The Small Noise Arbitrage Pricing Theory

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December 1998

This paper was written when the Author was visiting the University of Technology, Sydney.
He would like to thank A. D. Hall, J. Knight and S. J. Lin for their helpful advice and comments.

Abstract

This paper presents a small-noise version of the Arbitrage Pricing Theory (APT) which allows us to interpret the approximate linearity of the risk premia in terms of factor exposures for a fixed number of assets. The approximation becomes more accurate as the noise of the system decreases, even though the number of assets stays fixed.

JEL Codes: G12

Key Words: Arbitrage Pricing Theory, Linear Factor Models

Section 1 Introduction

Arbitrage Pricing Theory (APT) stresses the approximate linearity of the risk premia of assets in terms of the factor loadings. The basic assumption is that returns are generated by a linear factor model and that the above linear approximation becomes increasingly accurate as N, the number of assets, increases to infinity.

The original paper by Ross (1976) has been generalised in many directions. The linear factor model has been dispensed with by Bansal and Viswanathan (1993). The problem has been put in an equilibrium setting by Connor (1984) and has been given a very general factor structure by Chamberlain and Rothschild (1983) and Reisman (1992). Many other important papers have been put forward, but they all stress the approximate linearity of the risk premia for large N. It should be said that there is an interesting subquestion as to when the linear APT restriction holds exactly, see Connor (1993) and Huberman and Kandel (1987) among many others.

The contribution of this paper is to demonstrate that it is not just large economies that matter for the approximation; it is also the magnitude of idiosyncratic noise. Econometricians of a certain age will not be surprised by this assertion. In the simultaneous equation literature, asymptotic properties were analysed both as large sample calculations or, alternatively, as small sigma (noise) calculations, see Kadane (1971) for example. Our goal is to construct small noise bounds for the APT.

The consequences of this exercise have some empirical meaning. In particular, a market with a small number of assets may still be subject to the APT restrictions if most of the risk to assets in that market is factor risk. Likewise, large markets that have substantial idiosyncratic risk may have large deviations from the APT restrictions, a point that is already understood.

A discussion of relevant literature is presented in Section 2, together with definitions of a system noise parameter. A small noise APT theorem is presented in Section 3, together with discussion and conclusions.

Section 2

Assume the following linear model:

$$\tilde{x} = \tilde{E} + \tilde{B}\tilde{d} + \tilde{e} \qquad (2.1)$$

where $\tilde{x}$ is a (N x 1) vector of asset returns, $E(\tilde{x}) = \tilde{E}$, $E(\tilde{d}) = E(\tilde{e}) = 0$ and $\tilde{d}$ is a (k x 1) vector of factors, $\tilde{B}$ is an (N x k) matrix of loadings of rank k and $\tilde{e}$ has a diagonal (N x N) covariance matrix, $\tilde{\Sigma} = (s_{ii})$. It is further assumed that $\tilde{d}$ and $\tilde{e}$ are uncorrelated.

Before proceeding to discuss how to quantify system-wide idiosyncratic risk, we discuss some results in the literature that point to links between the exact APT restriction holding and the absence of idiosyncratic risk.

For this model, the exact APT restriction is defined as $E = l_0\tilde{i} + \tilde{B}\tilde{l}$ where $\tilde{i}$ is a (N x 1) vector of ones and l_0 and $\tilde{l}$ are ($k+1$) constants.

Firstly, it is well-known, see Ingersoll (1987), that if $\tilde{e} = 0$, then the exact APT restriction holds. Secondly, Connor (1984) presents an economy, called an insurable factor economy, where each investor's equilibrium portfolio consists of a linear combination of ($k+1$) mutual funds, see Corollory 2.1 (pg 22, op cit). These mutual funds are simply portfolios that fully replicate the factors and have zero idiosyncratic variance. He shows that in such an economy, see theorems 2 and 3, the exact APT restrictions holds for finite N. In this economy, although idiosyncratic risk exists, it is not present in equilibrium. Likewise Huberman and Kandel (1987) present results which show that the exact APT restriction holds if the vector of mutual funds corresponding to the factors intersect the mean-variance frontier of the assets, see Proposition 1. The rationale is the same. Again, the existence of the portfolio of factors/mutual funds on the frontier allows investors to hold an "optimal" portfolio that contains no idiosyncratic risk. Lastly, the infinite economy and limiting economy arguments all have the implication that the idiosyncratic risk should be reduced by diversification, ie, by holding a large enough number of assets. Taken together, these results suggest strongly that the arguments for the APT should be restructured in terms of noise/idiosyncratic risk, and that an alternative way to think about when the APT restriction holds can be rephrased in terms of idiosyncratic noise.

We now turn to issues concerned with the definition of a scalar measure of system noise. In equation (2.1), *S has been defined* to be diagonal with *i*th element s_{ii}. A useful system measure of noise should have the property that as it tends to zero, the s_{ii} should all tend to zero. Accordingly, define

$$s_{ii} = gd_{ii}, \quad i = 1, N \tag{2.2}$$

and set $d_{jj} = 1$, where $j = \arg\max(s_{ii})$.

It follows immediately that

$$\lim_{g \to 0} s_{ii} = 0 \qquad \forall i .$$

Since we are taking continuous limits here, rather than limiting on a sequence as is usual in the APT literature, consideration needs to be given to what a continuum of different *g* economies might be.

Define a *g* continuum of economies as

$$\tilde{r}^{g} = \tilde{E} + \tilde{B}^{g}\tilde{d}^{g} + \tilde{e}^{g}, \text{ for } g \in R^{+}$$

where R^{+} is the set of non-negative real numbers.

Now consider the concept of small *g* arbitrage. Consider a continuum of portfolios $\{ x^{g} \}$, $x^{g} \in R^{+}$. Each portfolio x^{g} has typical *i*th element x_i^{g} and is of length *N*, where *N* is fixed. Assume the following:

Assumption 2.1

(a) $$\sum_{i=1}^{N} x_i^{g} = 0$$

(b) $$\sum x_i^{g} E(r_i^{g}) \geq d > 0$$

(c) $$\sum \sum x_i^{g} x_j^{g} \operatorname{cov}(r_i^{g}, r_j^{g}) \to 0 \text{ as } \gamma \to 0.$$

The above assumptions tell us that; (a), x^{γ} is a hedge fund, (b) the expected return on the portfolio is bounded above zero, and (c) the variance of the portfolio tends to zero as γ tends to zero.

It is appropriate to give some rationale as to what a continuum of γ economies might mean. The usual assumptions in APT studies are that the return distributions discussed in (1) are exogenous. In the case of γ economies, these different distributions could be thought of as arising from various actions by regulators which inhibit idiosyncratic noise, or perhaps from the introduction of mutual funds that effectively mimic some of the risk factors, and whose introduction leads to new exogenous distributions through some external equilibrating process that leaves the noise component less noisy.

It is straightforward to demonstrate that an individual with an increasing concave expected utility function U will, ceterus paribus, prefer less system-wide noise to more.
Let the individual's optimized return be r_p where

$$r_p = e_p + \sum_{j=1}^{k} B_{p,j} f_j + \sqrt{g d_{pp}}\, e'_p \qquad (2.3)$$

In the above, e'_p is standardized idiosyncratic noise; the standard deviation of the noise being $\sqrt{g d_{pp}}$.
Equation (2.3) is a consequence of equation (2.1), the term d_{pp} is defined by (2.2).
Let the value of the expected utility be denoted by V = V( ) where we have written V in terms of the value of the system-wide noise. Initial wealth is denoted by W_0 which is set to 1. Thus,

$V(\gamma) = E(U(1+r_p))$

If $\partial V/\partial\gamma$ is computed, we see that

$$\frac{\partial V}{\partial g} = E\left(U'(1+r_p))\,\frac{\partial r_p}{\partial g}\right)$$

$$= \frac{1}{2} g^{-\frac{1}{2}} d_{pp}^{1/2} E(U'(1+r_p)e'_p)$$

It follows from the concavity of U and the positive dependence of r_p on e'_p that $\partial V/\partial\gamma$ must be negative. Thus, a decrease in γ leads to all investors being better off if their optimal portfolio remains factor equivalent to the previous optimum. That such a shift in γ is actually Pareto improving, however, does not follow from the above result without further argument.

Section 3

We now prove the following proposition about small γ arbitrage:

Proposition 3.1
In a γ continuum of economies with fixed N and no small γ arbitrage,

$$\lim_{g \to 0}\left(e_i^{g} - l_0^{g} - \sum_{j=1}^{K} B_{ij}^{g}\, l_j^{g} \right) = 0$$

Proof: The proof is standard and follows Ingersoll (page 173, op cit.).

Consider a best linear prediction of e_i^g , the ith element of E^g , on a constant and the k exposures B_{ij}^g , j = 1, k. Define the residual v_i (with the γ superscripts suppressed) as

$$v_i = e_i - l_0 - \sum_{j=1}^{K} B_{ij}\, l_j \,.$$

It follows that $\sum_{i=1}^{N} v_i = 0,$ and $\sum v_i B_{ij} = 0.$

We construct our portfolio as $w_i = v_i/\|v\|$ where $\|v\|$ is the Euclidean distance of $v = (v_1, v_2, \ldots, v_n)'$. The portfolio $w = (w_{1,} \ldots w_n)'$ has the following properties:

(i) $\sum w_i = 0.$

(ii) $\sum w_i e_i = \dfrac{\sum v_i e_i}{\|v\|} = \dfrac{\sum {v_i}^2}{\|v\|} = \|v\|$

(iii) $$\begin{aligned} \operatorname{var}(\textstyle\sum w_i x_i) &= \operatorname{var}(\textstyle\sum w_i e_i + \sum w_i e_i) \\ &= g \textstyle\sum {w_i}^2 d_{ii} \\ &= g \textstyle\sum v_i^2 d_{ii} / \sum v_i^2 \\ &\leq g \text{ since } d_{\mathrm{ii}} \leq 1. \end{aligned}$$

It follows that $\lim\limits_{g \to 0} \operatorname{var}(\sum w_i x_i) = 0$; this implies that $\lim E(\sum w_i x_i) = \lim\limits_{g \to 0} \|v\|$ = 0 to avoid small γ arbitrage Q.E.D.

In conclusion, we have shown that smaller γ economies with no small γ arbitrage lead to broadly the same results as when we increase the number of assets with no asymptotic arbitrage. However, the interpretation here is quite different and, we would argue, more natural. It further raises questions as to the Pareto improving nature of a reduction in system noise which it is hoped to address in further research. More practically, a market with a small number of stocks could be deemed to have risk premia computed by the APT in the in the above framework. Trying to assess this empirically for emerging markets should be an interesting exercise.

Bibliography

Ravi Bansal and S. Viswanathan "No Arbitrage and Arbitrage Pricing: A New Approach", *Journal of Finance*, vol 48, no. 4 (1993), 1231-1261.

Gary Chamberlain and M. Rothschild "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Markets", *Econometrica*, vol. 51 (1983), 1281-1304.

Gregory Connor "A Unified Beta Pricing Theory", *Journal of Economic Theory*, vol 34, (1984) 13-31.

Gur Huberman "A Simple Approach to Arbitrage Pricing Theory", *Journal of Economic Theory*, vol. 28 (1982), 183-191.

Gur Huberman and Shmuel Kandel "Mean-Variance Spanning", *Journal of Finance*, vol 42, no. 4 (September 1987), 873-888.

Johnathon E. Ingersoll "Theory of Financial Decision Making", 1987, Rowman and Littlefield, New Jersey.

J.B. Kadane "Comparison of K-Class Estimators when Disturbances are Small", *Econometrica*, (Sep), 1971, 206-213.

Haim Reisman "Reference Variables, Factor Structure, Multibeta Representation", *Journal of Finance*, vol 47, no. 4 (1992), 1303-1315.

Stephen A. Ross "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, vol. 13 (1976), 341-360.