A MULTICURRENCY EXTENSION OF THE LOGNORMAL INTEREST RATE MARKET MODELS

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Abstract. The *Market Models* of the term structure of interest rates, in which forward LIBOR or forward swap rates are modelled to be lognormal under the forward probability measure of the corresponding maturity, are extended to a multicurrency setting. If lognormal dynamics are assumed for forward LIBOR or forward swap rates in two currencies, the forward exchange rate linking the two currencies can only be chosen to be lognormal for one maturity, with the dynamics for all other maturities given by no–arbitrage relationships. Alternatively, one could choose forward interest rates in only one currency, say the domestic, to be lognormal and postulate lognormal dynamics for all forward exchange rates, with the dynamics of foreign interest rates determined by no-arbitrage relationships.

Since the seminal article of Black and Scholes (1973), their option pricing formula has been applied to a myriad of derivative financial instruments, sometimes on the basis of an arbitrage–free model and sometimes — especially in the hectic world of day–to–day derivatives trading — just on the basis of heuristic analogy. The formula has a strong intuitive appeal for practitioners and remains the most important tool of the financial engineer. Furthermore, market prices for a variety of option contracts are routinely communicated in terms of their Black/Scholes implied volatility, since all other determinants of an option's value, such as the price of the underlying asset and the relevant interest rate, are readily observed elsewhere.

In this context, theoretical work which embeds Black/Scholes–type formulae favored by practitioners in a consistent, arbitrage–free framework is particularly relevant. Miltersen, Sandmann and Sondermann (1997)[1] (MSS) showed that the practice of pricing interest rate cap and floor contracts by a formula which had hitherto only been justified by analogy to Black/Scholes is consistent with a term structure model satisfying the no-arbitrage constraints of the Heath, Jarrow and Morton (1992) framework. The critical assumption for this result is that *relative volatility*[2] *of forward rates such as LIBOR,* compounded according to market conventions, *is deterministic*. Brace, Gatarek and Musiela (1997) (BGM) resolved open questions in the construction of such a model, in particular concerning existence and measure relationships, and coined the term *Market Model*: It reflects market practice both in interest rate compounding and in the pricing of caps and floors. Given the deterministic volatility assumption, BGM explicitly identified forward LIBOR as lognormal martingales under the forward measure to the end of the respective accrual periods, an approach that was pursued further by Musiela and Rutkowski (1997a), who also present a particularly straightforward construction of the *Market Model*. The same methodology can be applied to forward swap rates to derive a model which supports the

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[1]See also Sandmann and Sondermann (1994) and Sandmann, Sondermann and Miltersen (1995).

[2]The relative volatility of a diffusion process $X(t)$ is σ if its quadratic variation is given by $X(t)^2\sigma^2dt$.

market practice of pricing swaptions instead of that of pricing caps and floors, as demonstrated by Jamshidian (1997).

The aim of this paper is to extend the *Market Models* to multiple currencies and to determine to what extent Black/Scholes–type valuation of interest rate caps and floors (or, alternatively, swaptions) can be reconciled with the application of the Black/Scholes formula to options on foreign exchange. We will focus on the core issues of the restrictions that the no-arbitrage requirement imposes on simultaneous lognormality assumptions. These restrictions become particularly clear when the relationships between the martingale measures associated with domestic and foreign numeraire assets are identified. Given a set of lognormality assumptions satisfying these restrictions, obtaining the corresponding closed form option pricing formulae is a straightforward application of the standard techniques of the Black/Scholes framework.[3]

The paper is organized as follows. Section 1 develops the relationships between domestic and foreign equivalent martingale measures and elaborates on the no-arbitrage conditions linking the volatilities of forward exchange and interest rates. The consequences of these conditions for the consistent choice of lognormality assumptions are discussed in section 2. Section 3 covers the continuous tenor case and the implications for the dynamics of the spot exchange rate. Extensions to *Market Model* simulation algorithms to accommodate a multicurrency model are presented in section 4, and section 5 concludes.

1. Measure Relationships

Given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T^*]}, \mathbf{P}_{T^*})$ satisfying the usual conditions, let $\{W_{T^*}(t)\}_{t\in[0,T^*]}$ denote a d–dimensional standard Wiener process and assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T^*]}$ is the usual $\mathbf{P}_{T^*}$–augmentation of the filtration generated by $\{W_{T^*}(t)\}_{t\in[0,T^*]}$.

The model is set up on the basis of assumptions **(BP.1)** and **(BP.2)** of Musiela and Rutkowski (1997a):

(BP.1) For any date $T \in [0,T^*]$, the price process of a zero coupon bond $B(t,T)$, $t \in [0,T]$ is a strictly positive special martingale under $\mathbf{P}_{T^*}$.

(BP.2) For any fixed $T \in [0,T^*]$, the *forward process*

$$F_B(t,T,T^*) = \frac{B(t,T)}{B(t,T^*)}, \qquad \forall\, t \in [0,T]$$

follows a martingale under $\mathbf{P}_{T^*}$.

Note that assumption **(BP.2)** means that $\mathbf{P}_{T^*}$ can be interpreted as the *time T^* forward measure* and implies that the bond price dynamics are arbitrage–free.

The objects to be modelled on the fixed income markets are the δ-compounded forward rates defined by

$$L(t,T) = \delta^{-1}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) \tag{1}$$

Since the compounding matches the market convention for rates such as the London Interbank Offer Rate, $L(t,T)$ is also referred to as forward LIBOR. Note that by assumption **(BP.2)**, $L(t,T^*-\delta)$ is a martingale under $\mathbf{P}_{T^*}$.

[3]Independently of the present paper, Mikkelsen (1999) constructs consistent LIBOR *Market Model* settings under various lognormality assumptions and derives a number of Black/Scholes–type formulae for standard products.

Initially, let us consider the discrete–tenor case, i.e. for each of the fixed income markets, the model is constructed in the manner described in section 4.1 of Musiela and Rutkowski (1997a). For notational simplicity, assume a horizon date T^* which is a multiple of δ, i.e. $T^* = N\delta$, $N \in \mathbb{N}$, and focus on a finite number of maturities $T_i = i\delta$, $i \in \{0, \dots, N\}$ (the *tenor structure* $\mathbb{T}$). The dynamics of the (domestic) forward LIBOR rate with the longest maturity under the (domestic) time T^* forward probability measure $\mathbf{P}_{T^*}$ are given by

$$dL(t, T_{N-1}) = L(t, T_{N-1})\lambda(t, T_{N-1})dW_{T^*}(t) \tag{2}$$

Analogously, **(BP.1)** and **(BP.2)** are assumed to hold for the foreign fixed income market and under the foreign time T^* forward probability measure $\tilde{\mathbf{P}}_{T^*}$, we have

$$d\tilde{L}(t, T_{N-1}) = \tilde{L}(t, T_{N-1})\tilde{\lambda}(t, T_{N-1})d\tilde{W}_{T^*}(t)$$

i.e. we use the tilde to denote values on the foreign market. Note that if $\lambda(t, T_{N-1})$ is a deterministic function of its arguments, $L(t, T)$ is a lognormal martingale. However, this is not a necessary condition for the model construction.

Consider now the bond price quotients $B(t, T)/B(t, T_i)$ for some $i \in \{1, \dots, N\}$ and any $T \in [0, T^*]$. Define the *time T_i forward measure* $\mathbf{P}_{T_i}$ as the measure under which these bond price quotients are martingales. Since $B(t, T)/B(t, T_i)$ can be interpreted as the price of $B(t, T)$ expressed in terms of units of $B(t, T_i)$, we say that $B(t, T_i)$ is a *numeraire* and $\mathbf{P}_{T_i}$ is the equivalent martingale measure associated with this numeraire. In fact, in order to guarantee the absence of arbitrage in the complete market setting we are considering, the price process of *any* asset expressed in terms of the numeraire must be a martingale under the associated martingale measure.[4]

As shown in Musiela and Rutkowski (1997a), the measures $\mathbf{P}_{T_i}$ exist and are linked by the Radon/Nikodym–derivatives given in terms of the Doléans exponential as

$$\frac{d\mathbf{P}_{T_i}}{d\mathbf{P}_{T_{i+1}}} = \mathcal{E}_{T_i}\left(\int_0^{\cdot} \gamma(u, T_i, T_{i+1}) \cdot dW_{T_{i+1}}(u)\right) \qquad \mathbf{P}_{T_{i+1}}\text{–a.s.} \tag{3}$$

with

$$\gamma(t, T_i, T_{i+1}) = \frac{\delta L(t, T_i)}{1 + \delta L(t, T_i)}\lambda(t, T_i) \qquad \forall\ t \in [0, T_i] \tag{4}$$

In particular, we have

$$dW_{T_i}(t) = dW_{T_{i+1}}(t) - \gamma(u, T_i, T_{i+1})dt \tag{5}$$

If, for all $i \in \{1, \dots, N-1\}$, $\lambda(t, T_i)$ is a deterministic function of its arguments, every process $L(t, T_i)$ is a lognormal martingale under the corresponding probability measure $\mathbf{P}_{T_{i+1}}$ (and analogously for $\tilde{L}(t, T_i)$ under $\tilde{\mathbf{P}}_{T_{i+1}}$).

Now let $X(t)$ denote the spot exchange rate in terms of units of domestic currency per unit of foreign currency. To satisfy technical regularity conditions, let us make the following assumption:

(X.1) The spot exchange rate process $X(t)$, $t \in [0, T^*]$, is a strictly positive special martingale under $\mathbf{P}_{T^*}$.

[4] This result goes back to a series of papers of Harrison and Kreps (1979), Harrison and Pliska (1981) and Harrison and Pliska (1983), whose seminal work has since then been extended and refined in several ways. For a more complete list of references, see any recent book on mathematical finance, such as Musiela and Rutkowski (1997b).

Note that $X(t)$ is not a tradeable asset in either market[5]; thus the spot exchange rate (discounted by the numeraire) will generally not be a martingale under any equivalent martingale measures associated with a numeraire asset. However, the foreign bond converted to domestic currency at the spot exchange rate is a domestic asset, and thus

$$X(t,T_i) := \frac{\tilde{B}(t,T_i)X(t)}{B(t,T_i)} \tag{6}$$

is a martingale under $\mathbf{P}_{T_i}$. Conversely,

$$\frac{1}{X(t,T_i)} = \frac{B(t,T_i)\frac{1}{X(t)}}{\tilde{B}(t,T_i)}$$

is a martingale under $\tilde{\mathbf{P}}_{T_i}$. $X(t,T_i)$ is the *time T_i forward exchange rate.*

By the Girsanov transformations (3), all domestic forward measures $\mathbf{P}_{T_i}$ are linked to $\mathbf{P}_{T^*}$ and all foreign forward measures $\tilde{\mathbf{P}}_{T_i}$ are linked to $\tilde{\mathbf{P}}_{T^*}$. Since these relationships are transitive, specifying the measure transformation linking $\mathbf{P}_{T_i}$ and $\tilde{\mathbf{P}}_{T_i}$ fixes the relationships between all forward measures, domestic and foreign.

Without loss of generality (in the sense that we could have chosen any other forward measure instead), let us specify the link for the terminal forward measures. Set

$$dX(t,T^*) = X(t,T^*)\sigma_X(t,T^*)\cdot dW_{T^*}(t) \tag{7}$$

Again, if $\sigma_X(t,T^*)$ is a deterministic function of its arguments, $X(t,T^*)$ is a lognormal martingale.

From general theory we know that in the complete, arbitrage–free market we are considering, there exist unique equivalent measures $\mathbf{P}_{T^*}$ and $\tilde{\mathbf{P}}_{T^*}$, such that all assets discounted by the time T^* domestic zero coupon bond $B(\cdot,T^*)$ are martingales under $\mathbf{P}_{T^*}$ and all assets discounted by the foreign bond $\tilde{B}(\cdot,T^*)$ are martingales under $\tilde{\mathbf{P}}_{T^*}$. Consequently we must have

$$\frac{d\tilde{\mathbf{P}}_{T^*}}{d\mathbf{P}_{T^*}} = \frac{X(T^*)\tilde{B}(T^*,T^*)B(0,T^*)}{X(0)\tilde{B}(0,T^*)B(T^*,T^*)} = \frac{X(T^*,T^*)}{X(0,T^*)}$$

and restricting $\mathbf{P}_{T^*}$, $\tilde{\mathbf{P}}_{T^*}$ to the information given at time t (i.e. for $\mathcal{F}_t$–measurable events, where $(\mathcal{F}_t)_{0\leq t\leq T^*}$ is the augmented filtration generated by the driving Brownian motion), we have

$$\left.\frac{d\tilde{\mathbf{P}}_{T^*}}{d\mathbf{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{X(t,T^*)}{X(0,T^*)}$$

Given the dynamics (7) we chose for the forward foreign/domestic exchange rate $X(t,T^*)$, $\mathbf{P}_{T^*}$ and $\tilde{\mathbf{P}}_{T^*}$ are therefore linked by

$$\frac{d\tilde{\mathbf{P}}_{T^*}}{d\mathbf{P}_{T^*}} = \mathcal{E}_{T^*}\left(\int_0^{\cdot}\sigma_X(u,T^*)du\right) \qquad \mathbf{P}_{T^*}\text{–a.s.}$$

By Girsanov's Theorem, Brownian motions under the two measures are related by

$$d\tilde{W}_{T^*}(t) = dW_{T^*}(t) - \sigma_X(t,T^*)dt \tag{8}$$

[5]In the context of the model under consideration, a tradeable asset is any semimartingale which can be represented as the value process of a selffinancing trading strategy in domestic and/or foreign zero coupon bonds. For a numeraire asset, we additionally require that the value process is strictly positive. $X(t)$ is only the time t *rate of conversion* from foreign to domestic currency, i.e. $X(t)$ multiplied by the value of an asset denoted in foreign currency gives the value of that asset denoted in domestic currency.

and by Ito's lemma,

$$d\left(\frac{1}{X(t,T^*)}\right) = \frac{1}{X(t,T^*)}\left(-\sigma_X(t,T^*)\cdot dW_{T^*}(t) + \|\sigma_X(t,T^*)\|^2 dt\right)$$

Thus the volatility of the forward domestic/foreign exchange rate $X(t,T^*)^{-1}$ is $\sigma_{\frac{1}{X}}(t,T^*) = -\sigma_X(t,T^*)$ and we can write

$$d\left(\frac{1}{X(t,T^*)}\right) = \frac{1}{X(t,T^*)}\sigma_{\frac{1}{X}}(t,T^*)d\tilde{W}_{T^*}(t)$$

Having linked the foreign and domestic fixed income markets by specifying the volatility of the time T^* forward exchange rate under the domestic and foreign time T^* forward measures, the volatilities of all other forward exchange rates under the respective forward measures are fixed, since the forward measures of different maturities are already linked by the specification of forward LIBOR volatilities. To derive the remaining forward exchange rate volatilities, we inductively make use of the relationship

$$\frac{X(t,T_i)}{X(t,T_{i+1})} = \frac{B(t,T_{i+1})}{B(t,T_i)}\frac{\tilde{B}(t,T_i)}{\tilde{B}(t,T_{i+1})} \tag{9}$$

For ease of notation, consider just the first step of the induction,

$$X(t,T_{N-1}) = X(t,T^*)\frac{B(t,T^*)}{B(t,T_{N-1})}\frac{\tilde{B}(t,T_{N-1})}{\tilde{B}(t,T^*)}$$

As shown in Musiela and Rutkowski (1997a), the dynamics of the forward bond price are given by

$$d\left(\frac{B(t,T_{N-1})}{B(t,T^*)}\right) = \frac{B(t,T_{N-1})}{B(t,T^*)}\gamma(t,T_{N-1},T^*)\cdot dW_{T^*}(t)$$

By Ito's lemma,

$$d\left(\frac{B(t,T^*)}{B(t,T_{N-1})}\right) = \frac{B(t,T^*)}{B(t,T_{N-1})}\left(-\gamma(t,T_{N-1},T^*)\cdot dW_{T^*}(t) + \|\gamma(t,T_{N-1},T^*)\|^2 dt\right)$$

and for the foreign forward bond, after switching to the corresponding domestic measure

$$d\left(\frac{\tilde{B}(t,T_{N-1})}{\tilde{B}(t,T^*)}\right) = \frac{\tilde{B}(t,T_{N-1})}{\tilde{B}(t,T^*)}\left(\tilde{\gamma}(t,T_{N-1},T^*)\cdot dW_{T^*}(t) - \tilde{\gamma}(t,T_{N-1},T^*)\cdot\sigma_X(t,T^*)dt\right)$$

Thus

$$\begin{aligned} d\left(\frac{\tilde{B}(t,T_{N-1})}{\tilde{B}(t,T^*)}\frac{B(t,T^*)}{B(t,T_{N-1})}\right) &= \frac{\tilde{B}(t,T_{N-1})}{\tilde{B}(t,T^*)}\frac{B(t,T^*)}{B(t,T_{N-1})}\Big((\tilde{\gamma}(t,T_{N-1},T^*) - \gamma(t,T_{N-1},T^*))\cdot dW_{T^*}(t) \\ &\quad + \big(\|\gamma(t,T_{N-1},T^*)\|^2 - \tilde{\gamma}(t,T_{N-1},T^*)\cdot\sigma_X(t,T^*) \\ &\quad - \gamma(t,T_{N-1},T^*)\cdot\tilde{\gamma}(t,T_{N-1},T^*)\big)dt\Big) \end{aligned}$$

and

$$\begin{aligned} dX(t,T_{N-1}) &= X(t,T_{N-1})\big((\tilde{\gamma}(t,T_{N-1},T^*) - \gamma(t,T_{N-1},T^*) + \sigma_X(t,T^*))\cdot dW_{T^*}(t) \\ &\quad - \gamma(t,T_{N-1},T^*)\cdot(\tilde{\gamma}(t,T_{N-1},T^*) - \gamma(t,T_{N-1},T^*) + \sigma_X(t,T^*))dt\big) \end{aligned}$$

Since $dW_{T_{N-1}} = dW_{T^*}(t) - \gamma(t,T_{N-1},T^*)dt$, we have

$$dX(t,T_{N-1}) = X(t,T_{N-1})\left((\tilde{\gamma}(t,T_{N-1},T^*) - \gamma(t,T_{N-1},T^*) + \sigma_X(t,T^*))\cdot dW_{T_{N-1}}(t)\right)$$

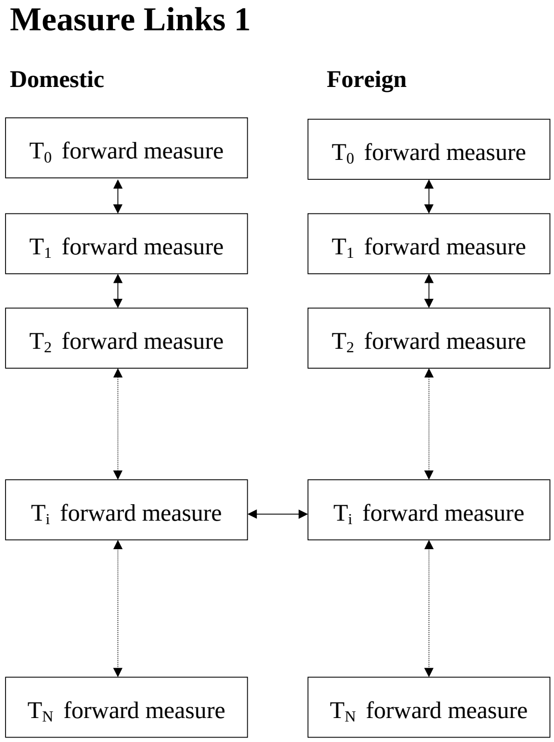

FIGURE 1. Lognormal LIBORs

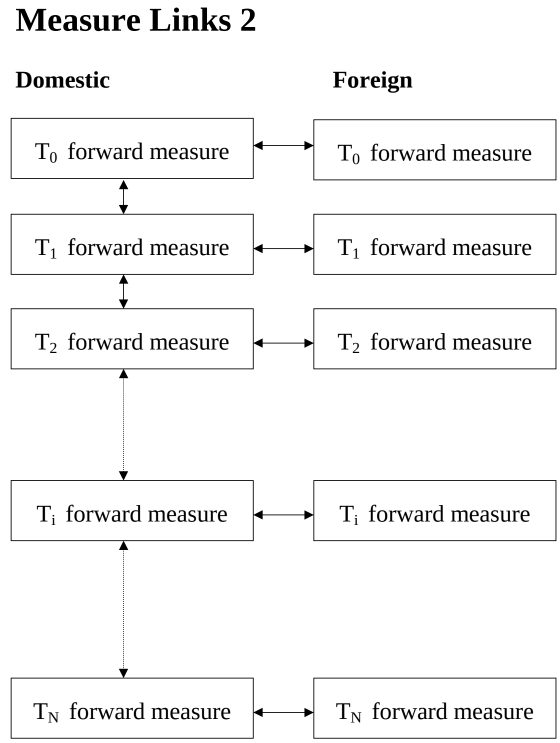

FIGURE 2. Lognormal exchange rates

Thus we must set

$$\sigma_X(t, T_{N-1}) = \tilde{\gamma}(t, T_{N-1}, T^*) - \gamma(t, T_{N-1}, T^*) + \sigma_X(t, T^*) \tag{10}$$

i.e. the forward exchange rate volatilities for all maturities are linked by

$$\sigma_X(t, T_{i-1}) = \tilde{\gamma}(t, T_{i-1}, T_i) - \gamma(t, T_{i-1}, T_i) + \sigma_X(t, T_i) \tag{11}$$

Note that the derivation of (11) does not depend on any of the assumptions of deterministic volatilities, but solely on the fact that σ_X, γ and $\tilde{\gamma}$ are, respectively, the volatility functions of the forward exchange rates, domestic and foreign forward bond prices.

Furthermore, if we choose to link the domestic and foreign market by a forward exchange rate to a maturity other than the terminal, we also solve (11) for $\sigma_X(t, T_i)$. In effect, we are using the domestic and foreign forward bond volatilities γ and $\tilde{\gamma}$ to move either forward or backward in maturity from the link.

2. Choices of Lognormality

As a consequence of (11), the choices of which underlying variables should follow a lognormal probability law are restricted. In the extremes, we have two cases, which can be mixed and matched to produce a variety of "hybrid" models. Consider figures 1 and 2. Each arrow denotes a measure relationship based on a *deterministic* volatility function, i.e. vertical arrows signify lognormal forward LIBORs while horizontal arrows signify lognormal forward exchange rates.

In figure 1, we have a discrete–tenor lognormal forward LIBOR model for both the domestic and foreign fixed income markets. Furthermore, one forward exchange rate is chosen to be lognormal. With that, all measure relationships are fixed and all remaining forward exchange rate volatilities are given by (11). By inserting (4) into (11), it becomes obvious that these volatilities depend on LIBOR levels and consequently only the forward exchange rate chosen to "link" the two fixed income markets can be lognormal.

More specifically, in this case the model is constructed as follows. We first set up a discrete tenor lognormal forward LIBOR model for each of the currencies as in Musiela and Rutkowski (1997a), by specifying d–dimensional volatility vectors $\lambda(t, T_i)$ and $\tilde{\lambda}(t, T_i)$ for all $0 \leq i < N$. Specifying these volatilities completely determines the discrete tenor term structure models in each of the currencies. The measure relationships between forward measures in each currency are constructed by backward induction using equations (3)-(5). Given these relationships, we can move backward and forward in maturity at will. Thus it is sufficient to link the two currencies by choosing a forward exchange rate volatility function $\sigma_X(\cdot, T_i)$ for some arbitrary T_i.

In figure 2, only domestic forward LIBORs are assumed to be lognormal, as well as *every* forward exchange rate. Note that lognormality of the forward exchange rate under the corresponding forward measure means that the Black/Scholes formula holds for a currency option of this maturity, irrespective of the interest rate dynamics. Again, the resulting (discrete–tenor) model is fully specified and condition (11) precludes the lognormality of foreign interest rates.

Here, the domestic discrete tenor model is constructed as in case 1. Furthermore, for all $0 \leq i \leq N$, forward exchange rate volatility functions $\sigma_X(t, T_i)$ are input into the model. Given these, (8) permits us to move from any domestic forward measure to the foreign forward measure of the same maturity, and indirectly between foreign forward measures of different maturities, thus all measure relationships are now fixed. By the input volatility function and the measure relationships[6], the continuous time dynamics of $L(t, T_i)$ and $X(t, T_j)$ are well determined for all $0 \leq i < N$ and $0 \leq j \leq N$, and by equation (9) the $\tilde{L}(t, T_i)$ are well determined also and their dynamics are easily derived by applying Ito's Lemma.

From a practical point of view, it may be attractive to mix the two cases. Consider, for example, a situation where currency options for shorter maturities are very actively traded, while for longer maturities there is very little implied volatility information available in the market. On the other hand, longer–dated interest rate options may be reasonably liquid. In such a case, one could model the "liquid" volatilities as deterministic, greatly facilitating model calibration. This means assuming lognormal forward exchange rates on the short end and moving to lognormal forward interest rates in both markets on the long end, in effect using the volatilities of longer–dated interest rates to extrapolate the volatilities of forward exchange rates.

3. Continuous Tenor and the Spot Exchange Rate

The discrete tenor version of the model discussed so far completely specifies stochastic dynamics only for rates (both foreign exchange and interest) maturing at dates in the tenor structure $\mathbb{T}$. Consequently, payoffs that occur at intermediate dates not in $\mathbb{T}$ or depend on intermediate rates cannot be valued. Extending the model to continuous tenor ensures that the value of the numeraire asset is well determined at any point in time and stipulates the dynamics of rates for all intermediate maturities. In particular, this is necessary in order to specify the dynamics of the *spot* exchange rate $X(t)$.

Case 1: In the continuous tenor version of the *Market Models* as it was originally proposed in the MSS/BGM papers, the assumption is that *all* forward LIBORs $L(t, T)$, i.e. for all $t \leq T$ and $T \leq T^* - \delta$, are lognormal martingales under the respective forward

[6] Note that in this context, we could use the terms *measure relationship* and *drift* synonymously.

measures. Then all that is missing are the volatilities of zero coupon bonds with time to maturity less than δ. BGM set this volatility to zero and derive the resulting dynamics of all bond prices $B(t,T)$ and of the continuously compounded short rate $r(t)$.

Case 2: Alternatively, as proposed in Schlögl (1999), one could set the volatility of all bonds maturing before the next date in the tenor structure[7] to zero and determine these bond prices by a deterministic interpolation scheme. This has the advantage of preserving the Markovian structure of the discrete tenor model and — contrary to the approach taken by MSS/BGM — guaranteeing the non-negativity of *all* interest rates. Furthermore, it turns out that arbitrage–free interpolation by day–count fractions becomes very tractable.

In either case, if $T_{\eta(t)}$ is the next date in the discrete tenor structure after t, the volatility of $B(t,T_{\eta(t)})$ is zero. This means that we can speak synonymously of the *spot LIBOR measure* of Jamshidian (1997) and the *risk neutral measure*:

3.1. Definition. Given a tenor structure $\mathbb{T} = \{T_i : i \in \{0,1,\dots,N\}\}$, the *spot LIBOR measure* $\mathbf{P}_\rho$ is the equivalent martingale measure associated with the numeraire

$$\rho(t) = B(t,T_{\eta(t)}) \prod_{i=0}^{\eta(t)-1} (1+\delta L(T_i,T_i))$$

where

$$\eta(t) = \max\{i \in \{0,\dots,N\} | T_{i-1} < t\}$$

Note that the spot LIBOR measure can be interpreted as a chain of transition probabilities given by conditional forward measures of consecutive maturities (cf. lemma A.1 in the appendix).

3.2. Definition. Given a continuously compounded short rate $r(t)$, the *risk neutral measure* $\mathbf{P}_\beta$ is the equivalent martingale measure associated with the numeraire

$$\beta(t) = \exp\left\{\int_0^t r(s)ds\right\}$$

3.3. Lemma. *If the volatility of $B(t,T_{\eta(t)})$ is zero for all $T_0 \le t \le T_N$, the spot LIBOR measure and the risk neutral measure coincide.*

Proof. See appendix. Note that when speaking of the risk neutral measure in this context, it is implicitly assumed that the continuously compounded short rate $r(t)$ exists. As argued in Schlögl (1999), if the short bond volatility is zero, the discrete tenor model can always be extended to continuous tenor in such a manner that the short rate exists on each of the open intervals $[T_i, T_{i+1}[$, which is all that is needed.

□

If domestic and foreign fixed income markets are described by continuous tenor models of either type 1 or 2, $B(t,T_{\eta(t)})$ and $\tilde{B}(t,T_{\eta(t)})$ are well defined and the dynamics of the spot exchange rate can be obtained (rather tediously) by applying Ito's Lemma to

$$X(t) = \frac{X(t,T_{\eta(t)})B(t,T_{\eta(t)})}{\tilde{B}(t,T_{\eta(t)})} \tag{12}$$

$X(t,T)$ can then be calculated for arbitrary T by inserting the $X(t)$ resulting from the above equation into (6).

[7] i.e. $B(t,T_{\eta(t)})$, with $\eta(t)$ defined as in definition 3.1 below

Note that given BGM's choice setting short bond volatilities to zero, the above relationship implies

$$\sigma_X(t,t) = \sigma_X(t,t+\delta) = \sigma_X(t,T) \qquad \forall\, t \leq T \leq t+\delta$$

while in the second case of extending discrete to continuous tenor we only have

$$\sigma_X(t,t) = \sigma_X(t,T_{\eta(t)}) = \sigma_X(t,T) \qquad \forall\, t \leq T \leq T_{\eta(t)} \leq t+\delta \tag{13}$$

Furthermore, since the spot exchange rate $X(t)$ is well defined in a continuous tenor framework, domestic and foreign fixed income markets could be linked by specifying the volatility of $X(t)$. However, stipulating deterministic volatility for the spot exchange rate to link fixed income markets described by continuous tenor MSS/BGM models of cases 1 or 2 is not sufficient to guarantee lognormality of $X(t)$ under the risk neutral measure (i.e. the equivalent martingale measure associated with the continuously compounded savings account), or under any equivalent martingale measure associated with a tradeable numeraire asset, for that matter. This can be seen as a consequence of the highly non-linear drift term relating forward measures of different maturities (cf. equation 5) and the fact that the spot exchange rate is not a tradeable asset[8] and therefore not a martingale under any forward measure or the risk neutral measure.

Nor is the lognormality of $X(t)$ a particularly desirable feature with a view to obtaining Black/Scholes–type formulae: For currency options under stochastic interest rates, it is easily verified that a Black/Scholes–type pricing formula results if the exchange rate forward to the maturity of the option is a lognormal martingale under the corresponding forward measure; thus in view of market practice it is more natural to assume deterministic volatilities for *forward* exchange rates.

If nevertheless one chooses to link fixed income markets in two currencies by a spot exchange rate with deterministic volatility, lognormality under the domestic risk neutral measure holds for the foreign savings account converted to domestic currency at the spot exchange rate: By definition of the risk neutral measure $\mathbf{P}_\beta$,

$$\frac{X(t)\tilde{\beta}(t)}{\beta(t)} = \frac{X(t)\exp\left\{\int_0^t \tilde{r}(s)ds\right\}}{\exp\left\{\int_0^t r(s)ds\right\}}$$

is a martingale under $\mathbf{P}_\beta$, where r and $\tilde{r}$ denote the domestic and foreign continuously compounded short rates, respectively. $\tilde{\beta}(t)$ and $\beta(t)$ are of finite variation, thus deterministic volatility of $X(t)$ implies lognormality of $X(t)\tilde{\beta}(t)/\beta(t)$ under $\mathbf{P}_\beta$. However, this only implies a Black/Scholes–type formula for the rather exotic option to exchange a foreign savings account for a domestic savings account, i.e. for time T payoffs of the type $[X(T)\tilde{\beta}(T) - K\beta(t)]^+$.

As in the discrete tenor version of the model, alternatively to lognormal forward LIBOR dynamics in both currencies, one could specify deterministic volatilities for domestic forward LIBORs only and link fixed income markets in the two currencies by stipulating the volatilities of all exchange rates, forward and spot. The dynamics of foreign zero coupon bonds are then well defined in terms of spot and forward exchange rates, and domestic zero coupon bonds, simply by rearranging (6). Foreign forward LIBOR volatilities are determined by (11), valid in the continuous tenor case for all maturities $t < T \leq T^* - \delta$:

$$\tilde{\gamma}(t,T,T+\delta) = \sigma_X(t,T) - \sigma_X(t,T+\delta) + \gamma(t,T,T+\delta)$$

[8] cf. footnote 5

Furthermore, (8) holds for every maturity, in particular

$$\begin{aligned} d\tilde{W}_\beta(t) &= dW_\beta(t) - \sigma_X(t,t)dt \\ d\tilde{W}_{T_{\eta(t)}}(t) &= dW_{T_{\eta(t)}}(t) - \sigma_X(t,T_{\eta(t)})dt \end{aligned}$$

Since the spot LIBOR and risk neutral measures coincide, this implies

$$d\tilde{W}_{T_{\eta(t)}}(t) = d\tilde{W}_\beta(t) + \sigma_X(t,t)dt - \sigma_X(t,T_{\eta(t)})dt$$

which is not surprising since rearranging (12) yields

$$\frac{\tilde{B}(t,T_{\eta(t)})}{B(t,T_{\eta(t)})} = \frac{X(t,T_{\eta(t)})}{X(t)}$$

and applying Ito's lemma to both sides of this equation reveals that the volatility of the foreign short bond $\tilde{B}(t,T_{\eta(t)})$ must be $\sigma_X(t,T_{\eta(t)}) - \sigma_X(t,t)$. In addition, it is easily verified that the drift of $X(t)$ under $\mathbf{P}_\beta$ is $r(t) - \tilde{r}(t)$.

4. Simulation Algorithms

As discussed in section 2, consistency with no-arbitrage conditions restricts the lognormality assumptions which can be made simultaneously, leaving some standard products and of course the more complex derivative instruments to be priced numerically or by approximate formulae. The non-linear drift terms relating Brownian motions under forward measures of different maturities preclude any simple Markovian structure of the model[9] and thus Monte Carlo simulation becomes the numerical method of choice in many cases. This section serves to briefly outline how two simulation algorithms proposed in the literature can be easily extended to accommodate the multicurrency case.

Glasserman and Zhao (1998) present an algorithm in which forward LIBORs (or forward swap rates) are first transformed into variables which are martingales under the forward measure chosen for simulation. These variables are then simulated and forward LIBORs are subsequently recovered from the simulated values. By simulating martingales, Glasserman and Zhao avoid the bias that results from the discretization of a stochastic drift term. To apply their approach, suitably transformed variables need to be specified.

First, let us consider a discrete tenor model[10] with both domestic and foreign lognormal forward LIBORs. Assume the simulation is to be carried out under the domestic time T_n forward measure (the case for a simulation under a foreign measure is covered by a symmetric argument). As in Glasserman/Zhao, the variable

$$V_n(t,T_j) = \begin{cases} L(t,T_j)\prod_{i=j+1}^{n-1}(1+\delta L(t,T_i)) & j=1,\ldots,n-1 \\ L(t,T_j)\prod_{i=n\vee\eta(t)}^{j}(1+\delta L(t,T_i))^{-1} & j=n,\ldots,N-1 \end{cases} \tag{14}$$

is a martingale under the T_n forward measure up to T_n, and under the spot LIBOR measure thereafter. Given $V_n(t,T_j)$ for all $j \in \{1,\ldots,N-1\}$, all domestic forward LIBORs $L(t,T_j)$ can be recovered.

Analogously defined, $\tilde{V}_n(t,T_j)$ is a martingale under the foreign time T_n forward measure up to T_n, and under the foreign spot LIBOR measure thereafter. Consequently, $X(t,T_{n\vee\eta(t)})\tilde{V}_n(t,T_j)$ is a martingale under the domestic measure, as is $X(t,T_{n\vee\eta(t)})$. Therefore, $X(t,T_{n\vee\eta(t)})$ is simulated directly, and given $X(t,T_{n\vee\eta(t)})$ and $X(t,T_{n\vee\eta(t)})\tilde{V}_n(t,T_j)$ for

[9] For a discussion of the Markovian properties of the lognormal *Market Models*, see Schlögl (1999).

[10] Glasserman and Zhao's algorithm is formulated in terms of the discrete tenor model. However, it is also directly applicable to the continuous tenor version of the model proposed in Schlögl (1999), since in that case all Markovian properties of the discrete tenor model are preserved.

all $j \in \{1, \dots, N-1\}$, all foreign forward LIBORs $\tilde{L}(t, T_j)$ can be recovered. Obviously, one would choose the simulation measure $\mathbf{P}_{T_n}$ to match the choice of deterministic forward exchange rate volatility, i.e. such that the link exchange rate $X(t, T_{n\vee\eta(t)})$ is lognormal under $\mathbf{P}_{T_n}$ up to T_n, and conditionally lognormal under $\mathbf{P}_{\eta(t)}$ given $\mathcal{F}_{T_{\eta(t)-1}}$ on each interval $[T_{\eta(t)-1}, T_{\eta(t)}]$ thereafter. Given $X(t, T_{n\vee\eta(t)})$, all $L(t, T_j)$ and $\tilde{L}(t, T_j)$, the remaining forward exchange rates can also be recovered.

For the case where all discrete tenor forward exchange rates are assumed to be lognormal under the respective forward measures, a slightly different transformation suggests itself. Set

$$U_n(t, T_j) = \begin{cases} X(t, T_j) \prod_{i=j}^{n-1} (1 + \delta L(t, T_i)) & j = 1, \dots, n-1 \\ X(t, T_j) \prod_{i=n\vee\eta(t)}^{j-1} (1 + \delta L(t, T_i))^{-1} & j = n, \dots, N \end{cases}$$

which is a martingale under the T_n forward measure up to T_n, and under the spot LIBOR measure thereafter. Simulating $U_n(t, T_j)$ for all $j \in \{1, \dots, N\}$ and $V_n(t, T_j)$ for all $j \in \{1, \dots, N-1\}$ yields all $L(t, T_j)$ and all $X(t, T_j)$, from which all $\tilde{L}(t, T_j)$ can then be recovered.

Alternatively to the algorithm by Glasserman/Zhao, one can make use of the relationships between domestic and foreign forward measures derived in section 1 to transform Brownian motion increments generated under the measure chosen for simulation into the corresponding Brownian motion increments for all other measures, following the approach proposed in Brace, Musiela and Schlögl (1998). Each of the rates for which deterministic volatility is assumed is then simulated as a lognormal martingale under the corresponding forward measure. Again, only these rates need to be simulated, as all others can be recovered by means of static relationships such as (6) and (9), i.e.

$$X(t, T_i) = X(t, T_{i+1}) \frac{1 + \delta \tilde{L}(t, T_i)}{1 + \delta L(t, T_i)}$$

5. Conclusion

Applying a Black/Scholes–type formula to price a derivative financial instrument relies on the lognormality of the underlying asset under the appropriate martingale measure. When simultaneously considering interest rate and currency risk, the choice of lognormality assumptions is restricted by no-arbitrage conditions resulting from the relationship between forward exchange rates and domestic and foreign interest rates. Postulating a lognormal *Market Model* for both the domestic and foreign fixed income markets leaves only the possibility of specifying a lognormal probability law for *one* forward exchange rate, while if only domestic interest rate dynamics are given, all forward exchange rates could be assumed to be lognormal under the respective forward measures. These two cases may be mixed and combined in a variety of ways into “hybrid” models. Conversely, choosing lognormality assumptions so as to facilitate calibration to at–the–money prices of liquidly traded instruments allows less liquid volatilities to be interpolated or extrapolated in a consistent manner, for example using implied volatilities of short–dated currency options and long–dated interest rate options to obtain prices for currency options of longer maturity.

Two further extensions of the model suggest themselves. They are simple (though in some cases tedious) applications of the basic approach discussed in this paper. For one, the present results easily scale to the case of arbitrarily many currencies. Secondly, for the domestic and/or foreign fixed income market, the lognormality assumption on forward LIBORs may be replaced by a forward swap rate model of the type developed by Jamshidian (1997).

Also, it is worthwhile to note that the measure relationships derived in section 1 do not depend on assumptions of deterministic volatility functions. Thus they are equally valid in any term structure model constructed along the lines of Heath, Jarrow and Morton (1992)[11] as well as in extensions of the *Market Models* to more general volatility functions, such as the model of level–dependent LIBOR volatilities proposed by Andersen and Andreasen (1998).

In summary, while this paper makes no attempt to develop a model to better explain the *empirical* dynamics of interest rates and exchange rates, we have identified the trade–offs between lognormality assumptions supporting the simultaneous use of Black/Scholes–type formulae for various derivative financial instruments, showing to what degree a model can reflect popular market practice while remaining consistent with no-arbitrage.

Appendix A.

Proof of lemma 3.3: By the definition of the two measures, we must have

$$\left.\frac{d\mathbf{P}_\rho}{d\mathbf{P}_\beta}\right|_{\mathcal{F}_t} = \frac{\rho(t)}{\rho(0)}\frac{\beta(0)}{\beta(t)} = \frac{\rho(t)}{\beta(t)} \qquad \mathbf{P}_\beta\text{–a.s.}$$

If the volatility of $B(t,T_{\eta(t)})$ is zero for all $T_0 \leq t \leq T_N$, we have[12]

$$B(t,T_{\eta(t)}) = \frac{\beta(t)}{\beta(T_{\eta(t)})} = \exp\left\{\int_t^{T_{\eta(t)}} r(s)ds\right\} \qquad \forall\ T_0 \leq t \leq T_N$$

and consequently

$$\begin{aligned}\frac{\rho(t)}{\beta(t)} &= \beta(t)^{-1}B(t,T_{\eta(t)})\prod_{i=0}^{T_{\eta(t)}-1} B(T_i,T_i+\delta)^{-1} \\ &= \beta(t)^{-1}\frac{\beta(t)}{\beta(T_{\eta(t)})}\prod_{i=0}^{T_{\eta(t)}-1}\frac{\beta(T_{i+1})}{\beta(T_i)} \\ &= \beta(T_0)^{-1} = 1\end{aligned}$$

Thus

$$\left.\frac{d\mathbf{P}_\rho}{d\mathbf{P}_\beta}\right|_{\mathcal{F}_t} = 1 \qquad \forall\ T_0 \leq t \leq T_N,\ \mathbf{P}_\beta\text{–a.s.}$$

and the two measures coincide.

□

The following lemma shows that the spot LIBOR measure may be interpreted as the measure obtained by "chaining" conditional probabilities of forward measures:

[11] The Gaussian model presented by Frey and Sommer (1996) implicitly satisfies the restrictions derived here. However, in their case *all* volatility functions in our equation (11) are deterministic, i.e. lognormal exchange rate dynamics and lognormal bond price dynamics in all currencies can consistently coexist in a multicurrency Gaussian HJM model.

[12] cf. remark 2.1 of Brace, Gatarek and Musiela (1997)

A.1. Lemma. *For all $\mathcal{F}_{T_{i+1}}$–measurable events, the conditional probabilities given $\mathcal{F}_{T_i}$ are identical for the spot LIBOR measure and the time T_{i+1} forward measure, i.e.*

$$\mathbf{P}_\rho\{A|\mathcal{F}_{T_i}\} = \mathbf{P}_{T_{i+1}}\{A|\mathcal{F}_{T_i}\} \qquad \forall\ i \in \{0,\ldots,N-1\},\ A \in \mathcal{F}_{T_{i+1}}$$

Proof. Let E_ρ and $E_{T_{i+1}}$ denote the expectation operators under the measures $\mathbf{P}_\rho$ and $\mathbf{P}_{T_{i+1}}$, respectively.

$$\begin{aligned}
\mathbf{P}_\rho\{A|\mathcal{F}_{T_i}\} &= E_\rho[I_A|\mathcal{F}_{T_i}] \\
&= \frac{E_{T_{i+1}}\left[I_A \cdot \left(\left.\frac{d\mathbf{P}_\rho}{d\mathbf{P}_{T_{i+1}}}\right|_{\mathcal{F}_{T_{i+1}}} \right) \middle| \mathcal{F}_{T_i} \right]}{E_{T_{i+1}}\left[\left(\left.\frac{d\mathbf{P}_\rho}{d\mathbf{P}_{T_{i+1}}}\right|_{\mathcal{F}_{T_{i+1}}} \right) \middle| \mathcal{F}_{T_i} \right]} \\
&= \frac{B(0,T_{i+1})\left(\prod_{j=0}^{i}(1+\delta L(T_j,T_j))\right) E_{T_{i+1}}[I_A|\mathcal{F}_{T_i}]}{B(0,T_{i+1})\left(\prod_{j=0}^{i}(1+\delta L(T_j,T_j))\right)} \\
&= E_{T_{i+1}}[I_A|\mathcal{F}_{T_i}] = \mathbf{P}_{T_{i+1}}\{A|\mathcal{F}_{T_i}\}
\end{aligned}$$

□

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