

# A GENERAL FRAMEWORK FOR THE CONSTRUCTION AND THE SMOOTHING OF FORWARD RATE CURVES

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**ABSTRACT.** This paper establishes a general theoretical and numerical framework for the construction and the smoothing of instantaneous forward rate curves. It is shown that if the smoothness of a curve is defined as an integral of a function in the derivatives of the curve, then the optimal curves are splines that satisfy certain ordinary differential equations. For such curves, an efficient numerical method is given for the determination of the spline parameters subject to mild assumptions.

The resulting forward rate curves do not generally possess the desired degree of smoothness due mainly to the constraints imposed on the curves by the various market observed prices. A partial solution to this problem is then introduced which achieves additional smoothing by taking into account the bid-ask ranges of each market rate. This eliminates much of the oscillatory patterns and the points of high curvature, and produces curves that are ideal for applications such as the estimation of interest rate models, and the pricing and risk management of interest rate derivatives, which are sensitive to forward rate curves.

## 1. INTRODUCTION

Forward rate curves play a key role in the study of term structure dynamics and in the valuation and the risk management of interest rate derivatives. From a theoretical viewpoint, they are the most natural and convenient objects to model, as demonstrated in the seminal paper of Heath, Jarrow and Morton (1992), and from a practical viewpoint they play an important role in the pricing and the risk management of interest rate derivatives. Consequently, numerous methods have been proposed in the literature for their construction, including McCulloch (1975), Vasiček and Fong (1982), Adams and van Deventer (1994), Frishling and Yamamura (1996) and Yekutieli (1999).

The methods for constructing the forward rate curves generally fall into two categories. The first is the parametric approach in which a particular functional form for the entire forward rate curve is assumed and the function parameters are adjusted to fit the observed market prices as closely as possible. The second is the smoothness based approach in which a measure of smoothness is chosen for the curves and the forward rate curve is

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constructed to minimise this measure subject to the constraints imposed by the observed market prices.

The functional form of the forward rate curves distinguish the various parametric approaches, while the various smoothness based approaches are distinguished by the form of the smoothness measure. Examples of the parametric methods include McCulloch (1975), Vasiček and Fong (1982), Chambers, Carleton and Waldman (1984), Nelson and Siegel (1987) and Svensson (1994), while the examples of smoothness based methods include Adams and van Deventer (1994), Frishling and Yamamura (1996) and Yekutieli (1999).

The parametric methods generally produce smoother curves and require less computational time. However, they suffer from the problems that the assumed functional form for the forward rate curves are difficult to justify from economic principles and that the resulting curves do not usually return the prices of the market instruments used in their construction. The smoothness based methods do not suffer from these problems, but they are not without problems of their own. Their main problems are that they are often not based on firm theoretical foundations so that, in particular, the nature of the curves are unclear, and that the resulting curves often contain oscillatory patterns and points of high curvature.

The purpose of this paper is to provide a general framework for the smoothness based approach to constructing forward rate curves. Firstly, the paper establishes the firm theoretical foundations for this approach by clarifying the nature of the resulting curves and providing the explicit differential equations satisfied by these curves. Secondly, the paper provides efficient numerical methods for the practical implementation of this framework, and, finally, a smoothing technique is introduced that eliminates much of the oscillatory patterns and the points of high curvature from these curves.

More precisely, it is shown that if the smoothness measure is given as an integral of a function in the derivatives of the forward rate curve, then the curve that minimises this measure subject to the imposed market constraints is a spline whose functional form is given as the solution of certain ordinary differential equations.

For constructing the curves in practice, it is shown that the problem can be transformed to the standard constrained optimisation problem. In the special cases where the smoothness measure leads to curves that are polynomial splines, an efficient construction method is given which computes the spline parameters in an iterative fashion.

In general, the constraints imposed on the forward rate curves are computed using the mid prices, the last traded prices, or some combination of other quoted prices. However, a priori there are no reasons for making such a choice and, in fact, it would appear that any price in the bid ask range would be as valid as any other. The smoothing technique introduced in this paper determines the optimal set of market prices in the sense that they minimise the given smoothness measure. This significantly improves the shape of the resulting forward rate curves, and in many cases the oscillatory patterns and the points of high curvature are completely eliminated.

The structure of the remainder of the paper is as follows. The theoretical foundations underlying the smoothness based construction of forward rate curves is established in Section 2 and a method for implementing the framework is presented in Section 3. The

smoothing technique is then presented in Section 4 along with some examples, and the paper finally concludes with Section 5.

## 2. CHARACTERISATION OF OPTIMAL FORWARD RATE CURVES

**2.1. Notation.** For any  $i \in \mathbb{N}$ ,  $U \subset \mathbb{R}$  and  $r: U \rightarrow \mathbb{R}$ , denote by  $r^{(i)}(x)$  the  $i$ -th derivative of  $r$  at  $x \in U$  if it exists. Furthermore, for any  $d \in \mathbb{N}$  denote by  $\mathcal{C}^d(U)$  the set of functions  $r: U \rightarrow \mathbb{R}$  such that  $r^{(i)}(x)$  exists and are continuous for all  $x \in U$  and  $i \leq d$ , and for any  $V \subset U$  write  $\mathcal{C}_V^d(U) = \{r \in \mathcal{C}^{d-1}(U): r|_V \in \mathcal{C}^d(V)\}$ , where  $r|_V$  is the restriction of  $r$  to  $V$ .

Next, given any  $d \in \mathbb{N}$  and  $F: \mathbb{R}^{d+2} \rightarrow \mathbb{R}$ , write  $d_F$  for the order of the highest derivative of  $r: \mathbb{R} \rightarrow \mathbb{R}$  that appears in the formal expression

$$\sum_{i=0}^d (-1)^i \frac{d^i}{dx^i} \left[ \frac{\partial F(x, r(x), r'(x), \dots, r^{(d)}(x))}{\partial r^{(i)}} \right]. \quad (2.1)$$

For example, if  $d = 1$  and  $F(x_1, x_2, x_3) = x_3^2$  then  $d_F = 2$ .

**2.2. Smoothness of Curves.** Let  $n \in \mathbb{N}$ ,  $\bar{x} \in \mathbb{R}_+$  and for any sequence

$$\mathcal{X} = (0 = x_0 < x_1 < \dots < x_n = \bar{x}) \subset [0, \bar{x}] \quad (2.2)$$

write  $\mathcal{X}^c = [0, \bar{x}] \setminus \mathcal{X}$ . Furthermore, let  $d \in \mathbb{N}_+$  and let  $F: \mathbb{R}^{d+2} \rightarrow \mathbb{R}_+$  be a function continuously differentiable in each argument.

**Definition 2.1.** Let  $d \in \mathbb{N}$ ,  $\bar{x} \in \mathbb{R}_+$  and let  $F: \mathbb{R}^{d+2} \rightarrow \mathbb{R}_+$  and  $\mathcal{X} \subset [0, \bar{x}]$  be defined as above. Then the map  $\mu_F: \mathcal{C}_{\mathcal{X}^c}^d[0, \bar{x}] \rightarrow \mathbb{R}_+$  given by

$$\mu_F(r) = \int_0^{\bar{x}} F(x, r(x), r'(x), r''(x), \dots, r^{(d-1)}(x), r^{(d)}(x)) dx. \quad (2.3)$$

will be called a *smoothness measure of order  $d$*  with respect to the pair  $(F, \mathcal{X})$ . Given  $r_1, r_2 \in \mathcal{C}^d[0, \bar{x}]$ , the curve  $r_1$  will be said to be *smoother* than  $r_2$  if  $\mu_F(r_1) < \mu_F(r_2)$ .

From now on fix  $d \in \mathbb{N}_+$ , a continuously differentiable function  $F: \mathbb{R}^{d+2} \rightarrow \mathbb{R}_+$  and a finite sequence  $\mathcal{X} \subset [0, \bar{x}]$  such that  $\{0, \bar{x}\} \subset \mathcal{X}$ . Furthermore, assume that  $d \leq d_F$ .

**Definition 2.2.** Let  $\mu_F$  be a smoothness measure of order  $d$  with respect to  $(F, \mathcal{X})$ , let  $m \in \mathbb{N}$  and let  $\mathcal{C}$  be a set of functions  $C_i: \mathcal{C}_{\mathcal{X}^c}^{d_F}[0, \bar{x}] \rightarrow \mathbb{R}$  for  $1 \leq i \leq m$ . Then a curve  $r \in \mathcal{C}_{\mathcal{X}^c}^{d_F}[0, \bar{x}]$  will be said to be  $(\mu_F, \mathcal{C})$ -optimal if

- (a)  $C_i(r) = 0$  for  $1 \leq i \leq m$ , and
- (b) for any  $h \in \mathcal{C}^d[0, \bar{x}]$ , the function  $\tilde{\mu}_{h,F}: \mathbb{R} \rightarrow \mathbb{R}_+$  defined by

$$\tilde{\mu}_{h,F}(\epsilon) = \mu_F(r + \epsilon h) \quad (2.4)$$

satisfies the condition  $\tilde{\mu}'_h(0) = 0$ , where the superscript  $'$  denotes differentiation with respect to  $\epsilon$ .

It should be noted that the set of smoothness measures of the form (2.3) is quite general, and includes most of the measures commonly used in the literature for constructing

forward rate curves. For example, Adams and van Deventer (1994) uses a curvature based smoothness measure where  $d = 2$  and

$$F_{AV}(x, r, r', r'') = \frac{1}{2}r''(x)^2, \quad (2.5)$$

Frishling and Yamamura (1996) uses a slope based smoothness measure where  $d = 1$  and

$$F_{FY}(x, r, r') = \frac{1}{2}r'(x)^2, \quad (2.6)$$

and Yekutieli (1999) introduces a length based smoothness measure where  $d = 1$  and

$$F_{YEK}(x, r, r') = \sqrt{1 + r'(x)^2}, \quad (2.7)$$

although  $F_{YEK}$  is approximated by  $F_{FY}$  in all the calculations.

In Adams and van Deventer (1994) and Yekutieli (1999) the theory of calculus of variations is applied to characterise the  $(\mu_F, \mathcal{C})$ -optimal curves for their specific smoothness measures and constraints. However, their methods do not readily extend to the general situation introduced above. For example, Adams and van Deventer (1994) assume that all the constraints are from zero coupon bond prices and uses this, along with the specific form of their smoothness measure, to transform the problem to a simpler one in terms of  $r''(x)$  which is easily solved. However, the same technique cannot be applied if the function  $F$  in (2.3) is not of the form  $F(x, r, r', r'', \dots, r^{(d)}) = G(r'')$ , for some  $G: \mathbb{R} \rightarrow \mathbb{R}_+$ , or if there are constraints other than those from zero coupon bond prices.

This paper extends and generalises these results by characterising the  $(\mu_F, \mathcal{C})$ -optimal curves for the general case in terms of differential equations and the boundary conditions that the curves must satisfy.

**2.3. Constraints.** In practice, the constraints imposed on the forward rate curves are due to securities of three main types:

- ( $C_b$ ) bills or zero coupon bonds,
- ( $C_f$ ) bill futures, and
- ( $C_s$ ) swaps or coupon bonds.

Let  $m_b$ ,  $m_f$  and  $m_s$  denote the number of constraints of type ( $C_b$ ), ( $C_f$ ) and ( $C_s$ ) respectively, and write

$$m = m_b + m_f + m_s. \quad (2.8)$$

The discount bills can be identified with pairs  $(\rho_i^b, x_i^b)$ , where  $1 \leq i \leq m_b$  and  $\rho_i^b$  is the discount rate with maturity  $x_i^b$ . The futures rates can be identified with triples  $(\rho_i^f, x_i^f, x_{i+1}^f)$ , where  $1 \leq i \leq m_f$  and  $\rho_i^f$  is the forward rate over the interval  $(x_i^f, x_{i+1}^f)$ . Finally, the swap rates can be identified with triples  $(\rho_i^s, x_i^s, \nu_i^s)$ , where  $1 \leq i \leq m_s$  and  $\rho_i^s$  is the fixed (coupon) rate on a  $x_i^s$ -maturity swap (bond) which is settled  $\nu_i^s$  per year. Note that although it is assumed, for simplicity, that the maturity of the bill underlying the  $x_i^f$ -maturity futures coincides with the maturity of the next futures contract, this can easily be relaxed. In order to ensure that the constraints are internally consistent, the following assumption is imposed.

**Assumption 2.3.** *The maturity times  $x_j^b, x_k^f, x_l^s$ , where  $1 \leq j \leq m_b$ ,  $1 \leq k \leq m_f$  and  $1 \leq l \leq m_s$ , are distinct.*

Henceforth, it will always be assumed that the above condition holds. Now, given a set of constraints  $\mathcal{C}$ , let  $\mathcal{X}_{\mathcal{C}} = (0 = x_0 < x_1 < \dots < x_n = \bar{x})$  be the minimal sequence in  $[0, \bar{x}]$ , where  $\bar{x} \in \mathbb{R}_+$ , such that

- (a)  $x_j^b \in \mathcal{X}_{\mathcal{C}}$  for  $1 \leq j \leq m_b$ ,
- (b)  $x_j^f, x_{j+1}^f \in \mathcal{X}_{\mathcal{C}}$  for  $1 \leq j \leq m_f$ ,
- (c)  $x_{jk}^s \in \mathcal{X}_{\mathcal{C}}$  for all the settlement times  $x_{jk}^s$  of the  $j$ -th swap and for all  $1 \leq j \leq m_s$ , where  $1 \leq k \leq \nu_j^s x_j^s$ .

That is, the sequence  $\mathcal{X}_{\mathcal{C}}$  consists of the cash flow times for the constraint imposing securities, and the endpoints  $x = 0$  and  $\bar{x}$ .

Given the sequence  $\mathcal{X}_{\mathcal{C}} \subset [0, \bar{x}]$  and a forward rate curve  $r: [0, \bar{x}] \rightarrow \mathbb{R}_+$ , define  $\gamma_j(r)$  as the price of the  $x_j$ -maturity zero coupon bond with unit face value so that

$$\gamma_j(r) = \exp \left[ - \int_0^{x_j} r(x) dx \right] \quad (2.9)$$

for  $0 \leq j \leq n$ , and define the indices  $i_j^b, i_j^f$  and  $i_{jk}^s$  so that

$$x_{i_j^b} = x_j^b, \quad x_{i_j^f} = x_j^f, \quad x_{i_{jk}^s} = x_{jk}^s. \quad (2.10)$$

Then the constraints  $C_j^b, C_j^f$  and  $C_j^s$  imposed by the bill rates, futures prices and the swap rates can be written

$$C_j^b(r) = \gamma_{i_j^b}(r) - \frac{1}{1 + \rho_j^b x_j^b}, \quad (2.11)$$

$$C_j^f(r) = \frac{\gamma_{i_{j+1}^f}(r)}{\gamma_{i_j^f}(r)} - \frac{1}{1 + \rho_j^f (x_{j+1}^f - x_j^f)}, \quad (2.12)$$

$$C_j^s(r) = \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} \gamma_{i_{jk}^s}(r) + \gamma_{i_{j, \nu_j^s x_j^s}^s}(r) - 1. \quad (2.13)$$

**2.4. Characterisation of Optimal Curves.** A characterisation of  $(\mu_F, \mathcal{C})$ -optimal curves is now given in terms of the differential equations that they must satisfy.

**Theorem 2.4.** *Let  $d \in \mathbb{N}_+$ ,  $F: \mathbb{R}^{d+2} \rightarrow \mathbb{R}$  be continuously differentiable in all arguments, and let  $\mathcal{C}$  be a set of constraints of the form (2.11)–(2.13). Then a  $(\mu_F, \mathcal{C})$ -optimal curve  $r \in \mathcal{C}_{\mathcal{X}_{\mathcal{C}}}^{d_F}[0, \bar{x}]$  satisfies the differential equation*

$$\begin{aligned} \sum_{j=0}^d (-1)^j \frac{d^j}{dx^j} \left( \frac{\partial F}{\partial r^{(j)}} \right) &= \sum_{j=1}^{m_b} \lambda_j^b \iota(x_j^b - x) + \sum_{j=1}^{m_f} \lambda_j^f \iota(x_{j+1}^f - x) \iota(x - x_j^f), \\ &+ \sum_{j=1}^{m_s} \lambda_j^s \left[ \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} \gamma_{i_{j,k}^s}(r) \iota(x_{i_{j,k}^s}^s - x) + \gamma_{i_{j, \nu_j^s x_j^s}^s}(r) \iota(x_{i_{j, \nu_j^s x_j^s}^s}^s - x) \right], \end{aligned} \quad (2.14)$$

for some  $\lambda_j^b, \lambda_j^f, \lambda_j^s \in \mathbb{R}$ , where

$$\iota(u - x) = \begin{cases} 1, & \text{if } u \geq x, \\ 0, & \text{otherwise,} \end{cases}$$

and satisfies the endpoint conditions

$$\sum_{k=j}^d (-1)^{k-j} \left[ \frac{d^{k-j}}{dx^{k-j}} \left( \frac{\partial F}{\partial r^{(k)}} \right) \right] = 0 \quad (2.15)$$

at  $x = 0+$  and  $x = \bar{x}-$ , for all  $1 \leq j \leq d$ .

*Proof.* See Appendix A. □

**Corollary 2.5.** *The functional form of the  $(\mu_F, \mathcal{C})$ -optimal curves is independent of the set of constraints imposed.*

*Proof.* Since the right-hand side of the ordinary differential equation (2.14) is constant, the functional form of the solution is obtained by solving the equation

$$\sum_{j=0}^d (-1)^j \frac{d^j}{dx^j} \left( \frac{\partial F}{\partial r^{(j)}} \right) = \alpha_i, \quad (2.16)$$

over each interval  $[x_{i-1}, x_i]$  where  $\alpha_i \in \mathbb{R}$ . It follows that the functional form of the solution is independent of the constraints. □

Since (2.16) is a differential equation of order  $d_F$ , its solution, assuming it exists, contains  $d_F + 1$  parameters, including the  $\alpha_i$  on the right-hand side. Substituting this solution into (2.15) imposes additional constraints on the parameters.

The next corollary gives a characterisation of polynomial splines as the  $(\mu_F, \mathcal{C})$ -optimal curves for the a certain class of smoothness measures.

**Corollary 2.6.** *Let  $d \in \mathbb{N}_+$  and let  $\mu_F$  be the smoothness measure corresponding to the function*

$$F(x, r, r', \dots, r^{(d)}) = \frac{1}{2} r^{(d)}(x)^2,$$

and let  $\mathcal{C}$  be a set of constraints. Then the  $(\mu_F, \mathcal{C})$ -optimal curves are polynomial splines of order  $2d$ .

*Proof.* In this case  $\partial F / \partial r^{(i)} = 0$  for  $i < d$  and  $\partial F / \partial r^{(d)} = r^{(d)}(x)$ . It follows from Corollary 2.5 that the  $(\mu, \mathcal{C})$ -optimal curves in this case satisfy the differential equation

$$\frac{d^d r^{(d)}(x)}{dx^d} = \alpha_i,$$

where  $\alpha_i \in \mathbb{R}$ , or equivalently  $r^{(2d)}(x) = a_{i,2d}$  over each interval  $[x_{i-1}, x_i]$  for some  $a_{i,2d} \in \mathbb{R}$ . Solving this differential equation gives

$$r(x) = \sum_{j=0}^{2d} a_{i,j} x^j$$

where  $a_{i,j} \in \mathbb{R}$ . □

The results obtained in Adams and van Deventer (1994) and Yekutieli (1999) are special cases of this corollary with  $d = 2$  and  $d = 1$  respectively.

**Corollary 2.7** (Adams and van Deventer (1994)). *Let  $\mu_F$  be the smoothness measure corresponding to  $F$  given by (2.5) and let  $\mathcal{C}$  be a set of constraints. Then the  $(\mu_F, \mathcal{C})$ -optimal curves are polynomial splines of degree 4.*

In proving their result, Adams and van Deventer (1994) make the assumption that all the constraints are of the type  $(C_b)$ .

**Corollary 2.8** (Yekutieli (1999)). *Let  $\mu_F$  be the smoothness measure corresponding to  $F$  given by (2.7) and let  $\mathcal{C}$  be a set of constraints. Then the  $(\mu_F, \mathcal{C})$ -optimal curves are quadratic splines.*

It should be noted that although there is a reference to quadratic splines in the paper, the result is not explicitly stated or proved in Yekutieli (1999).

The exponential splines can also be obtained as  $(\mu, \mathcal{C})$ -optimal curves for a suitable choice of  $F$ . For this, given a sequence  $\kappa = (\kappa_1 < \kappa_2 < \dots < \kappa_d) \subset \mathbb{R}_+$ , define  $\sigma_i(\kappa) \in \mathbb{R}$  for  $0 \leq i \leq d$  by the equation

$$p(x; \kappa) = \prod_{j=1}^d (x^2 - \kappa_j) = \sum_{j=0}^d (-1)^{d-j} \sigma_j(\kappa) x^{2j}. \quad (2.17)$$

Note that  $\sigma_j(\kappa) \geq 0$  for all  $j$  since  $\kappa_j \in \mathbb{R}_+$ . For example,  $\sigma_0(\kappa) = \prod_{j=1}^d \kappa_j$  and  $\sigma_{d-1}(\kappa) = \sum_{j=1}^d \kappa_j$ .

**Corollary 2.9.** *Let  $\mathcal{C}$  be a set of constraints,  $\kappa = (\kappa_1 < \kappa_2 < \dots < \kappa_d) \subset \mathbb{R}_+$ , and let  $\mu_F$  be the smoothness measure corresponding to the function  $F$  given by*

$$F(x, r, r', \dots, r^{(d)}) = \frac{1}{2} \sum_{j=0}^d \sigma_j(\kappa) r^{(j)}(x)^2, \quad (2.18)$$

where  $\sigma_j(\kappa)$  are as defined in (2.17). Then the  $(\mu_F, \mathcal{C})$ -optimal curves are exponential splines with exponents  $\pm\sqrt{\kappa_1}, \pm\sqrt{\kappa_2}, \dots, \pm\sqrt{\kappa_d}$ .

*Proof.* In this case,  $\partial F / \partial r^{(j)} = \sigma_j(\kappa) r^{(j)}(x)$ , and so it follows from (2.16) that the  $(\mu, \mathcal{C})$ -optimal curves satisfy the differential equation

$$\sum_{j=0}^d (-1)^{d-j} \sigma_j(\kappa) r^{(2j)}(x) = \alpha_i$$

over each interval  $[x_{i-1}, x_i]$ , where  $\alpha_i \in \mathbb{R}$ . Solving this differential equation gives the  $(\mu, \mathcal{C})$ -optimal curves

$$r(x) = a_{i,0} + \sum_{j=1}^d (a_{i,-j} e^{-\sqrt{\kappa_j}x} + a_{i,j} e^{\sqrt{\kappa_j}x})$$

where  $a_{i,0}, a_{i,\pm j} \in \mathbb{R}$ . □

Note that the non-negativity of the quantities  $\sigma_j(\kappa)$  ensures that the smoothness measure  $\mu_F$  corresponding to  $F$  in (2.18) is non-negative.

The next result shows that the curves  $r(x)$  that minimise the smoothness measure corresponding to  $F(x, r, r') = \sqrt{1 + r'(x)^2}$  are not the same as the curves that minimise the measure approximated by  $F(x, r, r') \approx r'(x)^2$  in Yekutieli (1999). It was shown in Corollary 2.8 that the curves that minimise the smoothness measure corresponding to the latter  $F$  are the quadratic splines.

**Corollary 2.10.** *Let  $\mu_F$  be the smoothness measure corresponding to the function  $F$  given by*

$$F(x, r, r') = \sqrt{1 + r'(x)^2}, \quad (2.19)$$

and let  $\mathcal{C}$  be a set of constraints. Then the  $(\mu_F, \mathcal{C})$ -optimal curves have the form

$$r(x) = \frac{1}{\lambda_i} \left[ \sqrt{1 - (\bar{\lambda}_i x + a_{i,1})^2} + \ln \left( \frac{1 + \sqrt{1 - (\lambda_i x + a_{i,1})^2}}{|\lambda_i x + a_{i,1}|} \right) \right] + a_{i,0}, \quad (2.20)$$

over each interval  $[x_{i-1}, x_i]$ , where  $a_{i,0}, a_{i,1} \in \mathbb{R}$ .

*Proof.* In this case,  $\partial F / \partial r' = (1 + r'(x)^2)^{-\frac{1}{2}}$ , and it follows that the  $(\mu_F, \mathcal{C})$ -optimal curves satisfy the differential equation

$$\frac{d}{dx} (1 + r'(x)^2)^{-\frac{1}{2}} = \alpha_i \quad (2.21)$$

over each interval  $[x_{i-1}, x_i]$ . Integrating this equation and rearranging the resulting expression gives

$$r'(x) = \pm \frac{\sqrt{1 - (\lambda_i x + a_{i,1})^2}}{\lambda_i x + a_{i,1}}, \quad (2.22)$$

for some  $a_{i,1} \in \mathbb{R}$ , which is easily solved to give (2.20).  $\square$

### 3. IMPLEMENTATION OF THE FRAMEWORK

This section considers a method for the practical implementation of the framework presented in Section 2. Recall that given a smoothness measure  $\mu_F$  of order  $d$ , a set of constraints  $\mathcal{C}$  and the corresponding set of cash flow times,  $\mathcal{X}_{\mathcal{C}}$ , the  $(\mu_F, \mathcal{C})$ -optimal curves are  $(d_F + 1)$ -parameter splines with knot points  $\mathcal{X}_{\mathcal{C}}$ . In particular, to specify a particular  $(\mu_F, \mathcal{C})$ -optimal curve, a total of  $n(d_F + 1)$  parameters must be determined. The equations that must be satisfied by the spline parameters are the continuity conditions at the knot points, the endpoint conditions, and the constraints from the set  $\mathcal{C}$ .

Unfortunately, these equations are insufficient to determine a global  $(\mu_F, \mathcal{C})$ -optimal curve in general. However, if the equations for the spline parameters are linear, which is the case for example when the splines are polynomials and the set  $\mathcal{C}$  consists entirely of constraints of type  $(C_b)$ , then Proposition 3.2 gives a simple necessary and sufficient condition for the existence of global  $(\mu_F, \mathcal{C})$ -optimal curves. In the general situation, however, the equations are sufficient only to determine local  $(\mu_F, \mathcal{C})$ -optimal curves.

Furthermore, the problem of determining the parameters for  $(\mu_F, \mathcal{C})$ -optimal curves is of dimension  $n(d_F + 1)$  which can be computationally prohibitive when  $d_F$  is large.

By taking advantage of the conditions satisfied by the spline parameters, the method presented in this section transforms the problem to an equivalent one of dimension at most  $n$ , which is more manageable computationally.

**3.1. A Special Case when the Constraints of Type  $(C_b)$ .** This subsection considers the special case in which the constraint set  $\mathcal{C}$  consists entirely of constraints of type  $(C_b)$ . This is the case considered in Adams and van Deventer (1994) for their smoothness measure corresponding to the function  $F$  in (2.5).

In this case, the differential equation that must be satisfied by  $(\mu_F, \mathcal{C})$ -optimal curves over each interval  $[x_j, x_{j+1}]$  is

$$\sum_{k=0}^d (-1)^k \frac{d^k}{dx^k} \left( \frac{\partial F}{\partial r^{(k)}} \right) = \sum_{k=1}^n \lambda_k^b \iota(x_k^b - x) := a_{j,d_F}, \quad (3.1)$$

since  $n = m = m_b$ . This is an ordinary differential equation of order  $d_F$ , and so the solution over each interval  $[x_j, x_{j+1}]$ , if it exists, involves  $d_F + 1$  parameters  $a_{j,k}$ , where  $0 \leq k \leq d_F$ . Now, suppose that the solution exists and denote the solution over the interval  $[x_j, x_{j+1}]$  by  $r_j(x; a)$ , noting that each  $r_j(x; a)$  contains the parameters  $a_{j,k}$  where  $0 \leq k \leq d_F$ . Since  $r_j(x; a)$  are solutions of (3.1), the functional form of  $r_j(x; a)$  are known. Hence, the determination of the curve  $r_j(x; a)$  is equivalent to the determination of the parameters  $a_{j,k}$  where  $0 \leq k \leq d_F$ , and consequently the determination of the  $(\mu_F, \mathcal{C})$ -optimal curve  $r(x; a)$  is equivalent to the determination of  $n(d_F + 1)$  parameters  $a_{j,k}$ , where  $0 \leq j \leq n - 1$  and  $0 \leq k \leq d_F$ .

Firstly, the requirement that  $r^{(i)}(x; a)$  is continuous on  $[0, \bar{x}]$  for  $0 \leq i \leq d_F - 1$  gives  $(n - 1)d_F$  equations

$$r_j^{(i)}(x_j; a) - r_{j+1}^{(i)}(x_j; a) = 0. \quad (3.2)$$

Next, the endpoint conditions give  $2d$  equations

$$\sum_{k=j}^d (-1)^{k-j} \left[ \frac{d^{k-j}}{dx^{k-j}} \left( \frac{\partial F}{\partial r^{(k)}} \right) \right] = 0 \quad (3.3)$$

at  $x = 0+$  and  $x = \bar{x}-$ , for all  $1 \leq j \leq d$ . Finally the constraint set  $\mathcal{C}$  gives  $n$  equations

$$\sum_{j=1}^i \int_{x_{j-1}}^{x_j} r_j(x; a) dx = \ln(1 + \rho_i^b x_i^b) \quad (3.4)$$

for  $1 \leq i \leq n$ . The total number of equations for the  $n(d_F + 1)$  spline parameters are hence

$$(n - 1)d_F + 2d + n. \quad (3.5)$$

**Proposition 3.1.** *Let  $\mu_F$  be a smoothness measure of order  $d$ , and suppose that  $\mathcal{C}$  consists entirely of constraints of the form  $(C_b)$  so that  $\mathcal{C} = \{C_j^b \mid 1 \leq j \leq n\}$ . Denote by  $\mathcal{X}_{\mathcal{C}}$  the corresponding sequence of knot points. If there exists a unique  $(\mu_F, \mathcal{C})$ -optimal curve, then  $d_F \leq 2d$ .*

*Proof.* If there is a unique  $(\mu_F, \mathcal{C})$ -optimal curve, then there exists a unique solution to the equations (3.2)–(3.4). But since these equations involve  $n(d_F + 1)$  variables, viz. spline parameters, the equations (3.2)–(3.4) must contain  $n(d_F + 1)$  linearly independent subset. From (3.5) it then follows that  $n(d_F + 1) \leq (n - 1)d_F + 2d + n$ , which is equivalent to the condition  $d_F \leq 2d$ .  $\square$

For the special case in which the equations (3.2)–(3.4) are linear, the above result can be sharpened using the standard results from linear algebra.

**Proposition 3.2.** *Suppose (3.2)–(3.4) are linear in the parameters  $a_{jk}$ , and denote the system of equations in the matrix form  $Ma = v$ , where  $M$  is the coefficient matrix,  $a$  is the vector of spline parameters, and  $v$  is the vector of the right-hand sides of equations (3.2)–(3.4). Then there exists a unique  $(\mu_F, \mathcal{C})$ -optimal curve if and only if*

$$\text{rank}(M) = n(d_F + 1) = \text{rank}(M|v).$$

*In particular, if there exists a unique  $(\mu_F, \mathcal{C})$ -optimal curve, then  $d_F \leq 2d$ .*

Note that the equations (3.2)–(3.3) corresponding to the function  $F$  in Corollary 2.6 are linear. So provided that the set  $\mathcal{C}$  consists entirely of constraints of the form  $(C_b)$ , there exists a unique polynomial spline satisfying the equations (3.2)–(3.4) if and only if the conditions of Proposition 3.2 are satisfied.

**3.2. The General Case.** Consider now the case where the set  $\mathcal{C}$  contains a constraint of the form  $(C_f)$  or  $(C_s)$ . From Corollary 2.5 and (2.14), the  $(\mu_F, \mathcal{C})$ -optimal curve, if it exists, must satisfy an ordinary differential equation of the form

$$\sum_{k=0}^d (-1)^k \frac{d^k}{dx^k} \left( \frac{\partial F}{\partial r^{(k)}} \right) = a_{j,d_F}, \quad (3.6)$$

over each interval  $[x_j, x_{j+1}]$  where  $a_{j,d_F} \in \mathbb{R}$ . In order to proceed, the following assumption must be made.

**Assumption 3.3.** *There exists a solution to (3.6) for all  $a_{j,d_F} \in \mathbb{R}$ .*

As above, denote by  $r_j(x; a)$  the solution to (3.6) over the interval  $[x_j, x_{j+1}]$  with corresponding parameters  $a_{j,k}$ , where  $0 \leq k \leq d_F$ . Then, once again, the problem of determining the curve  $r(x)$  is equivalent to determining the parameters  $a_{j,k}$  that satisfy the continuity equations, the endpoint conditions and the constraints from the set  $\mathcal{C}$ . In general, these equations are insufficient to determine the spline parameters uniquely, and consequently the global  $(\mu_F, \mathcal{C})$ -curves cannot be expected. Nevertheless, the local  $(\mu_F, \mathcal{C})$ -optimal curves can be obtained as follows.

For notational convenience, define  $\tilde{a}_k$ , for  $1 \leq k \leq n(d_F + 1)$ , by  $\tilde{a}_{id_F+j+1} = a_{i,j}$ , where  $0 \leq i \leq n - 1$  and  $0 \leq j \leq d_F$ . Note that not all the constraints in  $\mathcal{C}$  are of type  $(C_b)$  and hence expressible in the form (3.4). Let  $\mathcal{C}' \subset \mathcal{C}$  be the subset consisting of the constraints of type  $(C_b)$  and  $(C_f)$ , and choose an ordering for the equations (3.2)–(3.3) and the  $m_b + m_f$  constraint equations in  $\mathcal{C}'$ . Denote by  $G_i(\tilde{a})$  the functions  $\mathbb{R}^{n(d_F+1)} \rightarrow \mathbb{R}$  corresponding to these equations, for  $1 \leq i \leq (n - 1)d_F + 2d + m_b + m_f$ , so that

$$G_{(n-1)j+k}(\tilde{a}) = r^{(j)}(x_k; \tilde{a}) - r^{(j)}(x_{k+1}; \tilde{a}),$$

where  $0 \leq j \leq d_F - 1$  and  $1 \leq k \leq n - 1$ ,

$$G_{(n-1)d_F+j}(\tilde{a}) = \sum_{k=j}^d (-1)^{k-j} \left[ \frac{d^{k-j}}{dx^{k-j}} \left( \frac{\partial F}{\partial r^{(k)}} \right) \right]_{x=0+},$$

$$G_{(n-1)d_F+d+j}(\tilde{a}) = \sum_{k=j}^d (-1)^{k-j} \left[ \frac{d^{k-j}}{dx^{k-j}} \left( \frac{\partial F}{\partial r^{(k)}} \right) \right]_{x=\bar{x}-},$$

where  $1 \leq j \leq d$ ,

$$G_{(n-1)d_F+2d+j}(\tilde{a}) = \sum_{k=0}^{i_j^b-1} \int_{x_k}^{x_{k+1}} r_k(x; a) dx - \ln(1 + \rho_j^b x_j^b),$$

where  $1 \leq j \leq m_b$ , and

$$G_{(n-1)d_F+2d+m_b+j}(\tilde{a}) = \sum_{k=i_j^f}^{i_{j+1}^f-1} \int_{x_k}^{x_{k+1}} r_k(x; a) dx - \ln(1 + \rho_j^f (x_{j+1}^f - x_j^f)),$$

for  $1 \leq j \leq m_f$ . Let  $\{x_{i_j}\} = \mathcal{X}_C \setminus (\{x_0\} \cup \{x_{i_j^b}\} \cup \{x_{i_j^f}\})$ , so that  $1 \leq j \leq n - m_b - m_f$ , and define the functions  $G_{(n-1)d_F+2d+m_b+m_f+j}(\tilde{a})$  by

$$G_{(n-1)d_F+2d+m_b+m_f+j}(\tilde{a}) = \sum_{k=0}^{i_j-1} \int_{x_k}^{x_{k+1}} r_k(x; \tilde{a}) dx.$$

Let  $K = (n - 1)d_F + 2d + n$  and define  $G: \mathbb{R}^{n(d_F+1)} \rightarrow \mathbb{R}^K$  by  $G = (G_1, \dots, G_K)$ . Note that the vector of parameters  $\tilde{a}$  corresponding to  $(\mu_F, \mathcal{C})$ -optimal curves satisfy the equations

$$G_i(\tilde{a}) = 0, \quad \text{and} \quad C_j^s(\tilde{a}) = 0, \quad (3.7)$$

where  $C_j^s$  are the constraints corresponding to (2.13), expressed in terms of the spline parameters, and where  $1 \leq i \leq n(d_F + 1) + 2d + m_b + m_f$  and  $1 \leq j \leq m_s$ .

**Assumption 3.4.** Let  $K = (n - 1)d_F + 2d + n$  as above, and assume that the function  $G: \mathbb{R}^{n(d_F+1)} \rightarrow \mathbb{R}^K$  satisfies the condition

$$\text{rank} \left[ \frac{\partial(G_1, \dots, G_K)}{\partial(\tilde{a}_1, \dots, \tilde{a}_{n(d_F+1)})} \right] = n(d_F + 1), \quad (3.8)$$

for almost all<sup>1</sup>  $(\tilde{a}_1, \dots, \tilde{a}_{n(d_F+1)}) \in \mathbb{R}^{n(d_F+1)}$ .

The purpose of this assumption is to ensure that the function  $G = (G_1, \dots, G_K)$  is locally invertible. Assuming (3.8) is satisfied, let  $U \subset \mathbb{R}^{n(d_F+1)}$  be an open subset such that  $G|_U: U \rightarrow G(U)$  is a homeomorphism, and let

$$V = \{(G_{K-n+m_b+m_f+1}(\tilde{a}), \dots, G_K(\tilde{a})) \mid \tilde{a} \in U\} \subset \mathbb{R}_+^{n-m_b-m_f}. \quad (3.9)$$

<sup>1</sup>Almost all with respect to the Lebesgue measure on  $\mathbb{R}^{n(d_F+1)}$ . The purpose of this is to enable inversion of  $G$  by small perturbation about  $a$  if  $G$  is not invertible at  $a$ .

Then  $\xi = (\xi_1, \dots, \xi_{n-m_b-m_f}) \in V$  determines a unique  $\tilde{a}_\xi \in \mathbb{R}^{n(d_F+1)}$  that satisfies the equations  $G_i(\tilde{a}_\xi) = 0$ , for  $1 \leq i \leq K - n + m_b + m_f$ , by

$$\tilde{a}_\xi = G|_U^{-1}(0, \dots, 0, \xi_1, \dots, \xi_{n-m_b-m_f}).$$

This relationship allows the smoothness measure  $\mu_F(r)$  to be rewritten in terms of  $\xi \in V$  so that

$$\tilde{\mu}_F(\xi) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} r_k(x; G|_U^{-1}(0, \xi)) dx. \quad (3.10)$$

In view of (3.8) there is a bijection between the parameters  $\tilde{a} \in U$  corresponding to  $(\mu_F, \mathcal{C})$ -optimal curves and  $\xi \in V$  that minimise the smoothness measure  $\tilde{\mu}_F(\xi)$  subject to the constraints

$$\tilde{C}_j^s(\xi) = C_j^s(G|_U^{-1}(0, \xi)) = 0, \quad (3.11)$$

where  $1 \leq j \leq m_s$ . For any specific  $F$ , the main problem lies in the inversion of the function  $G$ . However, in the special case of polynomial splines, the function  $G$  is linear and  $G^{-1}$  is easily obtained. This special case is considered in detail in Section 3.3.

**3.3. The Special Case of Polynomial Splines.** In this subsection, fix  $d \in \mathbb{N}_+$  and assume that  $F(x, r, r', \dots, r^{(d)}) = \frac{1}{2}r^{(d)}(x)^2$ . Then Assumption 3.3 is satisfied and from Corollary 2.6, the  $(\mu_F, \mathcal{C})$ -optimal curves are polynomial splines of order  $d_F = 2d$  so that over each interval  $[x_j, x_{j+1}]$

$$r_j(x; a) = \sum_{k=0}^{2d} a_{j,k}(x - x_j)^k.$$

The functions  $G_i: \mathbb{R}^{n(2d+1)} \rightarrow \mathbb{R}^{n(2d+1)}$  are given by

$$G_{(n-1)j+k}(a) = \left[ \sum_{l=j}^{2d} l_{[j]}(x_k - x_{k-1})^{l-j} a_{k,l} \right] - j! a_{k+1,j},$$

for  $0 \leq j \leq 2d - 1$  and  $1 \leq k \leq n - 1$ , where  $l_{[j]} = l!/(l-j)!$ ,

$$G_{2(n-1)d+j}(a) = a_{0,d+j-1},$$

$$G_{2(n-1)d+d+j}(a) = \sum_{l=d+j-1}^{2d} l_{[d+j-1]}(x_n - x_{n-1})^{l-d-j+1} a_{n-1,l},$$

for  $1 \leq j \leq d$ ,

$$G_{2nd+j}(a) = \sum_{k=1}^{j-1} \sum_{l=0}^{2d} \frac{\Delta_k^{l+1}}{l+1} a_{k,l} - \ln(1 + \rho_j^b x_j^b),$$

for  $1 \leq j \leq m_b$ ,

$$G_{2nd+m_b+j}(a) = \sum_{k=i_j^f}^{i_{j+1}^f-1} \sum_{l=0}^{2d} \frac{\Delta_k^{l+1}}{l+1} a_{k,l} - \ln \left( 1 + \rho_j^f (x_{j+1}^f - x_j^f) \right),$$

for  $1 \leq j \leq m_f$ , and

$$G_{2nd+m_b+m_f+j}(a) = \sum_{k=0}^{i_j-1} \sum_{l=0}^{2d} \frac{\Delta_k^{l+1}}{l+1} a_{k,l},$$

for  $1 \leq j \leq n - m_b - m_f$ , where  $\Delta_j = x_{j+1} - x_j$  and the indices  $i_j$  are defined by  $\{x_{i_j}\} = \{x_j\} \setminus (\{x_0\} \cup \{x_j^b\} \cup \{x_j^f\})$ . In this case, the function  $G = (G_1, \dots, G_{n(2d+1)})$  is linear and its Jacobian is square, non-singular and independent of the spline parameters. Consequently, the map  $G: \mathbb{R}^{n(2d+1)} \rightarrow \mathbb{R}^{n(2d+1)}$  is a homeomorphism and Assumption 3.4 is satisfied. Since the parameters  $a \in \mathbb{R}^{n(2d+1)}$  for  $(\mu_F, \mathcal{C})$ -optimal curves satisfy the equations  $G_j(a) = 0$ , where  $1 \leq j \leq 2nd + m_b + m_f$ , the image of these parameters under the map  $G$  is a subset of  $0 \times \mathbb{R}^{n-m_b-m_f} \subset \mathbb{R}^{n(2d+1)}$ . Hence the parameters corresponding to the  $(\mu_F, \mathcal{C})$ -optimal curves form a subset of  $G^{-1}(0 \times \mathbb{R}^{n-m_b-m_f})$ . To determine the parameters for a particular constraint set  $\mathcal{C}$ , note firstly that the smoothness measure can be written as

$$\mu_F(a) = \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left( \sum_{k=d}^{2d} k_{[d]} a_{j,k} (x - x_j)^{k-d} \right)^2 dx.$$

Choosing a lexicographical ordering for the parameters  $a_{j,k}$ , it follows from the above discussion that

$$a_{j,k} = a_{j,k}(\xi) = G^{-1}(0, \dots, 0, \xi)_{(d+1)j+k},$$

where  $\xi \in \mathbb{R}^{n-m_b-m_f}$ . Note that in this case the parameters  $a_{j,k}(\xi)$  are linear in  $\xi$ . Now, the smoothness measure can be written

$$\tilde{\mu}_F(\xi) = \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left( \sum_{k=d}^{2d} k_{[d]} a_{j,k}(\xi) (x - x_j)^{k-d} \right)^2 dx, \quad (3.12)$$

in terms of  $\xi \in \mathbb{R}^{n-m_b-m_f}$ , and the corresponding  $(\mu_F, \mathcal{C})$ -optimal curves are obtained by minimising  $\tilde{\mu}_F(\xi)$  in (3.12) subject to the constraints

$$\tilde{C}_j^s(\xi) = \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} \tilde{\gamma}_{i_{jk}^s}(\xi) + \tilde{\gamma}_{i_{j,\nu_j^s x_j^s}^s}(\xi) - 1 = 0,$$

where  $\Delta_k = x_{k+1} - x_k$  and

$$\tilde{\gamma}_j(\xi) = \exp \left[ - \sum_{k=0}^{j-1} \sum_{l=0}^{2d} \frac{\Delta_k^{l+1}}{l+1} a_{k,l}(\xi) \right].$$

This is a standard constrained optimisation problem with respect to  $\xi \in \mathbb{R}^{n-m_b-m_f}$ , and can be solved.

**3.4. Example of Quadratic Splines.** This subsection considers the special case where  $F$  is given by (2.6) so that the  $(\mu_F, \mathcal{C})$ -optimal curves are quadratic splines. In this case, the curve over the interval  $[x_j, x_{j+1}]$  is given by the quadratic

$$r_j(x; a) = a_{j,0} + a_{j,1}(x - x_j) + a_{j,2}(x - x_j)^2,$$

where  $a_{j,k} \in \mathbb{R}$ . The continuity of  $r(x)$  and  $r'(x)$  at the knot points gives functions

$$\begin{aligned} G_{j+1}(a) &= a_{j,0} + a_{j,1}\Delta_j + a_{j,2}\Delta_j^2 - a_{j+1,0}, \\ G_{n+j}(a) &= a_{j,1} + 2a_{j,2}\Delta_j - a_{j+1,1}, \end{aligned}$$

for  $0 \leq j \leq n-2$ , where  $\Delta_j = x_{j+1} - x_j$ . It turns out that the inversion of the function  $G$  becomes simpler if the endpoint condition at  $x = 0+$  and the constraints  $(C_b)$  and  $(C_f)$  are used as constraints in the optimisation process rather than in the construction of the function  $G$ . More specifically, it is convenient to define

$$G_{2n-1}(a) = a_{n-1,1} + 2a_{j,2}\Delta_{n-1},$$

which is the endpoint condition at  $x = x_n-$ ,

$$G_{2n+j}(a) = a_{j,0}$$

for  $0 \leq j \leq n-1$ , and

$$G_{3n}(a) = a_{n-1,0} + a_{n-1,1}\Delta_{n-1} + a_{n-1,2}\Delta_{n-1}^2.$$

It is easy to show that  $G = (G_1, \dots, G_{3n}): \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$  is invertible, and since the parameters  $a = (a_{j,k})$  for  $(\mu_F, \mathcal{C})$ -optimal curves satisfy the equations  $G_j(a) = 0$ , where  $1 \leq j \leq 2n-1$ , these parameters belong to the subset  $G^{-1}(0 \times \mathbb{R}^{n+1})$ . In particular, each  $\xi \in \mathbb{R}^{n+1}$  determines a unique  $a_\xi \in G^{-1}(0 \times \mathbb{R}^{n+1})$  by the equation  $a_\xi = G^{-1}(0, \xi)$ . It is a simple exercise in linear algebra to show that  $a_\xi$  is given by

$$\begin{aligned} a_{j,0} &= \xi_j, \\ a_{j,1} &= 2 \sum_{k=j}^{n-1} \frac{(-1)^{k-j}}{\Delta_k} (\xi_{k+1} - \xi_k), \\ a_{j,2} &= \frac{1}{\Delta_j} \left[ -\frac{1}{\Delta_j} (\xi_{j+1} - \xi_j) + 2 \sum_{k=j+1}^{n-1} \frac{(-1)^{k-j+1}}{\Delta_k} (\xi_{k+1} - \xi_k) \right], \end{aligned}$$

where  $0 \leq j \leq n-1$ . The smoothness measure and the discount factors can be expressed in terms of  $\xi \in \mathbb{R}^{n+1}$  by substituting these expressions into the equations

$$\mu_F(a) = \sum_{j=0}^{n-1} \left[ a_{j,1}^2 \Delta_{j+1} + 2a_{j,1}a_{j,2}\Delta_{j+1}^2 + \frac{4}{3}a_{j,2}^2\Delta_{j+1}^3 \right], \quad (3.13)$$

$$\gamma_j(a) = \prod_{k=0}^{j-1} \exp \left[ -a_{k,0}\Delta_{k+1} - \frac{1}{2}a_{k,1}\Delta_{k+1}^2 - \frac{1}{3}a_{k,2}\Delta_{k+1}^3 \right]. \quad (3.14)$$

Hence, the problem of determining the spline parameters reduces to determining  $\xi \in \mathbb{R}^{n+1}$  that minimise the smoothness measure in (3.13), subject to the constraints (2.11)–(2.13) and the condition

$$a_{0,1} = 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{\Delta_k} (\xi_{k+1} - \xi_k) = 0,$$

which are also expressible in terms of  $\xi$ . This is a standard constrained optimisation problem and can be solved using the methods outlined in Press, Teukolsky, Vetterling

and Flannery (1996). The method used to obtain the figures in this paper, however, is an iterative procedure adapted from Frishling and Yamamura (1996) which provides a much faster solution. The time taken to obtain the optimal solution  $\xi$  is typically less than 2 seconds on a personal computer. Since little attempt was made to optimise the code, it should be possible to reduce this time further.

#### 4. THE SMOOTHING TECHNIQUE

Examples of the forward rate curves constructed using the method described in Section 3, for the smoothness measure corresponding to  $F(x, r, r') = \frac{1}{2}r'(x)^2$ , are shown in Figure 4.1. The oscillatory patterns and the points of high curvature exhibited by the curves are characteristic features that result from the nature of the market constraints rather than the particular choice of the smoothness measure. Considering the special case of quadratic splines, intuitively these constraints determine the vector  $\xi \in \mathbb{R}^{n+1}$ , which are the forward rates at the knot points, and the choice of the smoothness measure determines how these points are connected. Consequently, the oscillations and the points of high curvature cannot be eliminated by simply choosing a different smoothness measure.

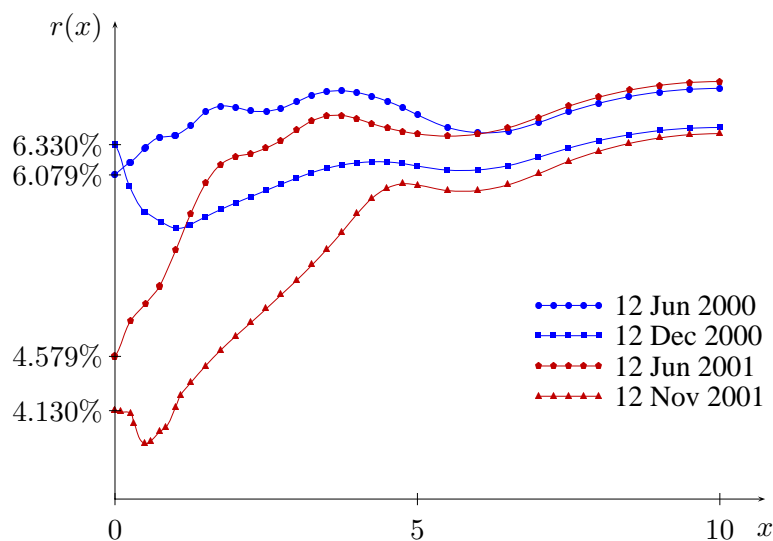


FIGURE 4.1. Forward rate curves constructed using the basic method of Section 3. The input data consisted of the overnight interbank borrowing rate, the four nearest bank bill futures prices, and 1–5, 7 and 10 year swap rates for the Australian market.

A common practice in calculating the constraints<sup>2</sup>  $C_j^d(r, \rho)$ ,  $C_j^f(r, \rho)$ , and  $C_j^s(r, \rho)$  in (2.11)–(2.13) is to use the mid rates, last traded rates, or some weighted average of the various values for  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$  that are quoted in the market. In constructing the curves in Figure 4.1, the mid rates were used. However, this involves a *subjective* choice for the set of market rates which is difficult to justify. In fact, it would appear that for each

<sup>2</sup>The constraints are written slightly differently in this section to emphasise their dependence of the market rates  $\rho_j^b$ ,  $\rho_j^f$  and  $\rho_j^s$ .

market rate  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$ , any value in the respective bid-ask range would be as valid as any other.

This observation suggests a method for smoothing the forward rate curves which, at the same time, also determines *objectively* the *optimal* set of market rates. These optimal rates are the  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$  that lie within their respective bid-ask ranges and minimise the smoothness measure  $\mu_F(r)$ , which depends on these rates. The details of this smoothing technique now follows.

Firstly, the smoothness measure will be written  $\mu_F(r, \rho)$  in this section to emphasise its dependence on the market rates  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$ . Next, denote by  $\rho_{j,b}^d$  and  $\rho_{j,a}^d$  the bid and ask rates for the  $j$ -th zero coupon bond respectively, and similarly write  $\rho_{j,b}^f$  and  $\rho_{j,a}^f$  for the futures, and  $\rho_{j,b}^s$  and  $\rho_{j,a}^s$  for the swaps. Then under the Assumptions 3.3 and 3.4, the determination of the  $(\mu_F, \mathcal{C})$ -optimal curves is equivalent to the determination of vectors  $\xi \in V$ , where  $V$  is given by (3.9) and minimise the smoothness measure  $\tilde{\mu}(\xi, \rho)$  in (3.10) subject to the constraints (3.11). The smoothing technique below simultaneously determines the vector  $\xi$  and the market rates  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$  that minimise the smoothness measure  $\tilde{\mu}_F(\xi, \rho)$ , subject to the constraints  $\tilde{C}_j^s(\xi, \rho)$  and

$$\rho_{j,b}^d \leq \rho_j^d \leq \rho_{j,a}^d, \quad 1 \leq j \leq m_d, \quad (4.1)$$

$$\rho_{j,b}^f \leq \rho_j^f \leq \rho_{j,a}^f, \quad 1 \leq j \leq m_f, \quad (4.2)$$

$$\rho_{j,b}^s \leq \rho_j^s \leq \rho_{j,a}^s, \quad 1 \leq j \leq m_s. \quad (4.3)$$

For notational convenience, let  $\rho \in \mathbb{R}_+^m$  denote the vector consisting of the market rates  $\rho_j^d$ ,  $\rho_j^f$  and  $\rho_j^s$ . Then the steps for a simple numerical procedure that solves this optimisation problem are as follows:

- (a) Set  $\rho_j = \frac{1}{2}(\rho_{j,b} + \rho_{j,a})$  for  $1 \leq j \leq m$ .
- (b) Numerically compute the gradient  $\nabla_\rho \tilde{\mu}_F(\xi, \rho)$  of  $\tilde{\mu}_F(\xi, \rho)$  with respect to  $\rho$ .
- (c) Let  $\mathcal{J} \subset \{1, 2, \dots, m\}$  consist of indices  $j$  such that

$$\begin{cases} \rho_j > \rho_{j,b}, & \text{if } \nabla_\rho \tilde{\mu}_F(\xi, \rho)_j > 0, \\ \rho_j < \rho_{j,a}, & \text{if } \nabla_\rho \tilde{\mu}_F(\xi, \rho)_j < 0. \end{cases} \quad (4.4)$$

Note that the set  $\mathcal{J}$  consists of the indices,  $j$ , for which the rate  $\rho_j$  can be moved in the negative direction of the corresponding component,  $\nabla_\rho \tilde{\mu}_F(\xi, \rho)_j$ , of the gradient without leaving the bid ask range.

- (d) Compute  $\alpha^* \in \mathbb{R}$  defined by the expression

$$\alpha^* = \min_{j \in \mathcal{J}} \left[ \max_{\alpha_j \in \mathbb{R}_+} \{ \rho_{j,b} \leq \rho_j - \alpha_j \nabla_\rho \tilde{\mu}_F(\xi, \rho)_j \leq \rho_{j,a} \} \right]. \quad (4.5)$$

Note that  $\alpha^*$  is the maximum factor by which  $\rho$  can be moved in the negative direction of  $\nabla_\rho \tilde{\mu}_F(\xi, \rho)$  while remaining in the bid-ask range of each  $\rho_j$ , ignoring those components  $j \notin \mathcal{J}$ .

(e) Let  $\tilde{\rho}_t = \rho - t\alpha^*\nabla_{\rho}\tilde{\mu}_F(\xi, \rho)$ , and compute

$$\mu_1 = \tilde{\mu}_F(\xi, \tilde{\rho}_1), \quad (4.6)$$

$$\mu_{0.5} = \tilde{\mu}_F(\xi, \tilde{\rho}_{0.5}), \quad (4.7)$$

$$\mu_0 = \tilde{\mu}_F(\xi, \tilde{\rho}_0), \quad (4.8)$$

where  $\tilde{\mu}_F(\xi, \tilde{\rho}_0)$  is the smoothness measure of the forward rate curve constructed from  $\tilde{\rho}_0$  using the method from Section 3, and similarly for  $\tilde{\mu}_F(\xi, \tilde{\rho}_{0.5})$  and  $\tilde{\mu}_F(\xi, \tilde{\rho}_1)$ .

(f) Fit a quadratic through the points  $(0, \mu_0)$ ,  $(0.5, \mu_{0.5})$  and  $(1, \mu_1)$ , and determine the critical point  $(t_c, \mu_c)$  of the quadratic.

(g) If  $0 \leq t_c \leq 1$  and  $(t_c, \mu_c)$  is the point where the *minimum* is achieved, then go back to step (b) with  $\rho \leftarrow \rho - t_c\alpha^*\nabla_{\rho}\tilde{\mu}_F(\xi, \rho)$ . Otherwise, if  $\mu_1 < \mu_0$  then go back to step (b) with  $\rho \leftarrow \rho - \alpha^*\nabla_{\rho}\tilde{\mu}_F(\xi, \rho)$ .

Note that each iteration through the steps (b)–(g) moves  $\rho$  to a new value, with each component within its bid-ask range, such that when used in the construction method from Section 3 gives a curve with a lower value for the smoothness measure. Since the value of the smoothness measure is bounded below, this procedure converges by the Monotone Convergence Theorem. However, in practice, a criterion such as the one that exits the iteration when the *reduction* in the smoothness measure is below some threshold value would be used to terminate the algorithm. The forward rate curves constructed from the data used for Figure 4.1 that incorporates the smoothing algorithm described above are shown in Figure 4.2. It is clearly evident from the figure that the smoothing technique eliminates much of the oscillatory patterns and the points of high curvature from these curves.

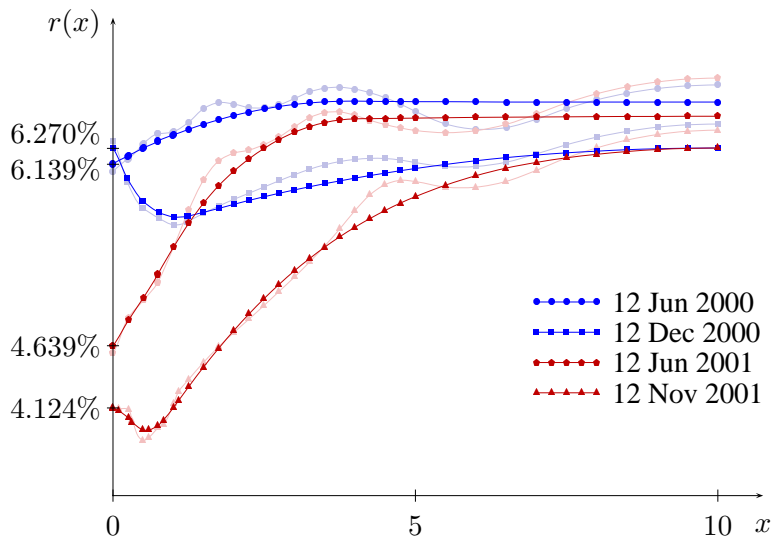


FIGURE 4.2. Smoothed forward rate curves constructed using the same data as in Figure 4.1. The bid-ask spread of 12 basis points was used for all the market rates.

The effect of the size of the bid ask spread on the shape of the smoothed forward rate curve is illustrated in Figure 4.3. The figure shows that even a small bid ask spread leads to a considerable improvement in the smoothness of the curves.

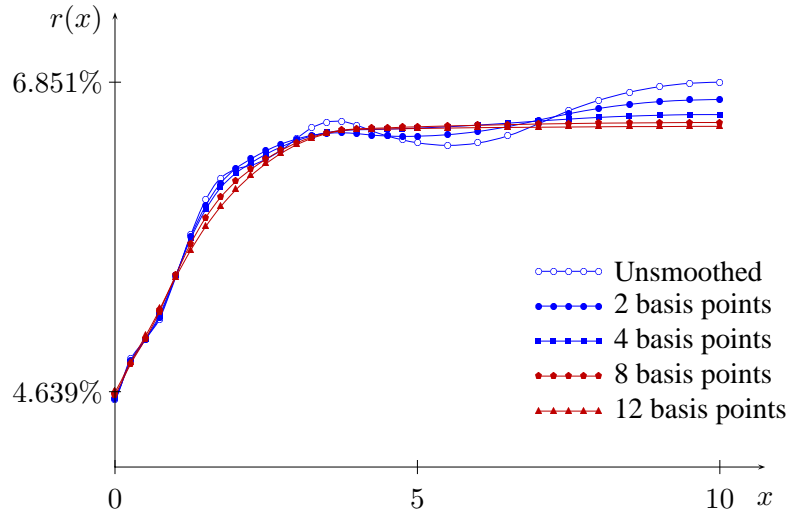


FIGURE 4.3. Forward rate curves for 12 June, 2001 smoothed with varying sizes of the bid ask spread.

## 5. CONCLUSION

This paper established a general theoretical and numerical framework for the smoothness based construction of instantaneous forward rate curves. In particular, an explicit characterisation of the optimal forward rate curves were given in terms of the ordinary differential equations they must satisfy. For constructing these curves in practice, an efficient numerical technique, capable of constructing the curves in real time, was provided. In the special case of quadratic splines, the method constructs curves in less than two seconds on a personal computer which makes it viable for real time applications.

The constraints imposed on the forward rate curve by the market observed rates often lead to curves that do not exhibit the desired degree of smoothness. This is due mainly to the somewhat arbitrary choice made on the market rates used. For example, the rates are usually taken as the mid rates, the last traded rates, or some other weighted average of the various quoted rates. An effective smoothing technique was introduced to obtain the optimal set of market rates, all within their bid-ask ranges, that produce the smoothest possible curve. This technique removes much of the oscillatory patterns and the points of high curvature from the forward rate curves.

The methods outlined in this paper produce a reliable set of forward rate curves that can be used, for example, in the parameter estimation of the various interest rates models, and in the valuation and the risk management of the various interest rate derivatives. The task of estimating the parameters for the various interest rate models, using the smoothed curves, is currently under investigation.

## APPENDIX A. PROOF OF THEOREM 2.4

The minimisation of the smoothness measure  $\mu_F$  in (2.3) with respect to  $r \in \mathcal{C}_{\mathcal{X}^c}^{d_F}[0, \bar{x}]$  subject to the constraints  $\mathcal{C}_i$ , where  $r$  satisfies the constraints  $C_j^b$ ,  $C_j^f$  and  $C_j^s$  is equivalent to the minimisation of the functional

$$\tilde{\mu}_F(r, \lambda) = \mu(r) + \sum_{j=1}^{m_b} \lambda_j^b C_j^b(r) + \sum_{j=1}^{m_f} \lambda_j^f C_j^f(r) - \sum_{j=1}^{m_s} \lambda_j^s C_j^s(r)$$

with respect to  $r \in \mathcal{C}_{\mathcal{X}^c}^{d_F}[0, \bar{x}]$  and the Lagrange multipliers  $\lambda_j^b, \lambda_j^f, \lambda_j^s \in \mathbb{R}$ .

Firstly, the constraints  $C_j^b$  and  $C_j^f$  can be rewritten equivalently in the form

$$\begin{aligned} C_j^b(r) &= \int_0^{x_j^b} r(x) dx - \ln(1 + \rho_j^b x_j^b), \\ C_j^f(r) &= \int_{x_j^f}^{x_{j+1}^f} r(x) dx - \ln(1 + \rho_j^f (x_{j+1}^f - x_j^f)), \end{aligned}$$

and the constraint  $C_j^s$  can be rewritten more explicitly as

$$C_j^s(r) = \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} e^{-\int_0^{x_{j,k}^s} r(x) dx} + e^{-\int_0^{x_{j,\nu_j^s}^s} r(x) dx} - 1.$$

Let  $(r, \lambda) \in \mathcal{C}_{\mathcal{X}^c}^{d_F}[0, \bar{x}] \times \mathbb{R}^m$  be a local minimum for  $\tilde{\mu}_F$ , and let  $h \in \mathcal{C}^d[0, \bar{x}]$ . Since the curve  $r$  is  $(\mu_F, \mathcal{C})$ -optimal, the function  $\bar{\mu}_{h,F}: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\bar{\mu}_{h,F}(\epsilon) = \tilde{\mu}_F(r + \epsilon h)$  satisfies the local minimum condition

$$\bar{\mu}'_{h,F}(0) = 0.$$

Noting that  $(r + \epsilon h)^{(i)}(x) = r^{(i)}(x) + \epsilon h^{(i)}(x)$ , the function  $\bar{\mu}_{h,F}(\epsilon)$  can be written more explicitly in the form

$$\begin{aligned} \bar{\mu}_{h,F}(\epsilon) &= \int_0^{\bar{x}} F(x, r + \epsilon h, r' + \epsilon h', \dots, r^{(d)} + \epsilon h^{(d)}) dx \\ &\quad + \sum_{j=1}^{m_b} \lambda_j^b C_j^b(r + \epsilon h) + \sum_{j=1}^{m_f} \lambda_j^f C_j^f(r + \epsilon h) - \sum_{j=1}^{m_s} \lambda_j^s C_j^s(r + \epsilon h). \end{aligned}$$

Differentiating  $\bar{\mu}_{h,F}(\epsilon)$  with respect to  $\epsilon$  and setting  $\epsilon = 0$  gives

$$\begin{aligned} \bar{\mu}'_{h,F}(0) &= \sum_{j=0}^d \int_0^{\bar{x}} h^{(j)}(x) \frac{\partial F}{\partial r^{(j)}} dx + \sum_{j=1}^{m_b} \lambda_j^b \int_0^{x_j^b} h(x) dx + \sum_{j=1}^{m_f} \lambda_j^f \int_{x_j^f}^{x_{j+1}^f} h(x) dx \\ &\quad + \sum_{j=1}^{m_s} \lambda_j^s \left[ \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} \gamma_{j,k}^s(r) \int_0^{x_{j,k}^s} h(x) dx + \gamma_{j,\nu_j^s}^s(r) \int_0^{x_{j,\nu_j^s}^s} h(x) dx \right]. \end{aligned}$$

Applying integration by parts to the summands in the first term gives

$$\int_0^{\bar{x}} h^{(j)}(x) \frac{\partial F}{\partial r^{(j)}} dx = \left[ \sum_{k=0}^{j-1} (-1)^k h^{(j-k-1)}(x) \frac{d^k}{dx^k} \left( \frac{\partial F}{\partial r^{(j)}} \right) \right]_0^{\bar{x}} \\ + (-1)^j \int_0^{\bar{x}} h(x) \frac{d^j}{dx^j} \left( \frac{\partial F}{\partial r^{(j)}} \right) dx$$

for  $0 \leq j \leq d$ . Substituting these expressions into the previous equation and collecting terms results in the equation

$$\sum_{j=1}^d \left[ \sum_{k=0}^{j-1} (-1)^k h^{(j-k-1)}(x) \frac{d^k}{dx^k} \left( \frac{\partial F}{\partial r^{(j)}} \right) \right]_0^{\bar{x}} + \int_0^{\bar{x}} h(x) \varphi(F, r) dx = 0, \quad (\text{A.1})$$

where

$$\varphi(F, r) = \sum_{j=1}^d (-1)^j \frac{d^j}{dx^j} \left( \frac{\partial F}{\partial r^{(j)}} \right) + \sum_{j=1}^{m_b} \lambda_j^b \iota(x_j^b - x) \\ + \sum_{j=1}^{m_f} \lambda_j^f \iota(x_{j+1}^f - x) \iota(x - x_j^f) \\ + \sum_{j=1}^{m_s} \lambda_j^s \left[ \frac{\rho_j^s}{\nu_j^s} \sum_{k=1}^{\nu_j^s x_j^s} \gamma_{i_j^s, k}^s(r) \iota(x_{i_j^s, k}^s - x) + \gamma_{i_j^s, \nu_j^s x_j^s}^s(r) \iota(x_{i_j^s, \nu_j^s x_j^s}^s - x) \right].$$

Since (A.1) must hold for any  $h \in \mathcal{C}^d[0, \bar{x}]$  it follows that  $\varphi(F, r) = 0$  subject to endpoint conditions

$$\sum_{k=i}^d (-1)^{k-j} \frac{d^{k-j}}{dx^{k-j}} \left( \frac{\partial F}{\partial r^{(k)}} \right) = 0$$

at  $x = 0+$  and  $x = \bar{x}-$  for  $1 \leq j \leq d$ . This completes the proof.

## REFERENCES

- Adams, K. and van Deventer, D. (1994), 'Fitting Yield Curves and Smooth Forward Rate Curves with Maximum Smoothness', *Journal of Fixed Income* **June**, 52–62.
- Chambers, D., Carleton, W. and Waldman, D. (1984), 'A New Approach to Estimation of the Term Structure of Interest Rates', *Journal of Financial and Quantitative Analysis* **19**(3), 223–269.
- Frishling, V. and Yamamura, J. (1996), 'Fitting a Smooth Forward Rate Curve to Coupon Instruments', *Journal of Fixed Income* **September**, 97–103.
- Heath, D., Jarrow, R. and Morton, A. (1992), 'Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation', *Econometrica* **60**(1), 77–105.
- McCulloch, J. (1975), 'The Tax-Adjusted Yield Curve', *Journal of Finance* **30**, 811–829.
- Nelson, C. and Siegel, A. (1987), 'Parsimonious Modeling of Yield Curves', *Journal of Business* **60**, 473–489.
- Press, W., Teukolsky, S., Vetterling, W. and Flannery, B. (1996), *Numerical Recipes in C - The Art of Scientific Computing*, second edn, Cambridge University Press, New York.
- Svensson, L. (1994), Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994, Discussion Paper 1051, Center for Economic Policy Research.

- Vasiček, O. and Fong, H. (1982), 'Term Structure Modeling Using Exponential Splines', *Journal of Finance* **32**, 339–356.
- Yekutieli, I. (1999), 'With Bond Stripping, the Curve's the Thing', *Bloomberg Magazine* **8**(3), 84–90.