A Variance Reduction Technique based on Integral Representations

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Abstract. Standard Monte Carlo methods can often be significantly improved with the addition of appropriate variance reduction techniques. In this paper a new and powerful variance reduction technique is presented. The method is based directly on the Itô calculus and is used to find unbiased variance reduced estimators for the expectation of functionals of Itô diffusion processes. The approach considered has wide applicability, for instance, it can be used as a means of approximating solutions of parabolic partial differential equations or applied to valuation problems that arise in mathematical finance. We illustrate how the method can be applied by considering the pricing of European style derivative securities for a class of stochastic volatility models, including the Heston model.

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1 Introduction

The application of Monte Carlo simulation to the pricing and hedging of derivative securities has over the last decade become a popular and widely used methodology. However, in its standard form the method can be very expensive in terms of computer resource usage. In this paper a new and powerful variance reduction technique is described. The method can be applied to a wide range of derivative security valuation problems or to approximate the solution of most types of parabolic partial differential equations (PDEs).

There are two main approaches for the valuation of derivative securities for continuous-time models where the state variables are specified by a scalar or vector stochastic differential equation. The first of these applies a martingale representation for the price process so that the valuation formula can be expressed via a conditional expectation. The second approach uses an appropriately defined PDE. Here it is assumed that the model is Markovian, so that the valuation formula can be expressed as a function of time and state variables. In the standard no-arbitrage pricing theory both approaches involve the construction of an equivalent risk neutral measure.

The application of Monte Carlo simulation to option pricing problems was first demonstrated in the seminal work of Boyle (1977). The basic method has now become standard and the associated literature has rapidly expanded. Some more recent publications, which extend the Monte Carlo method to include new variance reduced estimators, path dependent options or quasi random sequences, include Boyle, Broadie & Glasserman (1997), Broadie & Glasserman (1997), Fu (1995), Grant, Vora & Weeks (1997) and Joy, Boyle & Tan (1996). Recently, Longstaff & Schwartz (2001) have proposed a new least-square Monte Carlo method that can be applied to path dependent securities including American derivatives.

The use of measure transformations to construct a range of variance reduced estimators was first proposed by Milstein (1988) and has subsequently been used or extended by Kloeden & Platen (1999), Hofmann, Platen & Schweizer (1992), Heath (1995) and Fournie, Lasry & Touzi (1997). The Itô integral representation method has been applied by Newton (1994) and extended by Heath (1995). It provides a continuous time extension of the martingale control variates described by Clewlow & Carverhill (1992).

The technique described in this paper, called the *diffusion operator integral* (DOI) *method*, is closely related to the measure transformation and Itô integral representation methods for variance reduction. All three methods, in their basic form, provide a similar level of variance reduction. However, the approach described here is superior in that it can be applied to a wider range of valuation problems. It can also be used to develop iterative approximation methods and more easily adapted to or employed in conjunction with PDE methods.

In this paper we concentrate mainly on the pricing of European style derivatives.

However, the DOI method has been used successfully by the authors in practical applications for a range of path dependent securities, including American derivatives. It can also be adapted to improve the performance of the least-square Monte Carlo method.

2 The DOI Variance Reduction Method

In this section the main theoretical result of the paper is described. Let $T \in (0, \infty)$ be fixed and Γ be an open connected subset of $\Re^d$. Consider a general d-dimensional diffusion process $X^{s,x} = \{X_t^{s,x},\, t \in [s,T]\}$ which satisfies the stochastic differential equation (SDE)

$$dX_t^{s,x} = a(t, X_t^{s,x})\, dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})\, dW_t^j \tag{2.1}$$

for $t \in [s,T]$ with initial value $X_s^{s,x} = x \in \Gamma$ for $s \in [0,T]$.

Here the drift coefficient $a : [0,T] \times \Gamma \to \Re^d$ and diffusion coefficient $b^j : [0,T] \times \Gamma \to \Re^d$, $j \in \{1, 2, \ldots, m\}$, satisfy appropriate growth and Lipschitz conditions so that (2.1) admits a unique strong solution and is Markovian, see Kloeden & Platen (1999). The vector $W = \{W(t) = (W^1(t), \ldots, W^m(t))^\top,\, t \in [0,T]\}$ is an m-dimensional Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$, where $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t\in[0,T]}$ fulfills the usual conditions and $\mathcal{F}_0 = \{\phi, \Omega\}$.

Denote by $\tau : \Omega \to [s,T]$ the first exit time of $(t, X_t^{s,x})$ from $[s,T) \times \Gamma$, that is

$$\tau = \inf\{t \geq s :\, (t, X_t^{s,x}) \notin [s,T) \times \Gamma\}. \tag{2.2}$$

We consider here a stopping time formulation to ensure that our results can be applied to a range of path dependent options including American style derivatives.

Define the operators L^0 and L^j, $j \in \{1, 2, \ldots, m\}$ on a sufficiently smooth function $f : [0,T] \times \Gamma \to \Re$ by

$$\begin{aligned} L^0 f(t,x) &= \frac{\partial f}{\partial t}(t,x) + \sum_{i=1}^{d} a^i(t,x) \frac{\partial f}{\partial x^i}(t,x) \\ &\quad + \frac{1}{2} \sum_{i,k=1}^{d} \sum_{j=1}^{m} b^{i,j}(t,x)\, b^{k,j}(t,x) \frac{\partial^2 f}{\partial x^i\, \partial x^k}(t,x) \end{aligned} \tag{2.3}$$

and

$$L^j f(t,x) = \sum_{i=1}^{d} b^{i,j}(t,x) \frac{\partial f}{\partial x^i}(t,x) \tag{2.4}$$

for $(t,x) \in (0,T) \times \Gamma$.

Let $\partial\Gamma$ denote the boundary of Γ. That is $\partial\Gamma = \{x \notin \Gamma : \exists (x_n)_{n\in\{1,2,\dots\}}$ with $x_n \in \Gamma$, $n \in \{1,2,\dots\}$ and $\lim_{n\to\infty} \|x - x_n\| = 0\}$. Consider a payoff function $h : B \to \Re$, where

$$B = [0,T) \times \partial\Gamma \cup \{T\} \times (\Gamma \cup \partial\Gamma) \tag{2.5}$$

and a valuation function $u : [0,T] \times \Gamma \to \Re$ given by

$$u(t,x) = E\left(h(\tau, X_\tau^{t,x})\right) \tag{2.6}$$

for $(t,x) \in [0,T] \times \Gamma$. We assume that h satisfies appropriate integrability conditions so that the process $M = \{M_t,\, t \in [0,T]\}$ with

$$M_t = E\left(h(\tau, X_\tau^{0,x}) \,\middle|\, \mathcal{F}_t\right) \tag{2.7}$$

for $t \in [0,T]$ is a square integrable $(\underline{\mathcal{F}}, P)$-martingale. Using the martingale representation theorem, see Karatzas & Shreve (1988), together with the Markov property for X, it can be inferred that there exists an m-dimensional $\underline{\mathcal{F}}$-predictable integrand $\xi = \{\xi_t = (\xi_t^1, \dots, \xi_t^m)^\top,\, t \in [0,T]\}$ with

$$\begin{aligned} M_t &= u(t, X_{t\wedge\tau}^{0,x}) \\ &= u(0,x) + \sum_{j=1}^m \int_0^{t\wedge\tau} \xi_s^j \, dW_s^j \end{aligned} \tag{2.8}$$

for $t \in [0,T]$.

Our aim will be to find an unbiased variance reduced estimator for $u(0,x)$ given an approximation $\bar{u} : [0,T] \times \Gamma \to \Re$ to u. Examples will be provided in this and subsequent sections on how to construct such approximate functions. We assume that $\bar{u} \in \mathcal{C}^{1,2}([0,T] \times \Gamma)$ is from the class of functions $f : [0,T] \times \Gamma \to \Re$ with $\frac{\partial f}{\partial t}(\cdot, x)$ continuous on $(0,T)$ for $x \in \Gamma$ and $\frac{\partial^2 f}{\partial x^i\, \partial x^k}(t, \cdot)$ continuous on Γ for $t \in (0,T)$ and $i,k \in \{1,2,\dots,d\}$. We also assume that $L^j\, \bar{u}$ satisfies appropriate integrability conditions so that the process $\bar{M}^j = \{\bar{M}_t^j,\, t \in [0,T]\}$ with

$$\bar{M}_t^j = \int_0^{t\wedge\tau} L^j\, \bar{u}(s, X_s^{0,x})\, dW_s^j \tag{2.9}$$

for $t \in [0,T]$, $j \in \{1,2,\dots,m\}$ is a square integrable $(\underline{\mathcal{F}}, P)$-martingale. In addition, it is assumed that

$$\bar{u}(\tau, X_\tau^{0,x}) = u(\tau, X_\tau^{0,x}) = h(\tau, X_\tau^{0,x}). \tag{2.10}$$

That is $\bar{u}$ is an approximation to u which matches the payoff for u on B. Applying the Itô formula for semimartingales to $\bar{u}$ and using (2.1) yields

$$\bar{u}(\tau, X_\tau^{0,x}) = \bar{u}(0,x) + \int_0^\tau L^0\, \bar{u}(t, X_t^{0,x})\, dt + \sum_{j=1}^m \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j. \tag{2.11}$$

Consequently, by (2.6), (2.10), (2.11), Fubini's theorem and the martingale property of $\bar{M}^j$ one obtains

$$\begin{aligned} u(0,x) &= E\left(h(\tau, X_\tau^{0,x})\right) \\ &= E\left(\bar{u}(\tau, X_\tau^{0,x})\right) \\ &= \bar{u}(0,x) + E\left(\int_0^\tau L^0\,\bar{u}(t, X_t^{0,x})\,dt\right) \\ &= \bar{u}(0,x) + \int_0^T E\left(\mathbf{1}_{\{t<\tau\}} L^0\,\bar{u}(t, X_t^{0,x})\right) dt \end{aligned} \tag{2.12}$$

for $x \in \Gamma$. Here $\mathbf{1}_{\{t<\tau\}}$ denotes the indicator function applied to the event $\{t < \tau\}$. Due to (2.12) the random variable

$$\bar{Z}_\tau = \bar{u}(0,x) + \int_0^\tau L^0\,\bar{u}(t, X_t^{0,x})\,dt \tag{2.13}$$

is an *unbiased estimator* for $u(0,x)$. We will refer to this estimator as the DOI estimator.

To compute the variance of $\bar{Z}_\tau$ note that from (2.13), (2.11), (2.10) and (2.8) we have

$$\begin{aligned} \bar{Z}_\tau &= \bar{u}(\tau, X_\tau^{0,x}) - \sum_{j=1}^m \int_0^\tau L^j\,\bar{u}(t, X_t^{0,x})\,dW_t^j \\ &= u(\tau, X_\tau^{0,x}) - \sum_{j=1}^m \int_0^\tau L^j\,\bar{u}(t, X_t^{0,x})\,dW_t^j \\ &= u(0,x) + \sum_{j=1}^m \int_0^\tau \left(\xi_t^j - L^j\,\bar{u}(t, X_t^{0,x})\right) dW_t^j. \end{aligned} \tag{2.14}$$

Since $\bar{M}^j$ is square integrable for each $j \in \{1, 2, \ldots, m\}$ the variance of $\bar{Z}_\tau$ is therefore given by

$$\begin{aligned} \operatorname{Var}(\bar{Z}_\tau) &= E\left[\left(\sum_{j=1}^m \int_0^\tau \left(\xi_t^j - L^j\,\bar{u}(t, X_t^{0,x})\right) dW_t^j\right)^2\right] \\ &= \sum_{j=1}^m \int_0^T E\left(\mathbf{1}_{\{t<\tau\}} \left(\xi_t^j - L^j\,\bar{u}(t, X_t^{0,x})\right)^2\right) dt \end{aligned} \tag{2.15}$$

If the valuation function u is sufficiently smooth to permit an application of the Itô formula, then the integrands ξ^j in (2.8) take the form

$$\xi_t^j = L^j\,u(t, X_t^{0,x}) \tag{2.16}$$

for $j \in \{1, 2, \ldots, m\}$ and $t \in [0, T \wedge \tau]$. In this case the expression in (2.15) can be replaced by the formula

$$\mathrm{Var}(\bar{Z}_\tau) = \sum_{j=1}^{m} \int_0^T E\left(\mathbf{1}_{\{t<\tau\}}\left(\left(L^j\, u - L^j\, \bar{u}\right)(t, X_t^{0,x})\right)^2\right) dt. \tag{2.17}$$

This shows that if a good approximation $\bar{u}$ to u can be found, so that $L^j\, u$ is close to $L^j\, \bar{u}$ for each $j \in \{1, 2, \ldots, m\}$, then the variance of $\bar{Z}_\tau$ will be small. The form of (2.17) shows that finding a small variance for $\bar{Z}_\tau$ is related to finding suitable approximations to the hedge ratio for the underlying security.

Note that by (2.6), (2.11), (2.13) and since $\bar{M}^j$ is a square integrable martingale, the random variable

$$\begin{aligned}
\bar{Z}_{\tau,\alpha} &= \bar{u}(\tau, X_\tau^{0,x}) - \alpha \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j \\
&= \bar{Z}_\tau + (1-\alpha) \sum_{j=1}^{m} \int_0^\tau L^j\, \bar{u}(t, X_t^{0,x})\, dW_t^j
\end{aligned} \tag{2.18}$$

is an *unbiased estimator* for $u(0, x)$. This variance reduced estimator can be interpreted as a control variate based on an Itô integral representation, see Heath (1995). Inspection of (2.18) shows that for $\alpha = 1$ the DOI and the above unbiased estimator are the same. This observation therefore explains a relationship between the two variance reduction methods.

3 Extensions of the DOI Method

We now consider applications of the DOI variance reduction technique to the approximation of solutions of parabolic PDEs. In addition, some extensions of the method are presented.

In this section the domain Γ is also assumed to be a bounded subset of $\Re^d$ with $B \subseteq \Re^d$ as given by (2.5). Our task will be to find an approximation to the solution at an initial point $(0, x)$ with $x \in \Gamma$ to the PDE

$$L^0\, u(t, x) = 0 \tag{3.1}$$

for $(t, x) \in (0, T) \times \Gamma$ with boundary condition

$$u(t, x) = h(t, x) \tag{3.2}$$

for $(t, x) \in B$ and some measurable function $h :\, B \to \Re$, where L^0 is the diffusion operator in (2.3). The link between this PDE formulation and the stochastic representation (2.6) is provided by the well-known Feynman-Kac formula. Sufficient conditions ensuring that (2.6) is a solution to (3.1) - (3.2), are for example, provided by Friedman (1975), Krylov (1980) and Heath & Schweizer (2000). In this case, for any solution u of (3.1) - (3.2), an application of the Itô formula shows that

$$u(t,x) = u(\tau, X_\tau^{t,x}) - \sum_{j=1}^{m} \int_t^\tau L^j\, u(s, X_s^{t,x})\, dW_s^j \tag{3.3}$$

for $(t,x) \in [0,T] \times \Gamma$. The assumption that Γ is a bounded subset of $\Re^d$ implies that

$$\sup_{x\in\Gamma} |L^j\, u(t,x)| < \infty$$

a.s. for $t \in [0,T]$ and $j \in \{1, 2, \ldots, m\}$ and therefore

$$E\left(\sum_{j=1}^{m} \int_t^\tau L^j\, u(s, X_s^{t,x})\, dW_s^j\right) = 0.$$

Combining this result with (3.3) ensures that the representation (2.6) applies for u. Consequently, the solution to (3.1) - (3.2) must be unique.

Suppose there is some approximation $\bar{u} \in \mathcal{C}^{1,2}([0,T] \times \Gamma)$ with $\bar{u}(t,x) = h(t,x)$ for $(t,x) \in B$. Then the difference $u^* = u - \bar{u}$ satisfies the PDE

$$L^0\, u^*(t,x) + L^0\, \bar{u}(t,x) = 0 \tag{3.4}$$

with boundary condition

$$u^*(t,x) = 0 \tag{3.5}$$

for $(t,x) \in B$. If $\bar{u}$ is close to u it may be computationally much more efficient to solve (3.4) - (3.5) rather than (3.1) - (3.2). This provides a simple but powerful illustration of how the ideas used to produce variance reduced estimators can be directly applied in a PDE setting.

In some cases it may not be possible or convenient to use only PDE methods to solve (3.1) - (3.2). For example, the dimension d may be too high. A hybrid method, which combines features of a PDE approach together with the DOI variance reduction technique, would be to first apply a numerical solver to find an approximation $\bar{u}$ to u. Then by simulation, using the formula (2.12) with estimator (2.13), a better approximation of the solution $u(0,x)$, $x \in \Gamma$, at the point $(0,x)$, could be obtained. Typically, the approximate solution $\bar{u}$, output from say an implicit finite difference solver, would only be available at discrete points on a grid. Finally, some additional smoothing or multi-dimensional interpolation technique could be applied to ensure that $\bar{u} \in \mathcal{C}^{1,2}([0,T] \times \Gamma)$.

For many practical applications that arise in a financial modelling context a systematic way of obtaining an approximate solution $\bar{u}$ satisfying (2.6) is to first find an approximation to the diffusion process X. Let $\bar{X} = \{\bar{X}_t,\, t \in [s,T]\}$ be a d-dimensional diffusion process that is governed by the SDE

$$d\bar{X}_t^{s,x} = \bar{a}(t, \bar{X}_t^{s,x})\, dt + \sum_{j=1}^{m} \bar{b}^j(t, \bar{X}_t^{s,x})\, dW_t^j \tag{3.6}$$

for $t \in [s,T]$, $s \in [0,T]$ with coefficient functions $\bar{a} : [0,T] \times \Gamma \to \Re^d$ and $\bar{b}^j : [0,T] \times \Gamma \to \Re^d$, $j \in \{1,2,\dots,m\}$. As is the case for (2.1) it is assumed that (3.6) admits a unique strong solution which is Markovian. Denote by $\bar{L}^0$ the corresponding PDE operator, as given in (2.3), with $\bar{a}^i$ and $\bar{b}^{i,j}$, $i \in \{1,2,\dots,d\}$, $j \in \{1,2,\dots,m\}$ replacing a^i and $b^{i,j}$, respectively. Define the function $\bar{u} : [0,T] \times \Gamma \to \Re$ by replacing the diffusion process X appearing in (2.6) with $\bar{X}$, that is

$$\bar{u}(t,x) = E\left(h(\bar{X}_{\bar{\tau}}^{t,x})\right) \tag{3.7}$$

for $(t,x) \in [0,T] \times \Gamma$. Here $\bar{\tau} : \Omega \to [s,T]$ is the first exit time of $(t, \bar{X}_t^{s,x})$ from $[s,T) \times \Gamma$, that is

$$\bar{\tau} = \inf\left\{t \geq s : \left(t, \bar{X}_t^{s,x}\right) \notin [s,T) \times \Gamma\right\}. \tag{3.8}$$

If $\bar{u}$ is sufficiently smooth, then an application of the Kolmogorov backward equation yields the PDE

$$\bar{L}^0\, \bar{u}(t,x) = 0 \tag{3.9}$$

for $(t,x) \in [0,T] \times \Gamma$ with boundary condition

$$\bar{u}(t,x) = h(t,x) \tag{3.10}$$

for $(t,x) \in B$. Therefore, with this choice for $\bar{u}$ and using (3.9), the second last equation in (2.12) takes the form

$$\begin{aligned} u(0,x) &= \bar{u}(0,x) + E\left(\int_0^\tau L^0\, \bar{u}(t, X_t^{0,x})\, dt\right) \\ &= \bar{u}(0,x) + E\left(\int_0^\tau (L^0 - \bar{L}^0)\, \bar{u}(t, X_t^{0,x})\, dt\right) \end{aligned} \tag{3.11}$$

This shows that if $\bar{L}^0$ is close to L^0, then the variance of the unbiased estimator $\bar{Z}_\tau$ given by (2.13) will be small. The estimator $\bar{Z}_\tau$ is given by

$$\bar{Z}_\tau = \bar{u}(0,x) + \int_0^\tau (L^0 - \bar{L}^0)\, \bar{u}(t, X_t^{0,x})\, dt. \tag{3.12}$$

Consider an additional approximation function $\bar{\bar{u}} : [0,T] \times \Gamma \to \Re$ defined by

$$\bar{\bar{u}}(t,x) = \bar{u}(t,x) + z(t,x), \tag{3.13}$$

where

$$z(t,x) = E\left(\int_t^\tau L^0\, \bar{u}(s, \bar{X}_s^{t,x})\, ds\right) \tag{3.14}$$

for $(t,x) \in [0,T] \times \Gamma$. Let $\hat{z} : [0,T] \times \Gamma \times (0,\infty) \to \Re$ be given by

$$\hat{z}(t,x,y) = y + z(t,x) \tag{3.15}$$

for $(t,x,y) \in [0,T] \times \Gamma \times (0,\infty)$ with the process $Y^{s,y} = \{Y_t^{s,y},\, t \in [s,T]\}$ defined by the relation

$$Y_t^{s,y^*} = y + \int_s^t \mathbf{1}_{\{v<\tau\}}\, L^0\, \bar{u}(v, \bar{X}_v^{s,x})\, dv \tag{3.16}$$

for $t \in [s,T]$, $y \in (0,\infty)$, $x = (x_1, \ldots, x_d) \in \Gamma$, $y^* = (x_1, \ldots, x_d, y)$ and $s \in [0,T]$. Combining (3.14), (3.15) and (3.16) implies that

$$\hat{z}(t,x,y) = E\left(Y_T^{t,y^*}\right). \tag{3.17}$$

Let us define the extended operator $\hat{L}^0$ on sufficiently smooth functions $f : [0,T] \times \Gamma \times (0,\infty) \to \Re$ by

$$\begin{aligned} \hat{L}^0 f(t,x,y) \quad = \quad & \frac{\partial f}{\partial t}(t,x,y) + \sum_{i=1}^d a^i(t,x)\, \frac{\partial f}{\partial x^i}(t,x,y) \\ & + \frac{1}{2} \sum_{i,k=1}^d \sum_{j=1}^m b^{i,j}(t,x)\, b^{k,j}(t,x)\, \frac{\partial^2 f}{\partial x^i\, \partial x^k}(t,x,y) \\ & + L^0\, \bar{u}(t,x) \frac{\partial f}{\partial y}(t,x,y) \end{aligned} \tag{3.18}$$

for $(t,x,y) \in (0,T) \times \Gamma \times (0,\infty)$. Application of the Kolmogorov backward equation to $\hat{z}$ using (3.17) and the $(d+1)$-dimensional diffusion process consisting of X together with the component Y shows that

$$\hat{L}^0\, \hat{z}(t,x,y) = 0$$

for $(t,x,y) \in (0,T) \times \Gamma \times (0,\infty)$. Using (3.15) and (3.18) this equation can be rewritten in the form

$$\bar{L}^0\, z(t,x) + L^0\, \bar{u}(t,x) = 0 \tag{3.19}$$

for $(t,x) \in (0,T) \times \Gamma$. Combining (3.13), (3.19), (3.14) and (3.9) yields

$$\begin{aligned} L^0\,\bar{u}(t,x) &= L^0\,\bar{u}(t,x) + L^0\,z(t,x) \\ &= (L^0 - \bar{L}^0)\,z(t,x) \\ &= (L^0 - \bar{L}^0)\left(E\left(\int_t^\tau L^0\,\bar{u}(s,\bar{X}_s^{t,x})\,ds\right)\right) \\ &= E\left(\int_t^\tau (L^0 - \bar{L}^0)\,L^0\,\bar{u}(s,\bar{X}_s^{t,x})\,ds\right) \\ &= E\left(\int_t^\tau (L^0 - \bar{L}^0)\,\left[(L^0 - \bar{L}^0)\,\bar{u}(s,\bar{X}_s^{t,x})\right]ds\right). \end{aligned} \tag{3.20}$$

Consequently, these calculations provide the basis for the construction of an iterative series of approximations for u.

Let $\bar{u}$ be a smooth approximation function to u but which does not match the payoff or boundary values for u on the set B. In this case by (2.6) and (2.11) the DOI variance reduction equation (2.12) takes the form

$$\begin{aligned} u(0,x) &= E\left(h(\tau, X_\tau^{0,x})\right) \\ &= \bar{u}(0,x) + E\left(h(\tau,X_\tau^{0,x}) - \bar{u}(\tau,X_\tau^{0,x})\right) + E\left(\int_0^\tau L^0\,\bar{u}(t,X_t^{0,x})\,dt\right) \\ &= \bar{u}(0,x) + E\left(h(\tau,X_\tau^{0,x}) - \bar{u}(\tau,X_\tau^{0,x})\right) + \int_0^T E\left(\mathbf{1}_{\{t<\tau\}}\,L^0\,\bar{u}(t,X_t^{0,x})\right)dt. \end{aligned} \tag{3.21}$$

The corresponding unbiased estimator $\bar{Z}_\tau^h$ for $u(0,x)$ is given by

$$\bar{Z}_\tau^h = \bar{u}(0,x) + h(\tau,X_\tau^{0,x}) - \bar{u}(\tau,X_\tau^{0,x}) + \int_0^\tau L^0\,\bar{u}(t,X_t^{0,x})\,dt. \tag{3.22}$$

Note that (3.21) reduces to (2.12) if $h(\tau,X_\tau^{0,x}) = \bar{u}(\tau,X_\tau^{0,x})$.

4 An Example using the Heston Model

In this section the DOI variance reduction method is utilized to price European call options for the Heston stochastic volatility model, see Heston (1993). In the original treatment by Heston the prices of European style derivatives are computed via an integral representation using characteristic functions. The DOI method, described in this paper, is much more general and can be applied to a wide class of stochastic volatility models. The Heston model is a well-known example, which we have chosen to illustrate the DOI method. Consider the vector diffusion process $X = (S^{s,x^1}, v^{s,x^2}) = \{(S_t^{s,x^1}, v_t^{s,x^2}),\ t \in [s,T]\}$, whose real-world dynamics are governed by the system of SDEs

$$\begin{aligned} dS_t^{s,x^1} &= u\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, dW_t^1 \\ dv_t^{s,x^2} &= \kappa \left(\theta - v_t^{s,x^2}\right) dt + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, dW_t^1 + \sqrt{1-\varrho^2}\, dW_t^2\right) \end{aligned} \tag{4.1}$$

for $t \in [s,T]$ and $s \in [0,T]$ with initial values $S_s^{s,x^1} = x^1 > 0$ and $v_s^{s,x^2} = x^2 > 0$ and nonnegative constants μ, κ, θ, ξ and $\varrho \in [-1,1]$. The vector $W = \{W_t = (W_t^1, W_t^2),\ t \in [0,T]\}$ is a two-dimensional Wiener process. Here the symbols S^{s,x^1} and v^{s,x^2} denote the stock and squared volatility price processes, respectively. The savings account process $\bar{B} = \{\bar{B}_t,\ t \in [0,T]\}$ is given by $\bar{B}_t = e^{rt}$, where $r > 0$ is, for simplicity, a constant short rate.

For this type of model, which is incomplete if $|\varrho| \neq 1$, we need to choose a local equivalent martingale measure $\tilde{P}$, with $P \approx \tilde{P}$, such that the process $\frac{S}{B}$ is an $(\underline{\mathcal{F}}, \tilde{P})$-local martingale. This is sometimes done indirectly by specifying the market price of volatility risk. However, in this paper we choose $\tilde{P}$ to be the minimal equivalent local martingale measure, see for example, Föllmer & Schweizer (1991), Hofmann, Platen & Schweizer (1992) and Heath, Platen & Schweizer (2001). This measure has the property that it is both equivalent to P and such that any local $(\underline{\mathcal{F}}, \tilde{P})$-martingale, which is orthogonal to $\frac{S}{B}$, remains a local $(\underline{\mathcal{F}}, \tilde{P})$-martingale. Under this measure $\tilde{P}$ the dynamics for S^{s,x^1} and v^{s,x^2} becomes

$$\begin{aligned} dS_t^{s,x^1} &= r\, S_t^{s,x^1}\, dt + \sqrt{v_t^{s,x^2}}\, S_t^{s,x^1}\, d\tilde{W}_t^1 \\ dv_t^{s,x^2} &= \kappa \left(\tilde{\theta} - v_t^{s,x^2}\right) dt + \xi \sqrt{v_t^{s,x^2}} \left(\varrho\, d\tilde{W}_t^1 + \sqrt{1-\varrho^2}\, d\tilde{W}_t^2\right) \end{aligned} \tag{4.2}$$

for $t \in [s,T]$ and $s \in [0,T]$, where $\tilde{\theta} = \theta - \frac{\xi\, \varrho\, (\mu - r)}{\kappa}$ and

$$\begin{aligned} d\tilde{W}_t^1 &= \frac{\mu - r}{\sqrt{v_t}}\, dt + dW_t^1 \\ d\tilde{W}_t^2 &= dW_t^2. \end{aligned}$$

Here $\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^1, \tilde{W}_t^2),\ t \in [0,T]\}$ is a two-dimensional Wiener process under $\tilde{P}$. It is assumed that $\frac{\kappa \tilde{\theta}}{\xi^2} > \frac{1}{2}$ so that the process v^{s,x^2} remains a.s. strictly positive under $\tilde{P}$, see Karatzas & Shreve (1988). This condition also ensures that the stock price process S^{s,x^1} remains a.s. strictly positive under $\tilde{P}$.

We now describe a DOI variance reduction technique which can be utilized to approximate the option price

$$c(0,x) = e^{-rT}\, u(0,x), \tag{4.3}$$

where

$$u(0,x) = \tilde{E}\left((S_T^{0,x^1} - K)^+\right).$$

This means, we consider a European call with strike K under the Heston model with risk neutral dynamics given by (4.2) and $x = (x^1, x^2) = (S_0^{s,x^1}, v_0^{s,x^2})$. The main idea will be to use the estimate (3.11) with estimator (3.12). As already mentioned an approximation $\bar{X}$ for the diffusion process X needs to be found. A convenient choice, based on the Black-Scholes model, is as follows: Let $\bar{X} = (\bar{S}^{s,x^1}, \bar{v}^{s,x^2}) = \{(\bar{S}_t^{s,x^1}, \bar{v}_t^{s,x^2}), t \in [s,T]\}$ be the two-dimensional diffusion process which satisfies the SDE

$$\begin{aligned} d\bar{S}_t^{s,x^1} &= r\,\bar{S}_t^{s,x^1}\,dt + \sqrt{\bar{v}_t^{s,x^2}}\,\bar{S}_t^{s,x^1}\,d\tilde{W}_t^1 \\ d\bar{v}_t^{s,x^2} &= \kappa\left(\tilde{\theta} - \bar{v}_t^{s,x^2}\right)dt \end{aligned} \tag{4.4}$$

for $t \in [s,T]$ and $s \in [0,T]$ with initial values $\bar{S}_s^{s,x^1} = x^1 > 0$ and $\bar{v}_s^{s,x^2} = x^2 > 0$. For this system of SDEs the solution for $\bar{v}$ can be explicitly computed and is given by

$$\bar{v}_t^{s,x^2} = \tilde{\theta} + (x^2 - \tilde{\theta})\,e^{-\kappa(t-s)} \tag{4.5}$$

for $t \in [s,T]$ and $s \in [0,T]$. Denote by $BS(x^1, K, r, \sigma, T)$ the Black-Scholes price for a European call option with spot price x^1, strike K, short rate r, constant volatility σ and maturity T. Using (4.5) and setting $\tau = T$, the approximate function $\bar{u} : [0,T] \times (0,\infty)^2 \to \Re$ given in (3.7) takes the form

$$\begin{aligned} \bar{u}(t,x) &= \tilde{E}\left((\bar{S}_T^{t,x^1} - K)^+\right) \\ &= e^{r(T-t)}\,BS(x^1, K, r, \bar{\sigma}_t, T-t) \end{aligned} \tag{4.6}$$

for $(t,x) \in [0,T] \times (0,\infty)^2$, where $x = (x^1, x^2) = (\bar{S}_t^{s,x^1}, \bar{v}_t^{s,x^2})$ and

$$\begin{aligned} \bar{\sigma}_t &= \sqrt{\frac{1}{T-t}\int_t^T \bar{v}_z^{t,x^2}\,dz} \\ &= \sqrt{\tilde{\theta} - (x^2 - \tilde{\theta})\,\frac{e^{-\kappa(T-t)} - 1}{\kappa(T-t)}}. \end{aligned} \tag{4.7}$$

Evaluation of the expectation appearing in (3.11) requires the calculation of the values $(L^0 - \bar{L}^0)\bar{u}(t, X_t^{0,x})$ for $t \in [0,T]$. Using (4.2) and (4.4) and noting that the coordinates for $X = (X^1, X^2)$ and $\bar{X} = (\bar{X}^1, \bar{X}^2)$ correspond to the vectors (S^{s,x^1}, v^{s,x^2}) and $(\bar{S}^{s,x^1}, \bar{v}^{s,x^2})$, respectively, the operator $(L^0 - \bar{L}^0)$ can be determined. Thus, for sufficiently smooth functions $f : [0,T] \times (0,\infty)^2 \to \Re$ we have

$$(L^0 - \bar{L}^0)\,f(t,x) = \xi\,x^2\left(\varrho\,\frac{\partial^2 f}{\partial x^1\,\partial x^2}(t,x) + \frac{1}{2}\,\xi\,\frac{\partial^2 f}{\partial (x^2)^2}(t,x)\right) \tag{4.8}$$

for $(t,x) \in (0,T) \times (0,\infty)^2$, where $x = (x^1, x^2)$.

The corresponding values for $(L^0 - \bar{L}^0)\,\bar{u}(t,x)$ can now be computed in an explicit form because $\bar{u}(t,x)$ can be expressed as a scaled Black-Scholes price with volatility $\bar{\sigma}_t$, see (4.6) and (4.7). Substitution of the appropriate partial derivatives for $\bar{u}$ into (4.8) yields

$$\begin{aligned}(L^0 - \bar{L}^0)\,\bar{u}(t,x) &= \xi\, x^2\, e^{r(T-t)} \Bigg[\varrho\, \frac{\partial^2 BS}{\partial x^1\, \partial \bar{\sigma}_t}(x^1, K, r, \bar{\sigma}_t, T-t)\, \frac{\partial \bar{\sigma}_t}{\partial x^2} \\ &\quad + \frac{1}{2}\,\xi \left\{ \frac{\partial^2 BS}{\partial \sigma_t^2}(x^1, K, r, \bar{\sigma}_t, T-t) \left(\frac{\partial \bar{\sigma}_t}{\partial x^2} \right)^2 \right. \\ &\quad \left. + \frac{\partial BS}{\partial \sigma_t}(x^1, K, r, \bar{\sigma}_t, T-t)\, \frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2} \right\} \Bigg]. \end{aligned} \tag{4.9}$$

The partial derivatives $\frac{\partial BS}{\partial \bar{\sigma}_t}$, $\frac{\partial^2 BS}{\partial x^1\, \partial \bar{\sigma}_t}$, $\frac{\partial^2 BS}{\partial \bar{\sigma}_t^2}$, $\frac{\partial \bar{\sigma}_t}{\partial x^2}$ and $\frac{\partial^2 \bar{\sigma}_t}{\partial (x^2)^2}$ can be computed in a straightforward manner using the Black-Scholes formula together with the expression for $\bar{\sigma}_t$ given in (4.7). For example, $\frac{\partial BS}{\partial \bar{\sigma}_t}$ is often referred to as the Black-Scholes vega, see Hull (1993).

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be an equi-spaced time discretization of the interval $[0,T]$ with step size $\Delta = \frac{T}{N}$. For the simulation results described in this section a predictor-corrector method of weak order 1.0 proposed in Platen (1995) was employed. For a general d-dimensional diffusion process, which satisfies (2.1), this scheme takes the form

$$\begin{aligned} Y_{n+1} &= Y_n + \left\{ \alpha\, \hat{a}(t_{n+1}, \hat{Y}_{n+1}) + (1-\alpha)\, \hat{a}(t_n, Y_n) \right\} \Delta \\ &\quad + \sum_{j=1}^{m} \left\{ \eta\, b^j(t_{n+1}, \hat{Y}_{n+1}) + (1-\eta)\, b^j(t_n, Y_n) \right\} \Delta W_n^j \end{aligned} \tag{4.10}$$

for $n \in \{0, 1, \ldots, N-1\}$ with predictor

$$\hat{Y}_{n+1} = Y_n + a(t_n, Y_n)\,\Delta + \sum_{j=1}^{m} b^j\, \Delta W_n^j$$

and modified drift coefficient values

$$\hat{a}(t_n, Y_n) = a(t_n, Y_n) - \eta \sum_{i=1}^{d} \sum_{j=1}^{m} b^{i,j}(t_n, Y_n)\, \frac{\partial b^j}{\partial x^i}(t_n, Y_n).$$

Here $\alpha, \eta \in [0,1]$ and ΔW_n^j, $j \in \{1, 2, \ldots, m\}$, $n \in \{0, 1, \ldots, N-1\}$ are independent $N(0;\Delta)$ Gaussian random variables.

The corresponding discrete time approximation for the estimator $\bar{Z}_T$, given in (3.12) with $\tau = T$, can be obtained by adding this component to the system of equations (4.2). With this choice of components the numerical scheme (4.10) can be applied to construct an approximation $Y_N = (Y_N^1, Y_N^2, Y_N^3)$ to the random vector $(S_T^{0,x^1}, v_T^{0,x^2}, \bar{Z}_T)$ with initial values $Y_0^1 = x^1$, $Y_0^2 = x^2$ and $Y_0^3 = \bar{u}(0,x)$, $x = (x^1, x^2)$. The expectation $\tilde{E}(Y_N^3)$ therefore provides an approximation to the value $u(0,x)$, see (3.11), (4.2) and (4.3).

For all simulation experiments the results have been enhanced by the use of antithetic variates. This consisted of full reflection of both Wiener components. That is sample paths computed via the $2N$ outcomes $(\Delta W_n^1, \Delta W_n^2)$, $n \in \{0, 1, \ldots, N-1\}$ were combined with additional sample paths generated from the pairs $(\Delta W_n^1, -\Delta W_n^2)$, $(-\Delta W_n^1, \Delta W_n^2)$ and $(-\Delta W_n^1, -\Delta W_n^2)$, $n \in \{0, 1, \ldots, N-1\}$.

To provide a graphical illustration of the degree of variance reduction possible with the DOI method, Figure 4.1 displays a group of 10×4 simulated sample paths for the intrinsic value $(S_t^{0,x^1} - K)^+$, $t \in [0,T]$ with $N = 40$ discretization points. The results include sample paths generated from the antithetically produced outcomes as explained above. For this and subsequent plots the following default parameter values were used: $\kappa = 0.6$, $\tilde{\theta} = 0.04$, $\xi = 0.2$, $r = 0.04$, $T = 0.5$, $K = 100$ with initial values $S_0^{0,x^1} = 100$ and $v_0^{0,x^2} = 0.04$. Note that for these parameter values $\frac{\kappa \tilde{\theta}}{\xi^2} = 0.6 > 0.5$. These choices produce a strong stochastic volatility effect but sufficiently constrained to ensure that S^{0,x^1} and v^{0,x^2} remain a.s. strictly positive under $\tilde{P}$.

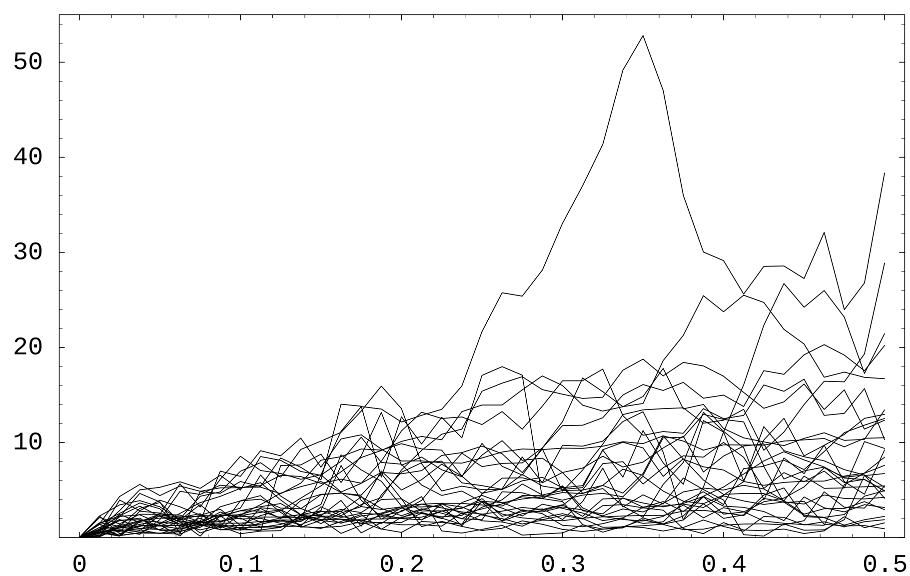

Figure 4.1: Simulated outcomes for the intrinsic value $(S_t^{0,x^1} - K)^+$, $t \in [0,T]$.

Figure 4.2 shows the same set of sample paths for the estimator $\bar{Z}_t$, $t \in [0,T]$. Note that simulated outcomes for $(S_t^{0,x^1} - K)^+$, $t \in [0,T]$ are spread over the wide interval $[0.00, 50.00]$ whereas those for $\bar{Z}_t$ are contained in the extremely small interval $[6.60, 6.77]$. For a larger sample of 100×4 trajectories the standard error for the means $\tilde{E}((S_T^{0,x^1} - K)^+)$ and $\tilde{E}(\bar{Z}_T)$ were 0.608 and 0.004, respectively. Comparison of the two figures therefore demonstrates the dramatic reduction in variance that can be obtained with the DOI technique.

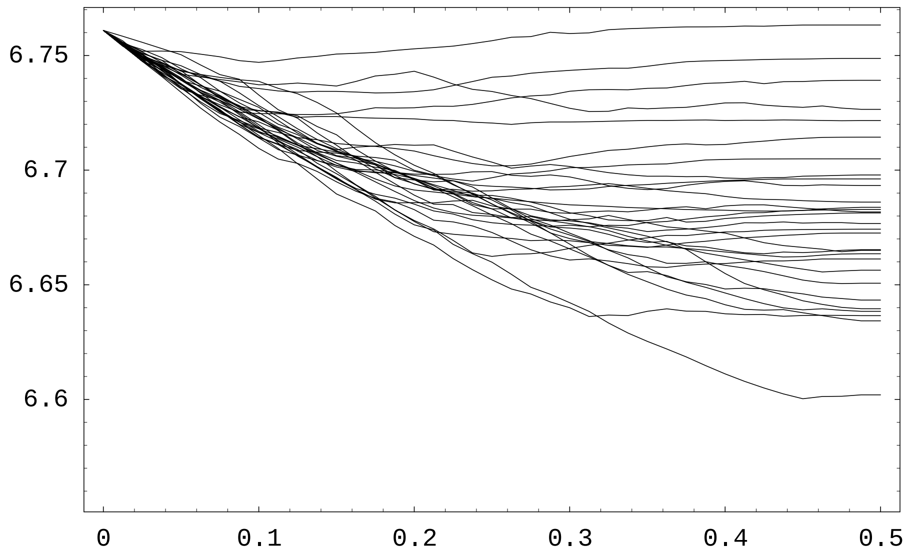

Figure 4.2: Simulated outcomes for the estimator $\bar{Z}_t$, $t \in [0, T]$.

The degree of variance reduction achieved with the DOI method is closely related to the variance of the integrand $(L^0 - \bar{L}^0)\,\bar{u}(t, X_t^{0,x})$ appearing in (3.12). Figure 4.3 shows the function values $(L^0 - \bar{L}^0)\,\bar{u}(t, x)$, for a fixed value of $t = 0.2$, using different values of the coordinates x^1 and x^2 corresponding to the components S^{0,x^1} and v^{0,x^2}, respectively.

The quantity $e^{-r(T-t)}\,(u(t,x) - \bar{u}(t,x))$ measures the difference in call option prices obtained from the Heston model and the corresponding Black-Scholes model with a time dependent deterministic volatility, see (4.3) and (4.6). Figure 4.4 visualizes these price differences as a function of time $t \in [0, T]$ and the strike K. A correlation coefficient of $\varrho = -0.15$ was used so that risky asset returns are negatively correlated with volatility. These results were obtained using 256×4 sample paths and $N = 20$ discretization points.

For $t = 0$, corresponding error bounds at a 99% confidence level are displayed in Figure 4.5. These results indicate that for a typical at-the-money call option the DOI method, with approximately 1000 sample paths and 20 discretization points, can be utilized to compute the corresponding price with a relative error of 0.04% at a 99% confidence level. For the same parameter values as used in Figure 4.4 the corresponding implied volatilities are displayed in Figure 4.6. This implied volatility surface shows a negative skew with some smile effects, a feature which is observed in many markets. For the Heston model choosing $\varrho > 0$ leads to a positive skew. Note that the choice $\varrho = 0$ produces a smile.

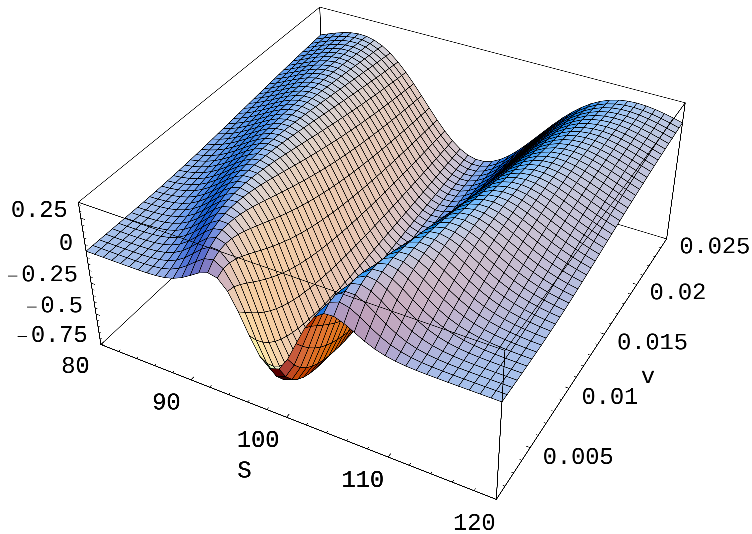

Figure 4.3: Diffusion operator values $(L - \bar{L}^0)\,\bar{u}$ as a function of asset price S and squared volatility v.

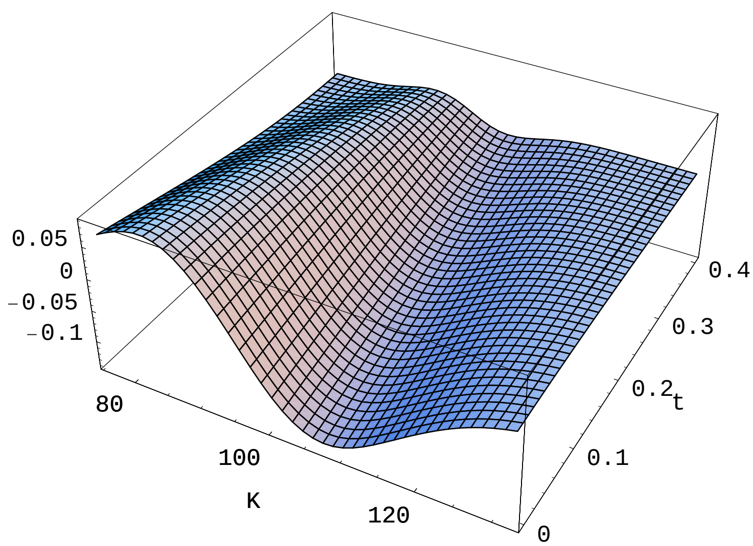

Figure 4.4: Price differences between the Heston and corresponding Black-Scholes model as a function of strike K and time t.

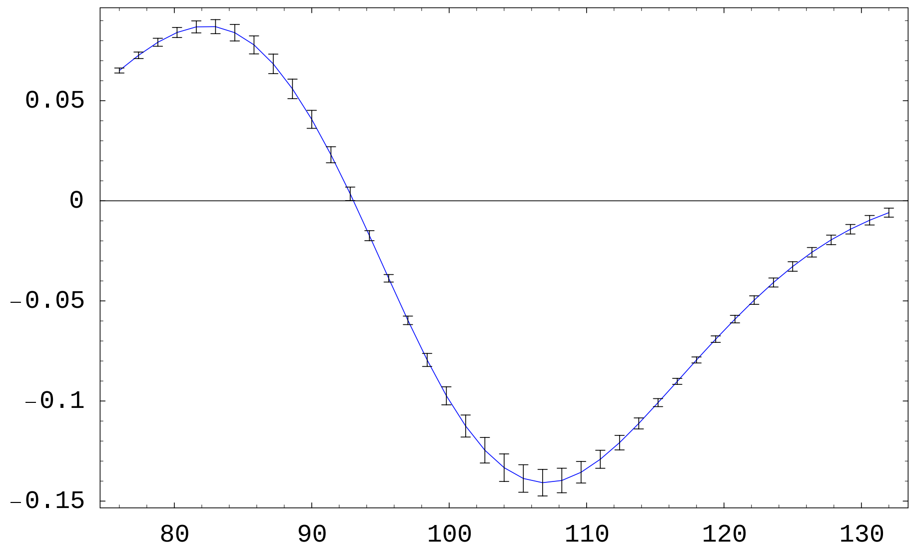

Figure 4.5: Prices and corresponding error bounds as a function of strike K.

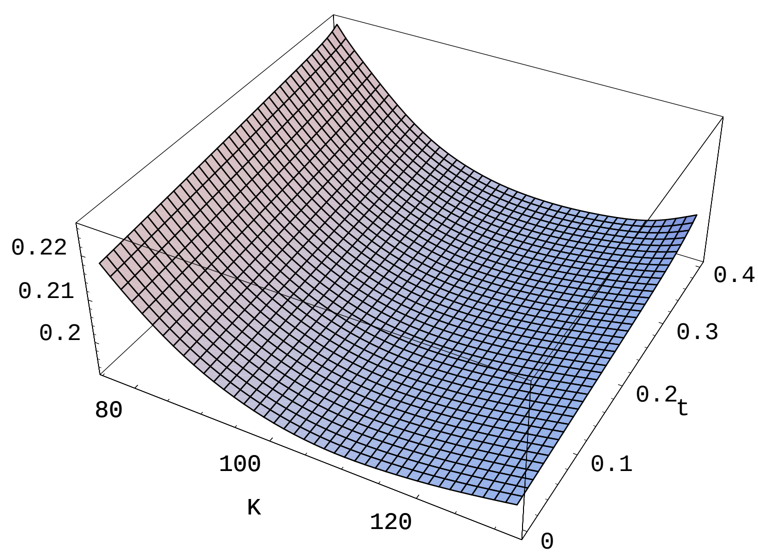

Figure 4.6: Implied volatility term structure for the Heston model.

Conclusion

This paper describes a new variance reduction technique, called the diffusion operator integral method, which can be used to find unbiased variance reduced estimators for the expectation of functionals of Itô diffusion processes. It can be applied to a wide range of derivative valuation problems and harnessed to improve the performance of PDE numerical solvers. Simulation experiments conducted for the Heston stochastic volatility model demonstrate that dramatic reductions in variance can be achieved. The method can be adapted for path dependent securities and used in conjunction with the least-square Monte Carlo approach as further research will demonstrate.

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