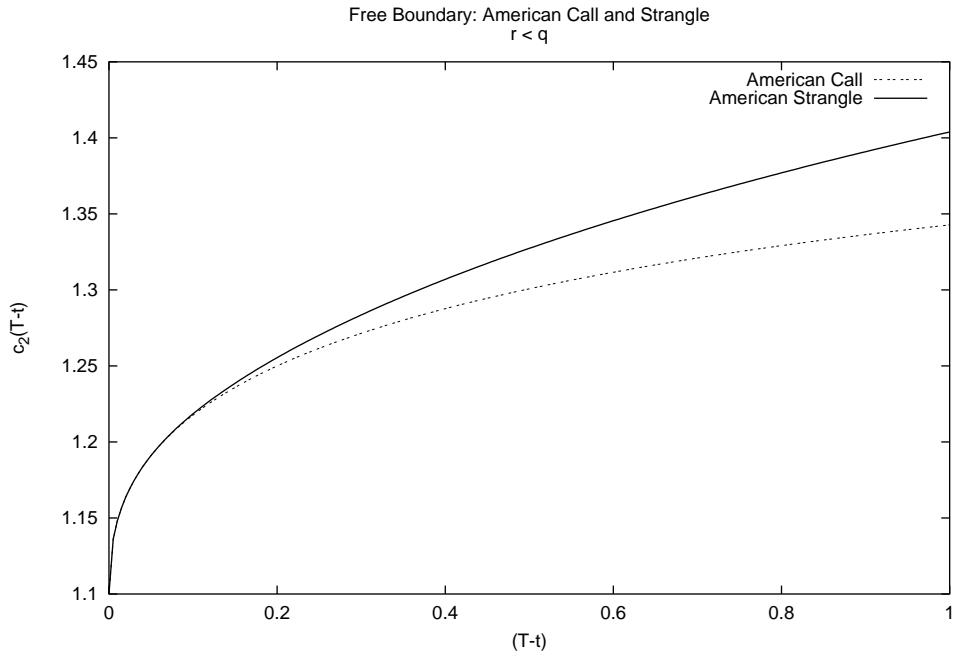


Figure 4: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 5\%, q = 10\%$

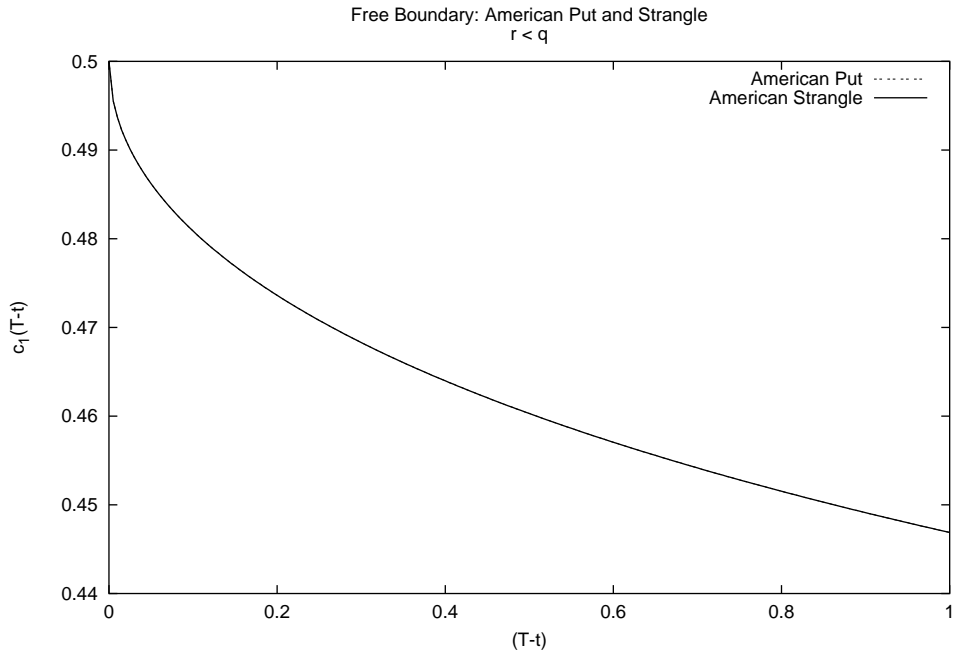


Figure 5: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 5\%, q = 10\%$

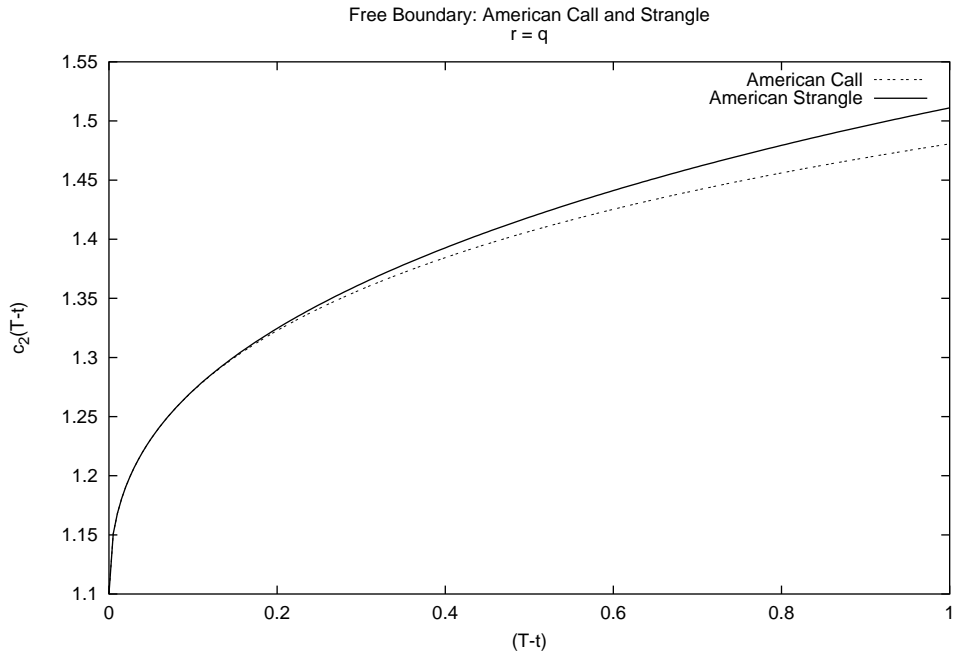


Figure 6: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 10\%, q = 10\%$

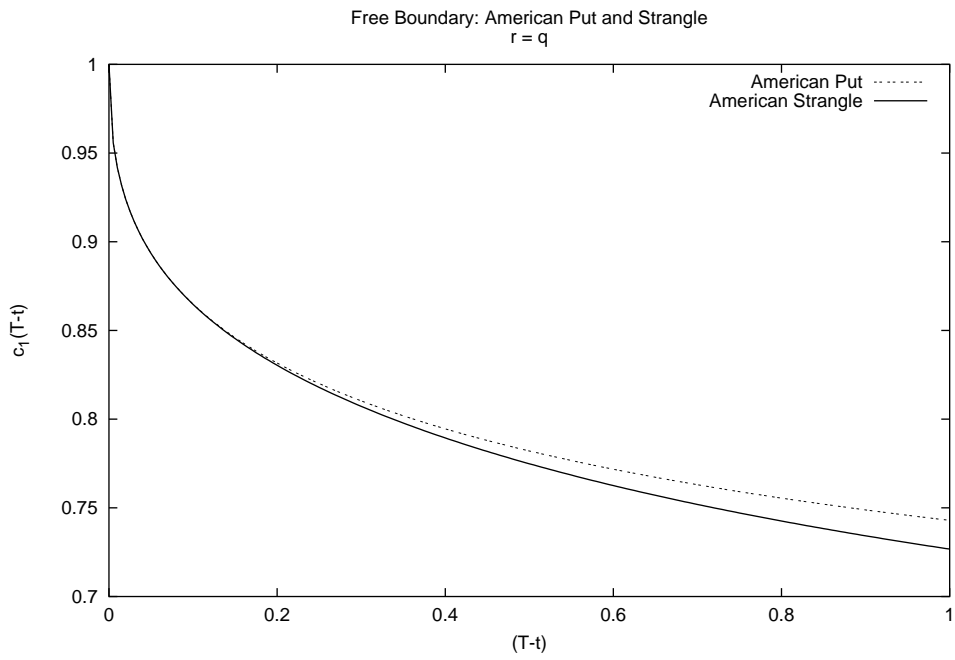


Figure 7: $K_1 = 1, K_2 = 1.1, \sigma = 20\%, r = 10\%, q = 10\%$

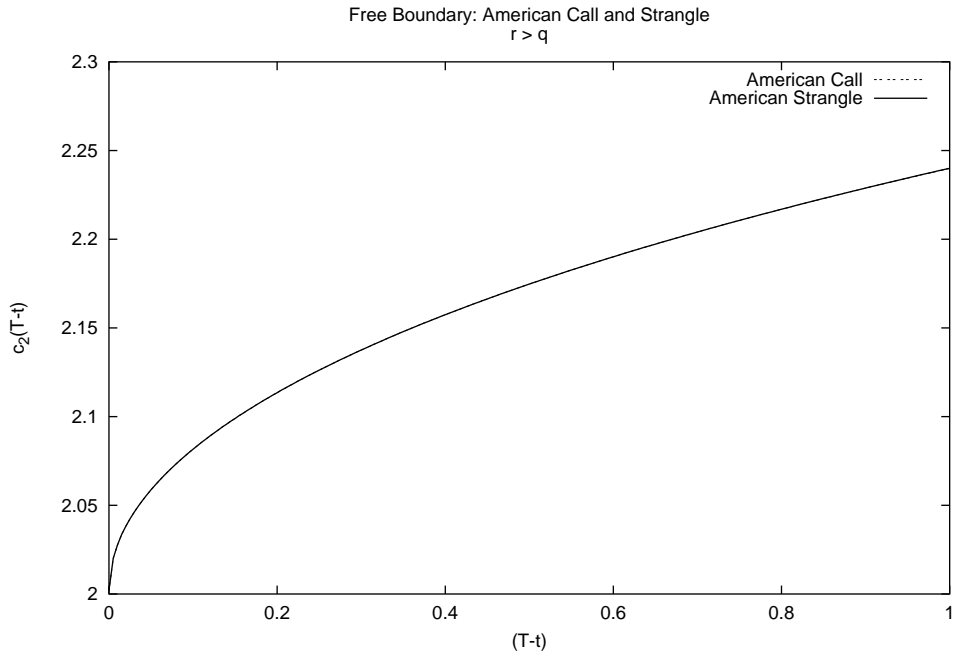


Figure 8: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 10\%, q = 5\%$

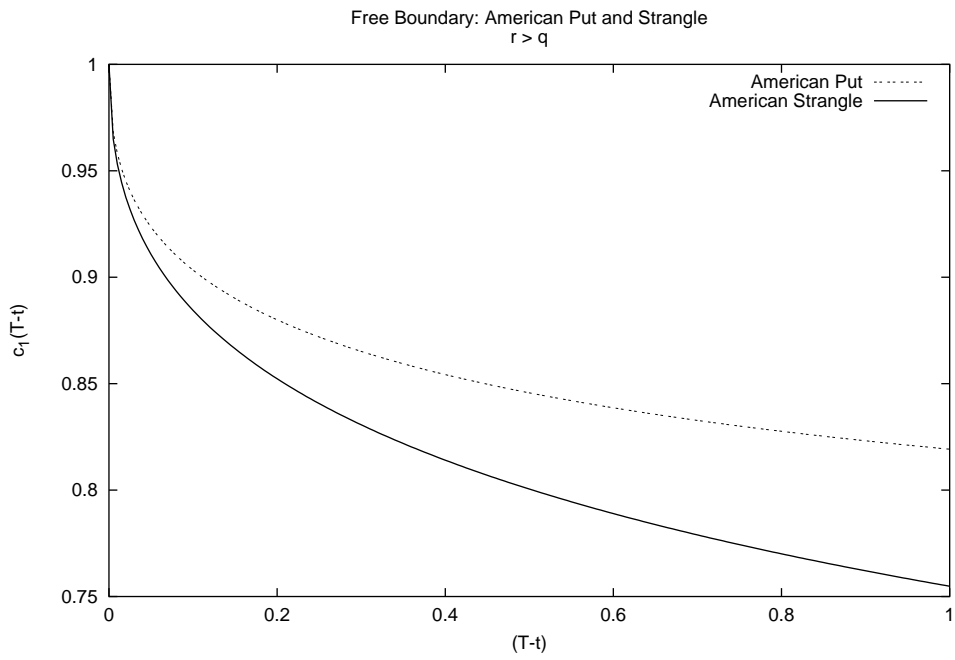


Figure 9: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 10\%, q = 5\%$

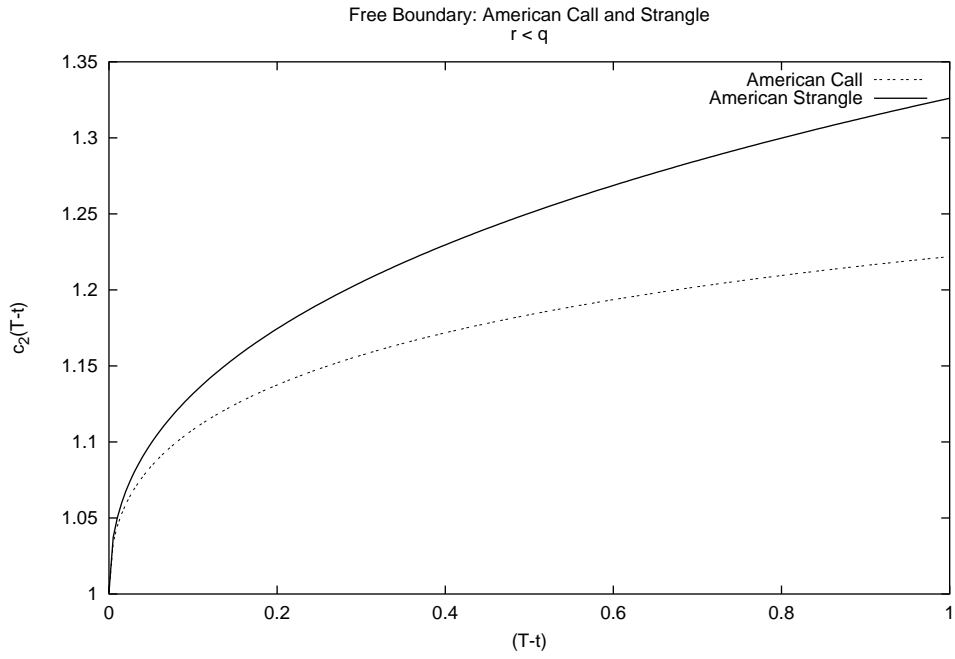


Figure 10: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 5\%, q = 10\%$

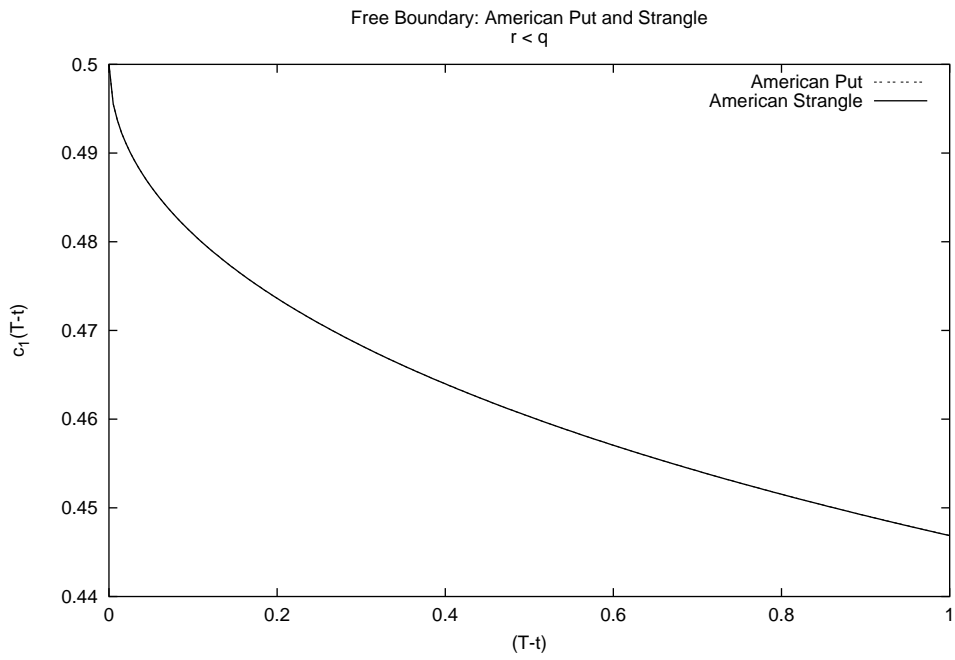


Figure 11: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 5\%, q = 10\%$

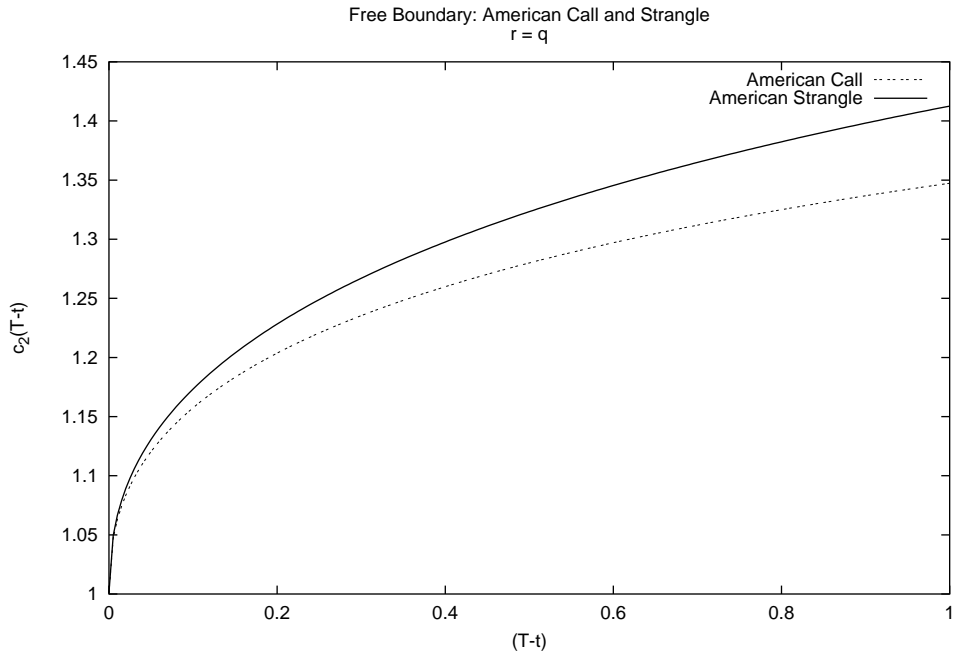


Figure 12: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 10\%, q = 10\%$

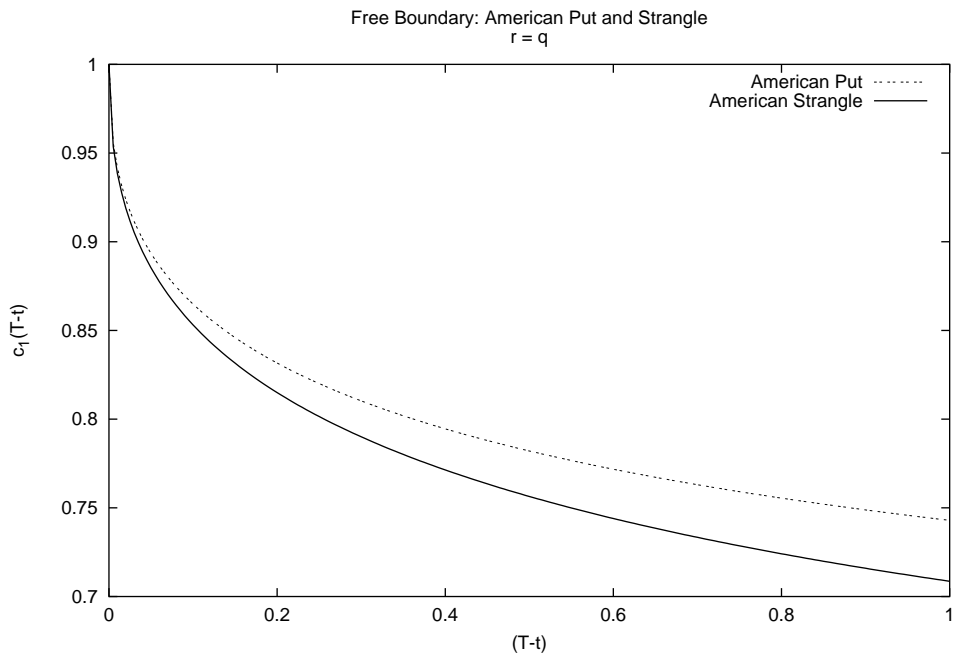


Figure 13: $K_1 = 1, K_2 = 1.001, \sigma = 20\%, r = 10\%, q = 10\%$

While it is clear that the early exercise boundaries for the strangle are not always equivalent to those of the component American call and put in the examples provided, the difference never exceeds 0.1, which in relative terms is no more than 10% of the put-side's strike price. Past research into American options, such as Ju (1998) and AitSahalia and Lai (2001), has found that the price of American call and put options is not greatly affected by the free boundary estimate used. While a 10% difference in the free boundary has obvious early exercise timing repercussions, it remains to be seen whether the price of the strangle using these free boundaries is far removed from that of a strangle priced simply using the sum of an American call and an American put. To explore the effect of these free boundary differences on the strangle's price, we compare the price of the American strangle against the "traditional" American call plus American put approach. The prices were found using Simpson's rule with 100 nodes (implying no need for interpolation when using our $c_1(\tau)$ and $c_2(\tau)$ estimates), and were compiled for a range of volatilities (tables 2-3) and call-side strikes (tables 4-5). In all cases, the prices were found for a range of underlying asset values, S , between 0 and 300,000. Thus, these results are indicative of a position in a contract involving several thousand American strangle contracts. The tables present only the prices for which the difference between the strangle and the call-put sum was greatest. The time to maturity is always set at 1 year.

It should be noted that in all cases, the American strangle price is always less than or equal to the sum of the corresponding American call and put prices. This is as expected, since the American strangle is equivalent to combining a long knock-out American call and a long knock-out American put, where the knock-out barriers are $c_1(\tau)$ and $c_2(\tau)$ for the call and put respectively. The decrease in the strangle's price reflects the presence of these implicit knock-out barriers, and hence the inability to separate the call and put sides in this new strangle position.

From table 2, we see that when $r > q$, the largest difference appears on the put-side, as one would expect. The difference remains around 1,000 for all the volatilities, but as the volatility increases, the relative difference decreases, and is at most between 5-6%. The greatest differ-

ences occur when the put is deep in-the-money, and this maximum occurs deeper in-the-money as the volatility increases.

σ (%)	Max Price Difference	S (000's)	Relative Difference
20	1,201.13	80	5.84%
40	1,156.51	60	2.85%
60	1,078.79	40	1.80%
80	1,137.30	30	1.62%

Table 2: Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of σ values. $r = 10\%$, $q = 5\%$, $T - t = 1$; $K_1 = 100,000$; $K_2 = 100,100$; prices of underlying range from $S = 0$ to $S = 300,000$ in steps of 10,000.

σ (%)	Max Price Difference	S (000's)	Relative Difference
20	1,478.68	130	4.93%
40	1,996.79	170	2.83%
60	2,776.88	240	1.98%
80	3,458.06	300	1.71%

Table 3: Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of σ values. $r = 5\%$, $q = 10\%$, $T - t = 1$; $K_1 = 100,000$; $K_2 = 100,100$; prices of underlying range from $S = 0$ to $S = 300,000$ in steps of 10,000.

K_2 (000's)	Max Price Difference	S (000's)	Relative Difference
100.01	1,207.49	80	5.87%
101.00	1,138.72	80	5.56%
110.00	630.92	80	3.14%
150.00	11.23	80	0.06%

Table 4: Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of K_2 values. $r = 10\%$, $q = 5\%$, $T - t = 1$; $K_1 = 100,000$; $\sigma = 20\%$; prices of underlying range from $S = 0$ to $S = 300,000$ in steps of 10,000.

When $r < q$, the maximum difference occurs on the call-side. Table 3 shows that this can exceed 3,000 for a large enough volatility, but as is the case in table 2, the smaller the volatility, the greater the relative price difference is. This difference never exceeds 5%, and the greatest differences arrive when the strangle is deep in-the-money on the call side. Thus the largest relative price deviations will occur for low volatilities. While this result appears counter-intuitive, it is important to note that “realistic” volatilities (e.g. 20%) produce the greatest relative price differences.

K_2 (000's)	Max Price Difference	S (000's)	Relative Difference
100.01	1,482.61	130	4.93%
101.00	1,433.60	130	4.92%
110.00	748.99	140	2.50%
150.00	15.16	180	0.05%

Table 5: Maximum price differences between the American Strangle and the same position formed using an American Call and an American Put for a range of K_2 values. $r = 5\%$, $q = 10\%$, $T - t = 1$; $K_1 = 100,000$; $\sigma = 20\%$; prices of underlying range from $S = 0$ to $S = 300,000$ in steps of 10,000.

Table 4 considers the maximum price differences for a range of call-side strikes, with $r > q$. As in table 2, the greatest differences occur deep in-the-money on the put-side, and become smaller as $K_2 - K_1$ increases. A similar result is shown in table 5, where $r < q$, and the difference is now greatest deep in-the-money on the call side. Once the call-side strike reaches 150,000, the relative price difference is at most less than 0.1%, while the largest relative differences, when the strangle is effectively a straddle, are still no more than 6%. Overall, it appears that a 10% difference in one of the early exercise boundaries will produce at most a 6% difference in the price, when comparing this American strangle with a position formed by going long in both an American call and an American put.

7 Conclusion

In this paper we have presented a generalisation of McKean's free boundary value problem for pricing American options. We have considered the example of an American strangle position, where exercising one side of the position early will knock-out the remaining side. McKean's integral equations for this strangle's price were derived, along with the integral equation system for its two free boundaries. It was shown that analytically, the free boundaries for the American strangle are not equal to those found when valuing independent American calls and puts.

Kim's form of the integral equation system was solved using a scheme typically applied to nonlinear Volterra integral equations. It was found that numerically, the early exercise bound-

ary of this strangle only differed significantly from the boundaries of corresponding American calls and puts for certain values of the risk-free rate and continuous dividend yield parameters. The differences became larger as the distance between the strangle's strikes was reduced, and as the time to expiry increased. Comparing the prices of this new strangle to those of a strangle formed using a long American call and a long American put, we showed that for several call-side strikes and volatilities, our strangle was cheaper than the "traditional" one by no more than 6%, and that these differences were most apparent when the strangle was deep in-the-money. Economically, this pricing difference can be viewed as the reduction in value caused by introducing the knock-out effect into the new strangle, and foregoing the freedom to separate the call and put sides.

The early exercise boundaries for our strangle required that the position be deeper in-the-money than a "traditional" strangle, to compensate the intrinsic value forgone on the out-of-the-money side. If one does not calculate these free boundaries correctly, there is the potential to exercise the American strangle presented in this paper too early. Despite these early exercise differences, the prices of the two strangles were usually very close. An investor interested in an American strangle position may be indifferent when choosing between this proposal and a "traditional" American strangle, since only a small premium is required for the added flexibility of the latter. Whether or not the reduced transaction costs from our self-closing strangle would benefit the investor is a matter we leave to future study.

One avenue for future research would be to consider other complex payoff types, such as an American butterfly (i.e. concave payoff), or an American bear/bull spread (i.e. monotonic payoff). These positions can be constructed with similar early exercise conditions to our American strangle, and can be evaluated using our generalisation of McKean's framework. The numerical method presented should be rigorously tested against existing techniques, such as binomial trees and finite differences. We are also exploring the potential to numerically solve McKean's integral equation in its original form.

Appendix A. The Incomplete Fourier Transform

Our aim is to prove that if

$$f(x, \tau) = H(b - x)H(x - a)g(x, \tau), \quad a \equiv a(\tau), \quad b \equiv b(\tau),$$

$$a < b \quad \forall \quad \tau \in [0, T], \text{ and } H(x) \equiv \text{Heaviside Function},$$

then application of the standard Fourier inversion theorem

$$f(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x, \tau) e^{i\eta x} dx \right] e^{-i\eta x} d\eta, \quad -\infty < x < \infty,$$

yields

$$g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b g(x, \tau) e^{i\eta x} dx \right] e^{-i\eta x} d\eta, \quad a < x < b,$$

which may be regarded as an inversion theorem for the incomplete Fourier transform.

Firstly,

$$\begin{aligned} RHS &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(b - x)H(x - a)g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^b H(b - x)H(x - a)g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \left[\int_b^{\infty} H(b - x)H(x - a)g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b H(b - x)H(x - a)g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \left[\int_{-\infty}^a H(b - x)H(x - a)g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b g(x, \tau)e^{i\eta x} dx \right] e^{-i\eta x} d\eta. \end{aligned}$$

Next consider

$$LHS = H(b - x)H(x - a)g(x, \tau)$$

$$= \begin{cases} g(x, \tau), & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$H(b-x)H(x-a)g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b g(x, \tau) e^{i\eta x} dx \right] e^{-i\eta x} d\eta, \quad -\infty < x < \infty$$

or alternatively,

$$g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b g(x, \tau) e^{i\eta x} dx \right] e^{-i\eta x} d\eta, \quad a < x < b$$

and

$$\frac{g(x, \tau)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^b g(x, \tau) e^{i\eta x} dx \right] e^{-i\eta x} d\eta, \quad x = a, b.$$

Appendix B. Properties of the Incomplete Fourier Transform

B.1. Proof of Proposition 1

Firstly consider

$$\begin{aligned} \mathcal{F}^c \left\{ \frac{\partial V}{\partial x} \right\} &= \int_{\ln c_1}^{\ln c_2} e^{i\eta x} \frac{\partial V}{\partial x} dx \\ &= \left[V e^{i\eta x} \right]_{\ln c_1}^{\ln c_2} - i\eta \int_{\ln c_1}^{\ln c_2} e^{i\eta x} V dx \\ &= V_{c_1, c_2}(\ln c_2, \tau) e^{i\eta \ln c_2} - V_{c_1, c_2}(\ln c_1, \tau) e^{i\eta \ln c_1} - i\eta \hat{V}_{c_1, c_2}(\eta, \tau). \end{aligned}$$

Finally by use of boundary conditions (11) and (12),

$$\mathcal{F}^c \left\{ \frac{\partial V}{\partial x} \right\} = (c_2 - K_2)e^{i\eta \ln c_2} - (K_1 - c_1)e^{i\eta \ln c_1} - i\eta \hat{V}.$$

Next consider

$$\begin{aligned} \mathcal{F}^c \left\{ \frac{\partial^2 V}{\partial x^2} \right\} &= \int_{\ln c_1}^{\ln c_2} e^{i\eta x} \frac{\partial^2 V}{\partial x^2} dx \\ &= \left[\frac{\partial V}{\partial x} e^{i\eta x} \right]_{\ln c_1}^{\ln c_2} - i\eta \int_{\ln c_1}^{\ln c_2} \frac{\partial V}{\partial x} e^{i\eta x} dx \\ &= \frac{\partial V_{c_1, c_2}(x, \tau)}{\partial x} \Big|_{x=\ln c_2} \cdot e^{i\eta \ln c_2} - \frac{\partial V_{c_1, c_2}(x, \tau)}{\partial x} \Big|_{x=\ln c_1} \cdot e^{i\eta \ln c_1} - i\eta \mathcal{F}^c \left\{ \frac{\partial V}{\partial x} \right\} \\ &= c_2 e^{i\eta \ln c_2} + c_1 e^{i\eta \ln c_1} - i\eta [(c_2 - K_2)e^{i\eta \ln c_2} - (K_1 - c_1)e^{i\eta \ln c_1} - i\eta \hat{V}], \end{aligned}$$

where the last equality follows by use of the boundary conditions (13) and (14), and the transform result (17). The last equation simplifies to

$$\mathcal{F}^c \left\{ \frac{\partial^2 V}{\partial x^2} \right\} = e^{i\eta \ln c_2} (c_2 - i\eta (c_2 - K_2)) - e^{i\eta \ln c_1} (-c_1 - i\eta (K_1 - c_1)) - \eta^2 \hat{V}.$$

Finally consider

$$\begin{aligned} \mathcal{F}^c \left\{ \frac{\partial V}{\partial \tau} \right\} &= \int_{\ln c_1}^{\ln c_2} e^{i\eta x} \frac{\partial V_{c_1, c_2}(x, \tau)}{\partial \tau} dx \\ &= \frac{\partial}{\partial \tau} \left[\int_{\ln c_1}^{\ln c_2} e^{i\eta x} V_{c_1, c_2}(x, \tau) dx \right] - \frac{c'_2}{c_2} e^{i\eta \ln c_2} V_{c_1, c_2}(\ln c_2, \tau) + \frac{c'_1}{c_1} e^{i\eta \ln c_1} V_{c_1, c_2}(\ln c_1, \tau) \\ &= \frac{\partial}{\partial \tau} \left[\mathcal{F}^c \{V\} \right] - \frac{c'_2}{c_2} e^{i\eta \ln c_2} V_{c_1, c_2}(\ln c_2, \tau) + \frac{c'_1}{c_1} e^{i\eta \ln c_1} V_{c_1, c_2}(\ln c_1, \tau), \end{aligned}$$

where

$$c'_j \equiv \frac{dc_j(\tau)}{d\tau}, \quad j = 1, 2.$$

Applying the boundary conditions (11) and (12) we have

$$\frac{\partial}{\partial \tau} \left[\mathcal{F}^c \{V\} \right] = \frac{c'_2}{c_2} e^{i\eta \ln c_2} (c_2 - K_2) - \frac{c'_1}{c_1} e^{i\eta \ln c_1} (K_1 - c_1) + \mathcal{F}^c \left\{ \frac{\partial V}{\partial \tau} \right\}.$$

Hence, finally

$$\mathcal{F}^c \left\{ \frac{\partial V}{\partial \tau} \right\} = \frac{\partial \hat{V}}{\partial \tau} - \frac{c_2'}{c_2} e^{i\eta \ln c_2} (c_2 - K_2) + \frac{c_1'}{c_1} e^{i\eta \ln c_1} (K_1 - c_1).$$

B.2. Proof of Proposition 2

Taking the incomplete Fourier transform of equation (9) with respect to x and using (17) - (19), we obtain:

$$\begin{aligned} \frac{\partial \hat{V}}{\partial \tau} - \frac{c_2'}{c_2} e^{i\eta \ln c_2} (c_2 - K_2) + \frac{c_1'}{c_1} e^{i\eta \ln c_1} (K_1 - c_1) \\ = \frac{1}{2} \sigma^2 [e^{i\eta \ln c_2} (c_2 - i\eta (c_2 - K_2)) - e^{i\eta \ln c_1} (-c_1 - i\eta (K_1 - c_1)) - \eta^2 \hat{V}] \\ + k [e^{i\eta \ln c_2} (c_2 - K_2) - e^{i\eta \ln c_1} (K_1 - c_1) - i\eta \hat{V}] - r \hat{V}, \end{aligned}$$

which may be re-arranged to

$$\begin{aligned} \frac{\partial \hat{V}}{\partial \tau} + \left(\frac{1}{2} \sigma^2 \eta^2 + k i \eta + r \right) \hat{V} \\ = e^{i\eta \ln c_2} \left[\frac{c_2'}{c_2} (c_2 - K_2) + \frac{1}{2} \sigma^2 (c_2 - i\eta (c_2 - K_2)) + k (c_2 - K_2) \right] \\ - e^{i\eta \ln c_1} \left[\frac{c_1'}{c_1} (K_1 - c_1) + \frac{1}{2} \sigma^2 (-c_1 - i\eta (K_1 - c_1)) + k (K_1 - c_1) \right]. \end{aligned}$$

It is a simple matter to rewrite this in terms of $F(\eta, \tau)$ to produce equations (20)-(21), and the initial condition is obtained by definition.

Appendix C. Derivation of the American Strangle Integral Equations

C.1. Proof of Proposition 4

In Appendix D we derive the convolution result that:

$$\begin{aligned} \mathcal{F}^{-1}\{\hat{F}(\eta, \tau_1)\hat{G}(\eta, \tau_2)\} &= \int_{-\infty}^{\infty} [f(x-u, \tau_1)H(\ln c_2(\tau_1) - (x-u))H((x-u) - \ln c_1(\tau_1))] \\ &\quad \times [g(u, \tau_2)H(\ln c_2(\tau_2) - u))H(u - \ln c_1(\tau_2))]du. \end{aligned}$$

Let

$$\hat{F}(\eta, \tau_1) = e^{-(\frac{1}{2}\sigma^2\eta^2 + k\eta + r)\tau}.$$

Hence

$$\begin{aligned} f(x, \tau_1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\frac{1}{2}\sigma^2\eta^2 + k\eta + r)\tau} e^{-i\eta x} d\eta \\ &= \frac{e^{-r\tau}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta(x+k\tau)} d\eta. \end{aligned}$$

Using the result that

$$\int_{-\infty}^{\infty} e^{-p\eta^2 - q\eta} d\eta = \sqrt{\frac{\pi}{p}} e^{\frac{q^2}{4p}}, \quad \text{Re}(p) \geq 0,$$

we have

$$\begin{aligned} f(x, \tau_1) &= \frac{e^{-r\tau}}{2\pi} \sqrt{\frac{2\pi}{\sigma^2\tau}} e^{\frac{[i(x+k\tau)]^2}{2\sigma^2\tau}} \\ &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{\frac{-(x+k\tau)^2}{2\sigma^2\tau}}. \end{aligned}$$

Next let $\hat{G}(\eta, \tau_2) = \hat{V}_{c_1, c_2}(\eta, 0)$. Hence we have

$$\begin{aligned}
g(x, \tau_2) &= V_{c_1, c_2}(x, 0) \\
&= \max(K_1 - e^x, 0) + \max(e^x - K_2, 0) \\
&= H(\ln K_1 - x)(K_1 - e^x) + H(x - \ln K_2)(e^x - K_2).
\end{aligned}$$

Thus

$$\begin{aligned}
V_{c_1, c_2}^{(1)}(x, \tau) &= \int_{-\infty}^{\infty} \left[\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} H(\ln c_2(\tau) - (x-u)) H((x-u) - \ln c_1(\tau)) \right] \\
&\quad \times [H(\ln K_1 - u)(K_1 - e^u) + H(u - \ln K_2)(e^u - K_2)] \\
&\quad \times H(\ln c_2(0) - u) H(u - \ln c_1(0)) du.
\end{aligned}$$

Consider the following Heaviside functions:

- i) $H(\ln K_1 - u) \Rightarrow u < \ln K_1$
- ii) $H(u - \ln K_2) \Rightarrow u > \ln K_2$
- iii) $H(\ln c_2(0) - u) \Rightarrow u < \ln c_2(0)$
- iv) $H(u - \ln c_1(0)) \Rightarrow u > \ln c_1(0)$
- v) $H(\ln c_2(\tau) - (x - u)) \Rightarrow u > x - \ln c_2(\tau); \ln c_1(\tau) < x < \ln c_2(\tau)$
- vi) $H((x - u) - \ln c_1(\tau)) \Rightarrow u < x - \ln c_1(\tau); \ln c_1(\tau) < x < \ln c_2(\tau)$.

It is known that

$$c_1(0) \leq K_1 \text{ and } c_2(0) \geq K_2.$$

Hence

$$\begin{aligned}
&H(\ln c_2(\tau) - (x - u)) H((x - u) - \ln c_1(\tau)) \\
&\quad \times H(\ln K_1 - u) H(\ln c_2(0) - u) H(u - \ln c_1(0))
\end{aligned}$$

$$= H(\ln K_1 - u)H(u - \ln c_1(0)),$$

and

$$\begin{aligned} & H(\ln c_2(\tau) - (x - u))H((x - u) - \ln c_1(\tau)) \\ & \times H(u - \ln K_2)H(\ln c_2(0) - u)H(u - \ln c_1(0)) \\ & = H(u - \ln K_2)H(u - \ln c_2(0)). \end{aligned}$$

Therefore

$$\begin{aligned} V_{c_1, c_2}^{(1)}(x, \tau) &= \int_{-\infty}^{\infty} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} H(\ln K_1 - u)H(u - \ln c_1(0)) e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} (K_1 - e^u) du \\ &+ \int_{-\infty}^{\infty} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} H(u - \ln K_2)H(\ln c_2(0) - u) e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} (e^u - K_2) du \\ &= \int_{\ln c_1(0)}^{\ln K_1} \frac{K_1 e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du - \int_{\ln c_1(0)}^{\ln K_1} \frac{e^u e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du \\ &+ \int_{\ln K_2}^{\ln c_2(0)} \frac{e^u e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du - \int_{\ln K_2}^{\ln c_2(0)} \frac{K_2 e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du \\ &= I_1 - I_2 + I_3 - I_4 \end{aligned}$$

To simplify $V_{c_1, c_2}^{(1)}(x, \tau)$ further, we shall re-express it in terms of the cumulative standard normal distribution, $N(y)$, where

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\alpha^2}{2}} d\alpha.$$

For the first term, I_1 :

$$I_1 = \frac{K_1 e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln c_1(0)}^{\ln K_1} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du.$$

Let

$$y = -\left(\frac{x - u + k\tau}{\sigma\sqrt{\tau}}\right);$$

therefore

$$dy = \frac{du}{\sigma\sqrt{\tau}}.$$

By defining

$$d_2(x, \tau; \beta) \equiv \frac{x - \ln \beta + k\tau}{\sigma\sqrt{\tau}},$$

the integral I_1 then becomes

$$\begin{aligned} I_1 &= \frac{K_1 e^{-r\tau}}{\sqrt{2\pi}} \int_{-d_2(x, \tau; c_1(0))}^{-d_2(x, \tau; K_1)} e^{-\frac{y^2}{2}} dy \\ &= K_1 e^{-r\tau} [N(-d_2(x, \tau; K_1)) - N(-d_2(x, \tau; c_1(0)))]. \end{aligned}$$

For the second term, I_2 :

$$\begin{aligned} I_2 &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln c_1(0)}^{\ln K_1} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} e^u du \\ &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln c_1(0)}^{\ln K_1} e^{-\frac{(u-[x+(k+\sigma^2)\tau])^2}{2\sigma^2\tau}} e^{x+k\tau+\frac{\sigma^2\tau}{2}} du. \end{aligned}$$

Recall that

$$k = r - q - \frac{1}{2}\sigma^2,$$

and let

$$y = -\left(\frac{-u + x + (k + \sigma^2)\tau}{\sigma\sqrt{\tau}}\right)$$

which in turn implies that

$$dy = \frac{du}{\sigma\sqrt{\tau}}.$$

By defining

$$d_1(x, \tau; \beta) \equiv \frac{x - \ln \beta + (k + \sigma^2)\tau}{\sigma\sqrt{\tau}},$$

the integral I_2 then becomes

$$\begin{aligned} I_2 &= e^x e^{-q\tau} \frac{1}{\sqrt{2\pi}} \int_{-d_1(x, \tau; c_1(0))}^{-d_1(x, \tau; K_1)} -e^{-\frac{y^2}{2}} dy \\ &= -e^x e^{-q\tau} [N(-d_1(x, \tau; K_1)) - N(-d_1(x, \tau; c_1(0)))]. \end{aligned}$$

Similarly it can be shown that

$$I_3 = e^x e^{-q\tau} [N(d_1(x, \tau; K_2)) - N(d_1(x, \tau; c_2(0)))],$$

and

$$I_4 = K_2 e^{-r\tau} [N(d_2(x, \tau; K_2)) - N(d_2(x, \tau; c_2(0)))].$$

Thus it is concluded that

$$\begin{aligned} V_{c_1, c_2}^{(1)}(x, \tau) &= [K_1 e^{-r\tau} N(-d_2(x, \tau; K_1)) - e^x e^{-q\tau} N(-d_1(x, \tau; K_1))] \\ &\quad + [e^x e^{-q\tau} N(d_1(x, \tau; K_2)) - K_2 e^{-r\tau} N(d_2(x, \tau; K_2))] \\ &\quad - [K_1 e^{-r\tau} N(-d_2(x, \tau; c_1(0))) - e^x e^{-q\tau} N(-d_1(x, \tau; c_1(0)))] \\ &\quad - [e^x e^{-q\tau} N(d_1(x, \tau; c_2(0))) - K_2 e^{-r\tau} N(d_2(x, \tau; c_2(0)))] \end{aligned}$$

C.2. Proof of Proposition 5

We begin by noting that

$$\begin{aligned} V_{c_1, c_2}^{(2)}(x, \tau) &= \mathcal{F}^{-1} \left\{ \int_0^\tau F_2(\eta, s) e^{-\left(\frac{\sigma^2 \eta^2}{2} + k i \eta + r\right)(\tau-s)} ds \right\} \\ &\quad - \mathcal{F}^{-1} \left\{ \int_0^\tau F_1(\eta, s) e^{-\left(\frac{\sigma^2 \eta^2}{2} + k i \eta + r\right)(\tau-s)} ds \right\}, \end{aligned}$$

where

$$F_2(\eta, s) = e^{i\eta \ln c_2(s)} \left[\frac{\sigma^2 c_2(s)}{2} + \left(\frac{c_2'(s)}{c_2(s)} - \frac{\sigma^2 i \eta}{2} + k \right) (c_2(s) - K_2) \right],$$

and

$$F_1(\eta, s) = e^{i\eta \ln c_1(s)} \left[\frac{\sigma^2 c_1(s)}{2} + \left(\frac{c_1'(s)}{c_1(s)} - \frac{\sigma^2 i \eta}{2} + k \right) (K_1 - c_1(s)) \right]$$

Following the approach outlined in Kucera and Ziogas (2002), $V_{c_1, c_2}^{(2)}(x, \tau)$ evaluates to:

$$\begin{aligned}
V_{c_1, c_2}^{(2)}(x, \tau) &= \int_0^\tau \left[\frac{\sigma^2 c_2(s)}{2} + \left(\frac{c_2'(s)}{c_2(s)} + \frac{1}{2} \left[k - \frac{(x - \ln c_2(s))}{(\tau - s)} \right] \right) (c_2(s) - K_2) \right] \\
&\quad \times \frac{e^{-g_2(x, s)}}{\sigma \sqrt{2\pi(\tau - s)}} ds \\
&\quad - \int_0^\tau \left[-\frac{\sigma^2 c_1(s)}{2} + \left(\frac{c_1'(s)}{c_1(s)} + \frac{1}{2} \left[k - \frac{(x - \ln c_1(s))}{(\tau - s)} \right] \right) (K_1 - c_1(s)) \right] \\
&\quad \times \frac{e^{-g_1(x, s)}}{\sigma \sqrt{2\pi(\tau - s)}} ds, \tag{52}
\end{aligned}$$

where we set

$$g_2(x, s) = \frac{(x - \ln c_2(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)} + r(\tau - s), \tag{53}$$

and

$$g_1(x, s) = \frac{(x - \ln c_1(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)} + r(\tau - s). \tag{54}$$

With a simple change of notation, equation (52) may be written as it appears in equations (24)-(26).

Appendix D. Convolution Integral for the Incomplete Fourier Transform

Using the definition for the incomplete Fourier transform of a function, let

$$\hat{F}(\eta, \tau_1) = \int_{a(\tau_1)}^{b(\tau_1)} f(y, \tau_1) e^{i\eta y} dy$$

and

$$\hat{G}(\eta, \tau_2) = \int_{a(\tau_2)}^{b(\tau_2)} g(u, \tau_2) e^{i\eta u} du.$$

Consider

$$\begin{aligned}
& \hat{F}(\eta, \tau_1)\hat{G}(\eta, \tau_2) \\
&= \int_{a(\tau_1)}^{b(\tau_1)} f(y, \tau_1)e^{i\eta y} dy \int_{a(\tau_2)}^{b(\tau_2)} g(u, \tau_2)e^{i\eta u} du \\
&= \int_{-\infty}^{\infty} H(b(\tau_1) - y)H(y - a(\tau_1))f(y, \tau_1)e^{i\eta y} dy \\
&\quad \times \int_{-\infty}^{\infty} H(b(\tau_2) - u)H(u - a(\tau_2))g(u, \tau_2)e^{i\eta u} du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(b(\tau_1) - y)H(y - a(\tau_1))H(b(\tau_2) - u)H(u - a(\tau_2)) \\
&\quad \times f(y, \tau_1)g(u, \tau_2)e^{i\eta(u+y)} dy du.
\end{aligned}$$

Let

$$y = x - u \quad \Rightarrow \quad dy = dx,$$

therefore

$$\begin{aligned}
& \hat{F}(\eta, \tau_1)\hat{G}(\eta, \tau_2) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(b(\tau_1) - (x - u))H((x - u) - a(\tau_1))H(b(\tau_2) - u)H(u - a(\tau_2)) \\
&\quad \times f(x - u, \tau_1)g(u, \tau_2)e^{i\eta x} dx du \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(b(\tau_1) - (x - u))H((x - u) - a(\tau_1))f(x - u, \tau_1) \right. \\
&\quad \left. \times H(b(\tau_2) - u)H(u - a(\tau_2))g(u, \tau_2) du \right] e^{i\eta x} dx \\
&= \mathcal{F} \left\{ \int_{-\infty}^{\infty} H(b(\tau_1) - (x - u))H((x - u) - a(\tau_1))f(x - u, \tau_1) \right. \\
&\quad \left. \times H(b(\tau_2) - u)H(u - a(\tau_2))g(u, \tau_2) du \right\}.
\end{aligned}$$

Appendix E. Value of the American Strangle Free Boundaries at Expiry

In deriving equations (40) and (41), it is necessary to analyse the limit of equations (38) and (39) as τ tends to zero. Using the method outlined by Kim (1990), we begin by considering

equation (39):

$$\begin{aligned}
K_1 - c_1(\tau) &= c_1(\tau)e^{-q\tau}[N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \\
&\quad - e^{-r\tau}[K_2N(d_2(c_1(\tau), \tau; K_2)) - K_1N(-d_2(c_1(\tau), \tau; K_1))] \\
&\quad + \int_0^\tau qc_1(\xi)e^{-q(\tau-\xi)}[N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) \\
&\quad \quad \quad - N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi \\
&\quad - \int_0^\tau re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) \\
&\quad \quad \quad - K_1N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi.
\end{aligned}$$

This equation can be factorised to produce

$$\begin{aligned}
&c_1(\tau) \left\{ 1 + e^{-q\tau}[N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \right. \\
&\quad \left. + \int_0^\tau qe^{-q(\tau-\xi)}[N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) - N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi \right\} \\
&= K_1 + e^{-r\tau}[K_2N(d_2(c_1(\tau), \tau; K_2)) - K_1N(-d_2(c_1(\tau), \tau; K_1))] \\
&\quad + \int_0^\tau re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) - K_1N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi,
\end{aligned}$$

which then yields the following implicit equation for $c_1(\tau)$:

$$\begin{aligned}
c_1(\tau) &= \left(K_1 + e^{-r\tau}[K_2N(d_2(c_1(\tau), \tau; K_2)) - K_1N(-d_2(c_1(\tau), \tau; K_1))] \right. \\
&\quad \left. + \int_0^\tau re^{-r(\tau-\xi)}[K_2N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) \right. \\
&\quad \quad \quad \left. - K_1N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi \right) \\
&\quad \times \left(1 + e^{-q\tau}[N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \right. \\
&\quad \left. + \int_0^\tau qe^{-q(\tau-\xi)}[N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) \right. \\
&\quad \quad \quad \left. - N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))]d\xi \right)^{-1}. \tag{55}
\end{aligned}$$

Before proceeding further, it should be noted that

$$K_1 < K_2, \quad c_1(\tau) \leq K_1, \quad \text{and} \quad c_2(\tau) \geq K_2. \tag{56}$$

To find the value of $c_1(0)$, we take the limit of equation (55) as τ tends to zero. In order to evaluate this limit, we need to find 4 limits involving d_1 and d_2 . The first to consider is

$$\begin{aligned}\lim_{\tau \rightarrow 0} d_2(c_1(\tau), \tau; K_2) &= \lim_{\tau \rightarrow 0} \frac{\ln \frac{c_1(\tau)}{K_2}}{\sigma \sqrt{\tau}} \\ &= -\infty, \text{ since } c_1(\tau) < K_2.\end{aligned}\tag{57}$$

Secondly, we have

$$\begin{aligned}\lim_{\tau \rightarrow 0} d_2(c_1(\tau), \tau; K_1) &= \lim_{\tau \rightarrow 0} \frac{\ln \frac{c_1(\tau)}{K_1}}{\sigma \sqrt{\tau}} \\ &= \begin{cases} 0, & c_1(0) = K_1 \\ -\infty, & c_1(0) < K_1. \end{cases}\end{aligned}\tag{58}$$

Similarly the following limits can be shown to be

$$\lim_{\tau \rightarrow 0} d_1(c_1(\tau), \tau; K_2) = -\infty, \text{ since } c_1(\tau) < K_2,\tag{59}$$

and

$$\lim_{\tau \rightarrow 0} d_1(c_1(\tau), \tau; K_1) = \begin{cases} 0, & c_1(0) = K_1 \\ -\infty, & c_1(0) < K_1. \end{cases}\tag{60}$$

Note also the following properties of the cumulative normal density function:

$$N(-\infty) = 0, \quad N(0) = 0.5, \quad N(\infty) = 1.\tag{61}$$

Given that the limits (58) and (60) depend on the value of $c_1(0)$ relative to K_1 , there are two cases to consider when finding the limit of equation (55). Consider the first case where $c_1(0) = K_1$. Taking the limit of equation (55) as τ tends to zero, and using the results from equations (57) - (61), we obtain

$$\lim_{\tau \rightarrow 0} c_1(\tau) = K_1.\tag{62}$$

Now consider the second case, where $c_1(0) < K_1$. The limit as τ tends to zero of equation

(55) is now of the form $\frac{0}{0}$, and L'Hopital's rule can therefore be applied. Firstly, let

$$\lim_{\tau \rightarrow 0} c_1(\tau) = \lim_{\tau \rightarrow 0} \frac{\hat{N}_1(\tau)}{\hat{D}_1(\tau)}$$

where

$$\begin{aligned} \hat{N}_1(\tau) \equiv & K_1 + e^{-r\tau} [K_2 N(d_2(c_1(\tau), \tau; K_2)) - K_1 N(-d_2(c_1(\tau), \tau; K_1))] \\ & + \int_0^\tau r e^{-r(\tau-\xi)} [K_2 N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) \\ & \quad - K_1 N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))] d\xi, \end{aligned}$$

and

$$\begin{aligned} \hat{D}_1(\tau) \equiv & 1 + e^{-q\tau} [N(d_1(c_1(\tau), \tau; K_2)) - N(-d_1(c_1(\tau), \tau; K_1))] \\ & + \int_0^\tau q e^{-q(\tau-\xi)} [N(d_1(c_1(\tau), \tau - \xi; c_2(\xi))) \\ & \quad - N(-d_1(c_1(\tau), \tau - \xi; c_1(\xi)))] d\xi. \end{aligned}$$

To apply L'Hopital's rule, we must differentiate both $\hat{N}_1(\tau)$ and $\hat{D}_1(\tau)$, and take their limit as τ tends to zero. For $\hat{N}_1(\tau)$ we have

$$\begin{aligned} \hat{N}'_1(\tau) = & -r e^{-r\tau} [K_2 N(d_2(c_1(\tau), \tau; K_2)) - K_1 N(-d_2(c_1(\tau), \tau; K_1))] \\ & + e^{-r\tau} \left[K_2 N'(d_2(c_1(\tau), \tau; K_2)) \frac{\partial d_2(c_1(\tau), \tau; K_2)}{\partial \tau} \right. \\ & \quad \left. + K_1 N'(-d_2(c_1(\tau), \tau; K_1)) \frac{\partial d_2(c_1(\tau), \tau; K_1)}{\partial \tau} \right] \\ & + r [K_2 N(d_2(c_1(\tau), 0; c_2(\tau))) - K_1 N(-d_2(c_1(\tau), 0; c_1(\tau)))] \\ & + r \int_0^\tau \{-r e^{-r(\tau-\xi)} [K_2 N(d_2(c_1(\tau), \tau - \xi; c_2(\xi))) \\ & \quad - K_1 N(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))] \\ & \quad + e^{-r(\tau-\xi)} [K_2 N'(d_2(c_1(\tau), \tau - \xi; c_2(\xi)))\theta_1 \\ & \quad + K_1 N'(-d_2(c_1(\tau), \tau - \xi; c_1(\xi)))\theta_2]\} d\xi, \end{aligned}$$

where

$$\theta_1 = \frac{\partial d_2(c_1(\tau), \tau - \xi; c_2(\xi))}{\partial \tau}$$

and

$$\theta_2 = \frac{\partial d_2(c_1(\tau), \tau - \xi; c_1(\xi))}{\partial \tau}.$$

Note that

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and

$$\lim_{x \rightarrow \infty} N'(x) = 0$$

at a faster rate than any other terms observed in $\hat{N}'_1(\tau)$ (see Kim, 1990). We also note that

$$\lim_{\xi \rightarrow \tau} d_2(c_1(\tau), \tau - \xi; c_1(\tau)) = 0$$

and

$$\lim_{\xi \rightarrow \tau} d_2(c_1(\tau), \tau - \xi; c_2(\tau)) = -\infty.$$

Combining all these limit results, it is concluded that

$$\begin{aligned} \lim_{\tau \rightarrow 0} \hat{N}'_1(\tau) &= -r[K_2 N(-\infty) - K_1 N(\infty)] + 0 + r[K_2 N(-\infty) - K_1 N(0)] + 0 \\ &= \frac{r}{2} K_1. \end{aligned} \tag{63}$$

Similarly for $\hat{D}'_1(\tau)$ it can be shown that

$$\lim_{\tau \rightarrow 0} \hat{D}'_1(\tau) = \frac{q}{2}. \tag{64}$$

Thus it is concluded that

$$\lim_{\tau \rightarrow 0} c_1(\tau) = \frac{r}{q} K_1. \tag{65}$$

Recalling that this result only holds when $c_1(0) < K_1$, it follows that we must have $r < q$.

Combining the results from equations (62) and (65) we find that

$$\lim_{\tau \rightarrow 0} c_1(\tau) = \min \left(K_1, \frac{r}{q} K_1 \right)$$

which is equation (40). Similarly the process can be repeated for equation (39), yielding

$$\lim_{\tau \rightarrow 0} c_2(\tau) = \max \left(K_2, \frac{r}{q} K_2 \right)$$

which is equation (41).

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