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On the Pricing and Hedging of Long Dated Zero Coupon Bonds

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Abstract. The pricing and hedging of long dated derivative contracts is a challenging area of research. As a result of utility indifference pricing for general payoffs the growth optimal portfolio turns out to be the appropriate numeraire or benchmark with the real world probability measure as corresponding pricing measure. This concept of real world pricing can be applied for valuing long dated derivatives. An equivalent risk neutral probability measure does not need to exist under this benchmark approach. This paper develops a parsimonious model for a stock index dynamics, which is based on a time transformed squared Bessel process. It uses a diversified world stock index as proxy for the growth optimal portfolio. Surprisingly low prices result for long dated zero coupon bonds that can be replicated using historical data. Such prices and hedges are difficult to explain under the prevailing risk neutral approach.

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1 Introduction

The *growth optimal portfolio* (GOP) was discovered in Kelly (1956) and is the portfolio that maximizes expected logarithmic utility from terminal wealth. It appears in a stream of literature including, for instance, Long (1990), Karatzas & Shreve (1998), Becherer (2001), Platen (2002, 2004), Goll & Kallsen (2003) and Karatzas & Kardaras (2006). Collectively, this literature demonstrates that the GOP plays a unifying role in derivative pricing and portfolio optimization. Under the prevailing *arbitrage pricing theory*, see for instance Ross (1976), Harrison & Kreps (1979), Long (1990), Constatinides (1992), Delbaen & Schachermayer (1994), Rogers (1997), Cochrane (2001) and Duffie (2001), several authors refer for the pricing of assets under the real world probability to the closely related *numeraire portfolio, state price density*, *pricing kernel*, *deflator* or *stochastic discount factor*. In a risk neutral setting the numeraire portfolio equals the GOP, see Bajeux-Besnainou & Portait (1997), Becherer (2001), Platen (2004) and Karatzas & Kardaras (2006). By using the GOP as numeraire and the real world probability as pricing measure, the *real world pricing* concept, see, for instance, Platen (2002) and Platen & Heath (2006), does not require the existence of an equivalent risk neutral probability measure.

The paper establishes via utility indifference pricing, see Davis (1997), the real world pricing concept for nonreplicable payoffs. To apply real world pricing effectively, in practice, it is of great importance that the GOP can be directly observed and its dynamics realistically modeled. This paper will argue that the world stock portfolio can be used as a proxy for the GOP. When discounted, it will be modeled by a time transformed squared Bessel process. By assuming a deterministic time transformation this yields a parsimonious model, the *minimal market model* (MMM), see Platen (2001, 2002), for the world stock index with the long term net growth rate of the discounted GOP as the main parameter. This model does not admit an equivalent risk neutral probability measure. The resulting surprisingly low prices for long dated zero coupon bonds and their demonstrated hedge are difficult to explain under the standard risk neutral approach. This paper demonstrates that the richer class of models that becomes available under the *benchmark approach* is essential for realistic modeling of the long term dynamics of the market.

The paper is structured as follows. Section 2 introduces a financial market model. Section 3 discusses utility maximization, utility indifference pricing and the market portfolio in relation to the GOP. Section 4 derives the minimal market model and hedges long dated zero coupon bonds.

2 Financial Market Model

The modeling of the financial market is based on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, with filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$, satisfying the usual conditions, see Karatzas & Shreve (1991). We consider a market where trading activity is modeled by the market time $\tau = \{\tau_t,\ t \in [0,T]\},\ T \in [0,\infty)$, which is a potentially random nondecreasing adapted process. The market time τ_t is assumed to be such that $\tau_T < \infty$ almost surely. A random market time allows subordination in the sense of Clark (1973).

The trading uncertainty is modeled by standard Wiener processes $W^k = \{W^k_\tau, \tau \in [0,\infty)\},\ k \in \{1, 2, \ldots, d\}$, which evolve in market time. Since market time is permitted to jump randomly one can model event driven trading uncertainties that could generate significant dependencies of jumps in asset prices. It allows also the modeling of Lévy process driven asset price dynamics as suggested, for instance, in Carr et al. (2003). A wide class of random market time processes can be used, however, we do not specify here any further the market time.

The financial market comprises $d+1$ primary security accounts, $d \in \{1, 2, \ldots\}$, that securitize all investable wealth over the finite time horizon $[0,T]$. These include a savings account, which is locally riskless and whose value at market time τ_t is given by $S^0_{\tau_t} = \exp\left\{\int_0^t r_s\, ds\right\}$ for $t \in [0,T]$, where r_t denotes the adapted, almost surely finite short rate at time t. They also include d nonnegative, savings account discounted, risky primary security accounts $\bar{S}^j = \{\bar{S}^j_\tau,\ \tau \in [0, \tau_T]\},\ j \in \{1, 2, \ldots, d\}$. Each of these evolves in market time and contains only units of one type of security, typically shares of stocks, with all proceeds reinvested.

To specify the dynamics of the jth discounted risky primary security account we assume that its value $\bar{S}^j_\tau,\ j \in \{1, 2, \ldots, d\}$, satisfies the stochastic differential equation (SDE)

$$d\bar{S}^j_\tau = \bar{S}^j_\tau \left(p^j_\tau\, d\tau + \sum_{k=1}^{d} b^{j,k}_\tau\, dW^k_\tau \right) \qquad (1)$$

for $\tau \in [0, \tau_T]$ with $S^j_0 > 0$. Here $b^{j,k}_\tau$ denotes the *volatility* of the jth primary security account with respect to the kth Wiener process W^k and p^j_τ the corresponding *risk premium*. The risk premia and volatilities are assumed to form adapted left continuous processes in market time, satisfying the integrability conditions

$$\int_0^{\tau_T} \sum_{j=1}^{d} \sum_{k=1}^{d} \left(b^{j,k}_\tau\right)^2 d\tau < \infty \quad \text{and} \quad \int_0^{\tau_T} \sum_{j=1}^{d} |p^j_\tau|\, d\tau < \infty$$

almost surely. Note that in the given incomplete market the market time, risk premia, short rate and volatilities can be influenced by uncertainties that are not modeled by the Wiener processes $W^1, W^2, \ldots, W^d$.

We call a predictable stochastic process $\boldsymbol{\delta} = \{\boldsymbol{\delta}_\tau = (\delta^0_\tau, \delta^1_\tau, \ldots, \delta^d_\tau)^\top,\ \tau \in [0, \tau_T]\}$ a *strategy* if for all $j \in \{0, 1, \ldots, d\}$ and $\tau \in [0, \tau_T]$ the Itô stochastic integral

$\int_0^\tau \delta_s^j \, d\bar{S}_s^j$ exists. Here δ_τ^j, $j \in \{0, 1, \ldots, d\}$, denotes the number of units of the jth primary security account held at market time $\tau \in [0, \tau_T]$ in the discounted portfolio $\bar{S}_\tau^\delta$. Then $\bar{S}_\tau^\delta = \sum_{j=0}^d \delta_\tau^j \, \bar{S}_\tau^j$ is the value of the corresponding discounted portfolio at the market time τ. A strategy δ and the corresponding $\bar{S}^\delta$ are said to be *self-financing* if

$$d\bar{S}_\tau^\delta = \sum_{j=0}^{d} \delta_\tau^j \, d\bar{S}_\tau^j \tag{2}$$

for $\tau \in [0, \tau_T]$. In what follows we consider only self-financing strategies and portfolios and will, therefore, omit the phrase "self-financing".

To avoid obvious arbitrage we assume that the volatility matrix $\boldsymbol{b}_\tau = [b_\tau^{j,k}]_{j,k=1}^d$ is invertible with inverse matrix $\boldsymbol{b}_\tau^{-1} = [b_\tau^{-1\,j,k}]_{j,k=1}^d$ for all $\tau \in [0, \tau_T]$. We now introduce the *market price of risk* θ_τ^k at the market time τ with respect to W^k, $k \in \{1, 2, \ldots, d\}$, as a component of the vector

$$\boldsymbol{\theta}_\tau = \left(\theta_\tau^1, \theta_\tau^2, \ldots, \theta_\tau^d\right)^\top = \boldsymbol{b}_\tau^{-1} \boldsymbol{p}_\tau \tag{3}$$

for $\tau \in [0, \tau_T]$, with $\boldsymbol{p}_\tau = (p_\tau^1, p_\tau^2, \ldots, p_\tau^d)^\top$ denoting the vector of risk premia.

It will be convenient to characterize a nonzero portfolio in terms of the fractions of its wealth invested in the primary security accounts. The jth fraction is given at market time τ by

$$\pi_{\delta,\tau}^j = \delta_\tau^j \, \frac{\bar{S}_\tau^j}{\bar{S}_\tau^\delta} \tag{4}$$

for $j \in \{0, 1, \ldots, d\}$ and $\tau \in [0, \tau_T]$. Some of these fractions may be negative, but they always sum to one such that $\sum_{j=0}^d \pi_{\delta,\tau}^j = 1$ for all $\tau \in [0, \tau_T]$. In terms of fractions a strictly positive, discounted portfolio satisfies then the SDE

$$d\bar{S}_\tau^\delta = \bar{S}_\tau^\delta \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,\tau}^j \, b_\tau^{j,k} \left(\theta_\tau^k \, d\tau + dW_\tau^k\right) \tag{5}$$

for $\tau \in [0, \tau_T]$.

We will demonstrate that a key to the understanding of the long term market dynamics will be obtained from the study of the *growth optimal portfolio* (GOP), which was introduced by Kelly (1956). One can demonstrate in various mathematical manifestations that the GOP outperforms all other strictly positive portfolios. For instance, in the long run its path outperforms almost surely that of any other strictly positive portfolio, see Platen (2004). Consequently, it represents a natural benchmark for investment management. To identify a GOP let $\bar{S}^\delta$ be a strictly positive discounted portfolio. By the Itô formula we obtain for its logarithm the SDE

$$d\ln(\bar{S}_\tau^\delta) = g_\tau^\delta \, d\tau + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,\tau}^j \, b_\tau^{j,k} \, dW_\tau^k, \tag{6}$$

where the *growth rate* g^δ_τ at market time τ is given by

$$g^\delta_\tau = \sum_{k=1}^d \left(\sum_{j=1}^d \pi^j_{\delta,\tau}\, b^{j,k}_\tau\, \theta^k_\tau - \frac{1}{2} \left(\sum_{j=1}^d \pi^j_{\delta,\tau}\, b^{j,k}_\tau \right)^2 \right) \tag{7}$$

for $\tau \in [0, \tau_T]$. A strictly positive, discounted portfolio $\bar{S}^{\delta_*}$ is called a *discounted GOP* if for all strictly positive, discounted portfolios $\bar{S}^\delta$ the inequality $g^{\delta_*}_\tau \geq g^\delta_\tau$ holds almost surely, for all $\tau \in [0, \tau_T]$.

The first order conditions associated with the maximization of (7) yield the *optimal fractions* $\pi^j_{\delta_*,\tau} = \sum_{k=1}^d \theta^k_\tau\, b^{-1\,j,k}_\tau$ for all $\tau \in [0, \tau_T]$ and $j \in \{1, 2, \ldots, d\}$. Thus, by (5) the SDE for a discounted GOP in market time is

$$d\bar{S}^{\delta_*}_\tau = \bar{S}^{\delta_*}_\tau \sum_{k=1}^d \theta^k_\tau \left(\theta^k_\tau\, d\tau + dW^k_\tau \right) \tag{8}$$

for $\tau \in [0, \tau_T]$. We fix the strictly positive initial value $\bar{S}^{\delta_*}_0 > 0$ and call $\bar{S}^{\delta_*}$ *the* discounted GOP.

Now, we refer to any security expressed in units of the GOP as *benchmarked*. By the Itô formula the benchmarked value

$$\hat{S}^\delta_\tau = \frac{\bar{S}^\delta_\tau}{\bar{S}^{\delta_*}_\tau} \tag{9}$$

of the portfolio with strategy δ satisfies the SDE

$$d\hat{S}^\delta_\tau = \sum_{j=0}^d \delta^j_\tau\, \hat{S}^j_\tau \sum_{k=1}^d \left(b^{j,k}_\tau - \theta^k_\tau \right) dW^k_\tau \tag{10}$$

for $\tau \in [0, \tau_T]$. Since there is no drift term in (10), $\hat{S}^\delta$ is a local martingale and one can prove the following fundamental property, see Platen (2002):

Lemma 2.1 *Any benchmarked nonnegative portfolio is an* $(\underline{\mathcal{A}}, P)$*-supermartingale.*

This supermartingale property allows to preclude the following weak form of arbitrage which resonates the real life constraint of *limited liability*: For any nonnegative benchmarked portfolio $\hat{S}^\delta$ with zero initial capital its supermartingale property yields the relation $0 = \hat{S}^\delta_0 \geq E\left(\hat{S}^\delta_{\tau_T} \,\middle|\, \mathcal{A}_0 \right) \geq 0$, and, therefore, the equality $P(\hat{S}^\delta_{\tau_T} > 0) = 0$. In other words, nonnegative portfolios are absorbed at zero whenever they reach zero. In economic language this can be interpreted as the absence of a weak form of arbitrage, see Loewenstein & Willard (2000) and Platen (2002). Note that the only ingredient for excluding such weak arbitrage is the existence of the GOP, which itself is a consequence of the invertibility of the volatility matrix. We emphasize that an equivalent risk neutral probability measure does not need to exist in the given framework.

We call a portfolio or price process *fair* if when benchmarked forms an $(\underline{\mathcal{A}}, P)$-martingale. The fair price process $U_H = \{U_H(\tau),\, \tau \in [0, \tau_T]\}$ of a replicable nonnegative payoff H with delivery at maturity T satisfies then at time $t \in [0, T]$ the *real world pricing formula*

$$U_H(\tau_t) = S^{\delta_*}_{\tau_t}\, E\left(\frac{H}{S^{\delta_*}_{\tau_T}} \,\Big|\, \mathcal{A}_t \right) < \infty. \tag{11}$$

Here the GOP $S^{\delta_*}_\tau = \bar{S}^{\delta_*}_\tau S^0_\tau$ is used as numeraire, see Long (1990) and Platen (2002), and the real world probability as pricing measure. From an economic point of view the fair portfolio that replicates a given replicable payoff provides the correct price for this claim since its benchmarked value is a martingale and, thus, by Lemma 2.1 the minimal replicating nonnegative portfolio. It is straightforward to show that when an equivalent risk neutral probability measure exists, the real world pricing formula (11) coincides with the standard risk neutral pricing formula, see Platen (2002).

3 Utility Indifference Pricing

In the given incomplete market it is important to have a rationale for the consistent pricing of nonreplicable payoffs. For this purpose we will employ utility indifference pricing and, therefore, consider expected utility maximization. We consider a twice differentiable, strictly increasing and strictly concave utility function $U : [0, \infty) \to \Re$, whose derivative U' is invertible with inverse U'^{-1}, and where $U'(0) = \infty$ and $U'(\infty) = 0$. Let us maximize for the time horizon $T \in (0, \infty)$ the finite expected utility from discounted terminal wealth

$$v^{\tilde{\delta}} = \max_{\bar{S}^\delta \in \bar{\mathcal{V}}^+_x} E\left(U\left(\bar{S}^\delta_{\tau_T}\right) \,\Big|\, \mathcal{A}_0 \right) < \infty. \tag{12}$$

Here the maximum is taken over the set $\bar{\mathcal{V}}^+_x$ of all discounted, strictly positive, fair portfolios $\bar{S}^\delta$ with initial capital $\bar{S}^\delta_0 = x > 0$. We emphasize that the minimal nonnegative hedge portfolio for a nonnegative replicable payoff is the corresponding fair portfolio and it makes not much sense to consider other portfolios than fair portfolios.

We prove in Appendix A the following result.

Theorem 3.1 *If the benchmarked savings account* $\hat{S}^0 = \{\hat{S}^0_\tau = \frac{1}{\bar{S}^{\delta_*}_\tau},\, \tau \in [0, \tau_T]\}$ *is a scalar diffusion process in market time, and*

$$\hat{u}(\tau_t, \hat{S}^0_{\tau_t}) = E\left(U'^{-1}\left(\lambda\, \hat{S}^0_{\tau_T}\right) \hat{S}^0_{\tau_T} \,\Big|\, \mathcal{A}_t \right) < \infty \tag{13}$$

for all $t \in [0,T]$, with λ such that $x = \frac{\hat{u}(0,\hat{S}^0_0)}{\hat{S}^0_0}$, then the discounted portfolio $\bar{S}^{\tilde{\delta}}_{\tau_t} = \bar{S}^{\delta_*}_{\tau_t}\, \hat{u}(\tau_t, \hat{S}^0_{\tau_t})$ maximizes the above expected utility from discounted terminal wealth. This portfolio has the fraction

$$\frac{1}{J^{\tilde{\delta}}_{\tau}} = 1 - \frac{\hat{S}^0_{\tau}}{\hat{u}(\tau, \hat{S}^0_{\tau})} \frac{\partial \hat{u}(\tau, \hat{S}^0_{\tau})}{\partial \hat{S}^0} \tag{14}$$

invested at market time $\tau \in [0, \tau_T]$ in the GOP and holds the remainder of its wealth in the savings account.

Under the alternative stock index model that we will derive below, $\hat{S}^0$ is a scalar diffusion process in market time, as requested by the above theorem. This theorem can be interpreted as a two-fund separation theorem in the spirit of Tobin (1958). The quantity $J^{\tilde{\delta}}_{\tau}$ in (14) plays the role of a risk aversion coefficient in the sense of Pratt (1964).

Under the assumptions of Theorem 3.1 let us consider a market participant who uses the utility function U with time horizon $T \in (0, \infty)$. Let us consider a nonnegative, potentially nonreplicable payoff H that is $\mathcal{A}_T$-measurable with $E(\frac{H}{S^{\delta_*}_T}) < \infty$, delivered at time T. We now determine for the payoff H and an expected utility maximizing market participant a price that is consistent with her or his utility function U. To solve this problem we apply *utility indifference pricing* in the sense of Davis (1997). This will yield the price at which the market participant is indifferent between entering the derivative contract or investing according to the expected utility maximizing strategy $\tilde{\delta}$.

Consider now a contract which delivers the nonnegative payoff H at time T with candidate price V at time $t = 0$. Let the market participant buy a very small positive fraction $\varepsilon > 0$ of the contract at time $t = 0$ for the amount εV and invest $x - \varepsilon V$ units of her or his wealth according to the expected utility maximizing strategy $\tilde{\delta}$ obtained in Theorem 3.1. Similarly as in (12) we then introduce the expected utility function

$$v^{\tilde{\delta}}_{\varepsilon,V} = E\left(U\left(\frac{(x - \varepsilon\, V)\, \bar{S}^{\tilde{\delta}}_{\tau_T}}{x} + \varepsilon \bar{H} \right) \Bigg| \mathcal{A}_0 \right) \tag{15}$$

with discounted payoff $\bar{H} = \frac{H}{S^0_{\tau_T}}$. We call the value V in (15) the *utility indifference price* of the payoff H if

$$\lim_{\varepsilon \to 0} \frac{v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V}}{\varepsilon} = 0 \tag{16}$$

almost surely. Based on this definition we derive in Appendix B the following result:

Theorem 3.2 *Under the assumptions of Theorem 3.1 the utility indifference price* $U_H(0)$ *at time* $t = 0$ *of a nonnegative, potentially nonreplicable, nonnegative payoff* H *that is delivered at time* T*, satisfies the real world pricing formula* (11) *provided that for all* $\mathcal{A}_T$*-measurable* $\kappa \in (0,1)$ *and all* $\tilde{V}$ *in a neighborhood of the value* $S_0^{\delta_*} E(\frac{H}{S_{\tau_T}^{\delta_*}} \mid \mathcal{A}_0) < \infty$ *the expectation*

$$\left| E\left(U''\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_{\tau_t}^{\delta_*}}\right)\left(1-\kappa\,\frac{\tilde{V}}{x}\right)+\kappa\,\bar{H}\right)\left(\bar{H}-\frac{\tilde{V}}{x}\,U'^{-1}\left(\frac{\lambda}{\bar{S}_{\tau_T}^{\delta_*}}\right)\right)^2 \,\middle|\, \mathcal{A}_0\right)\right| \leq K \tag{17}$$

is uniformly bounded by some constant $K < \infty$.

This theorem makes the important statement that the utility indifference price of a nonreplicable payoff does not depend on the utility function and forms a fair price process.

The technical condition (17) is not restrictive. It is satisfied for a wide range of utility functions and payoffs under the stock index model that we will derive below.

4 An Stock Index Model

Above we have shown that the GOP plays a crucial role as benchmark in investment management and as numeraire in derivative pricing. Therefore, it is of major interest to identify this portfolio in the market and to model its dynamics.

In Platen (2005) a diversification theorem is derived which is, in principle, model independent and only requires some regularity property of the market. It states that any portfolio, where all fractions get smaller with increasing number d of primary securities, is a proxy for the GOP. Consequently, if one assumes that the *market portfolio* (MP) of investable stocks is such a diversified portfolio, then one deals with a proxy of the GOP.

Furthermore, it has been shown in Platen (2006), that Sharpe ratio maximization leads to two fund separation into the GOP and the savings account. Such type of two fund separation is also obtained under expected utility maximization in Theorem 3.1. Now, let us discuss the situation when each market participant has her or his investable wealth at all times invested with some fraction in the GOP and the remainder in the savings account. The discounted value $\bar{S}_\tau^{\delta_{\mathrm{MP}}}$ of the MP satisfies in this case an SDE of the type

$$d\bar{S}_\tau^{\delta_{\mathrm{MP}}} = \bar{S}_\tau^{\delta_{\mathrm{MP}}} \left(J_\tau^{\delta_{\mathrm{MP}}}\right)^{-1} |\theta_\tau| \left(|\theta_\tau|\, d\tau + dW_\tau\right), \tag{18}$$

for $\tau \in [0, \tau_T]$, with total market price of risk $|\theta_\tau| = \sqrt{\sum_{k=1}^d |\theta_\tau^k|^2}$ and the stochastic differential $dW_\tau = \frac{1}{|\theta_\tau|} \sum_{k=1}^d \theta_\tau^k \, dW_\tau^k$ of the Wiener process W, see Platen &

Heath (2006). Obviously, if the MP has all wealth invested in the GOP, then the MP equals the GOP, which corresponds to an ideal risk aversion coefficient of $J_\tau^{\delta_{\mathrm{MP}}} = 1$. This situation is consistent with the above mentioned diversification theorem, where the fractions in all primary security accounts, including the savings account, are becoming smaller with increasing number of stocks in the market. For simplicity, we assume from now on a constant risk aversion coefficient $J_\tau^{\delta_{\mathrm{MP}}} = J^{\delta_{\mathrm{MP}}} > \frac{1}{2}$.

We plot in Figure 1 the logarithm of a reconstructed discounted MP for the

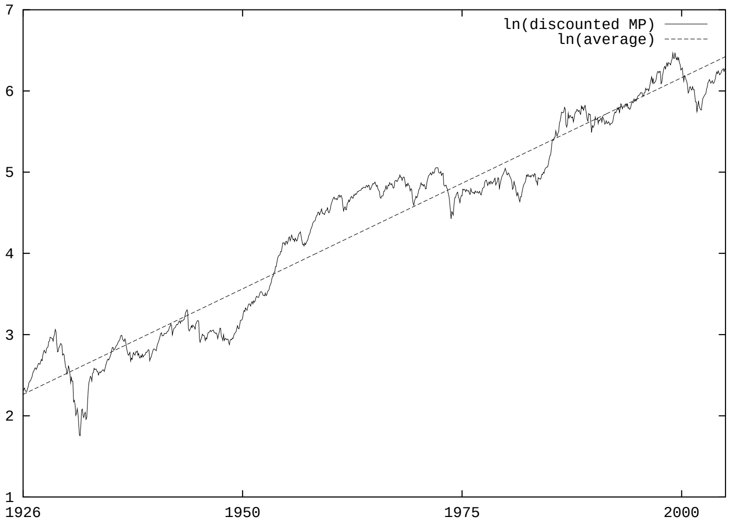

Figure 1: Logarithm of the discounted market portfolio.

world stock market for the period from January 1926 until March 2006, based on monthly data, provided by Global Financial Data. This discounted MP is denominated in units of the US dollar savings account and can be interpreted as a world stock index. For simplicity, we set in the remainder the market time equal to calendar time, that is $\tau_t = t$ for $t \in [0, T]$. One notes in Figure 1 that the logarithm of the discounted MP fluctuates around a linearly regressed line, which increases with some net growth rate η with respect to calendar time. In Figure 1 we estimate $\eta = 0.052$. In the following we aim to capture the dynamics of $\bar{S}^{\delta_{\mathrm{MP}}}$ in a parsimonious stock index model.

On the basis of the identified average long term exponential growth in Figure 1 we parameterize the SDE (18) of the discounted MP by its drift in terms of the simple exponential drift function

$$\alpha_t^{\delta_{\mathrm{MP}}} = \bar{S}_t^{\delta_{\mathrm{MP}}} \left(J^{\delta_{\mathrm{MP}}}\right)^{-1} |\theta_t|^2 = \alpha \exp\{\eta \, \tau\} \tag{19}$$

$t \in [0, T]$, with an initial scale parameter $\alpha > 0$. This parametrization of the SDE (18) is a departure from the usual parametrization in asset price modeling where the volatility

$$\frac{|\theta_t|}{J^{\delta_{\mathrm{MP}}}} = \sqrt{\frac{\alpha_t^{\delta_{\mathrm{MP}}}}{\bar{S}_t^{\delta_{\mathrm{MP}}} \, J^{\delta_{\mathrm{MP}}}}} \tag{20}$$

is typically taken as key parameter process. By substituting (19) and (20) into the SDE (18) we obtain

$$d\bar{S}_t^{\delta_{\mathrm{MP}}} = \alpha_t^{\delta_{\mathrm{MP}}}\, dt + \sqrt{\frac{\bar{S}_t^{\delta_{\mathrm{MP}}}\, \alpha_t^{\delta_{\mathrm{MP}}}}{J^{\delta_{\mathrm{MP}}}}}\, dW_t \tag{21}$$

for all $t \in [0, T]$. It is evident from (21) that the discounted MP is a time transformed squared Bessel process of dimension $4J^{\delta_{\mathrm{MP}}}$, see Revuz & Yor (1999). This observation is practically very useful, since much is known about the distributional properties of squared Bessel processes.

To simplify our analysis even further we choose the ideal risk aversion $J^{\delta_{\mathrm{MP}}} = 1$. For identifying the initial scale parameter α from historical data let us analyze $\sqrt{\bar{S}_t^{\delta_*}}$. By the Itô formula one obtains the SDE

$$d\sqrt{\bar{S}_t^{\delta_{\mathrm{MP}}}} = \frac{3\,\alpha_t^{\delta_{\mathrm{MP}}}}{8\sqrt{\bar{S}_t^{\delta_{\mathrm{MP}}}}}\, dt + \frac{1}{2}\sqrt{\alpha_t^{\delta_{\mathrm{MP}}}}\, dW_t.$$

Therefore, the quadratic variation of $\sqrt{\bar{S}_t^{\delta_{\mathrm{MP}}}}$ yields the expression

$$\left[\sqrt{\bar{S}^{\delta_{\mathrm{MP}}}}\right]_t = \frac{1}{4}\int_0^t \alpha_s^{\delta_{\mathrm{MP}}}\, ds = \frac{1}{4}\,\frac{\alpha}{\eta}\,(\exp\{\eta\, t\} - 1).$$

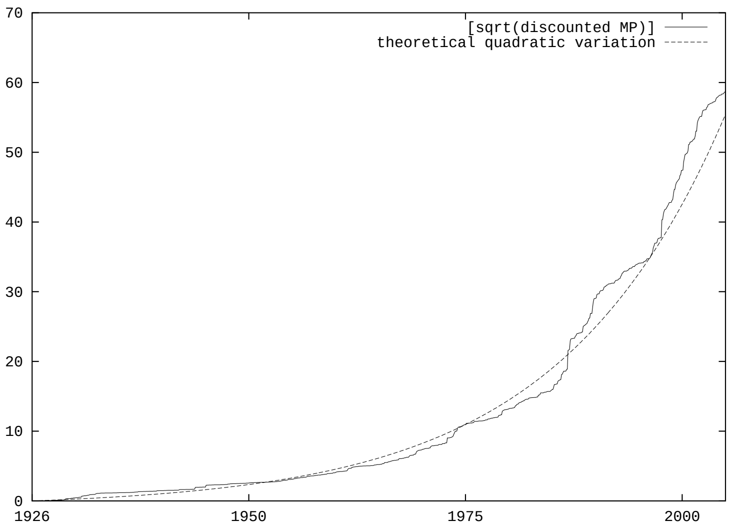

Figure 2: Observed quadratic variation $[\sqrt{\bar{S}^{\delta_{\mathrm{MP}}}}]_t$ and its theoretical value.

We plot in Figure 2 the observed quadratic variation and its theoretical value when setting $\alpha = 0.183$ and $\eta = 0.052$, which provides a good fit.

We refer to the above model as the *minimal market model* (MMM), see Platen (2001). The key parameters of this parsimonious model are the net growth rate $\eta > 0$ and the initial scale parameter $\alpha > 0$ if we consider the short rate as given.

The structure of this stock index model captures several stylized empirical features of stock market indices. For instance, with its negatively correlated volatility (20) it explains the well-documented leverage effect, see Black (1976), but also the observed Student t distributed log-returns of the MP with degrees of freedom four, see Fergusson & Platen (2006). Thus, we obtained a model for a stock market index with two easily estimated parameters.

Recall that $\hat{S}^0_t$ denotes the benchmarked savings account. When interpreting $S^{\delta_{\rm MP}}$ as the GOP in a complete market, the Radon-Nikodym derivative for the candidate risk neutral measure equals $\Lambda_t = \frac{\hat{S}^0_t}{\hat{S}^0_0}$ for $t \in [0,T]$. We show Λ_t in Figure 3, assuming that the discounted MP equals the discounted GOP. The

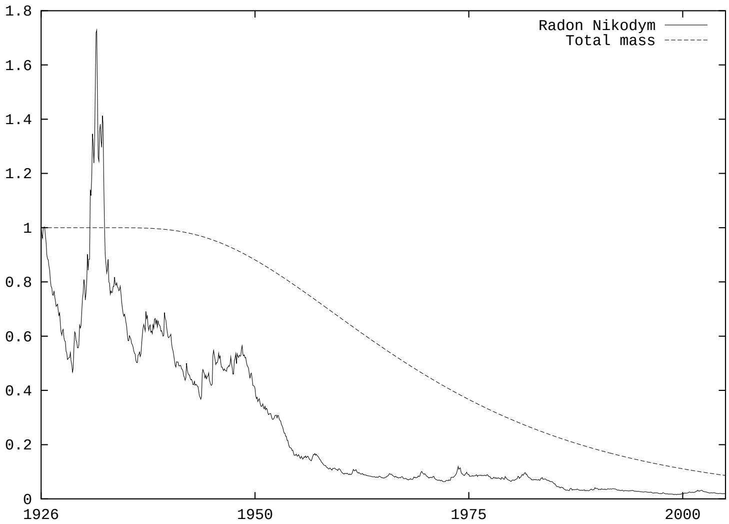

Figure 3: Radon-Nikodym derivative and total mass of the candidate risk neutral measure.

observed systematic average decline is due to the long term outperformance of the savings account by the world stock index, see also Dimson, Marsh & Staunton (2002).

Since under the stylized MMM $\Lambda = \{\Lambda_t,\, t \in [0,T]\}$ is the inverse of a time transformed squared Bessel process of dimension four, it follows from Revuz & Yor (1999) that Λ is an $(\underline{\mathcal{A}}, P)$-strict supermartingale. This is consistent with the observation of a systematically declining trajectory in Figure 3. The candidate risk neutral measure is under the MMM *not* equivalent to the real world probability measure P. Using the above estimated parameters we show in Figure 3 also the total mass

$$E\left(\Lambda_t \,\middle|\, \mathcal{A}_0\right) = 1 - \exp\left\{ -\frac{2\,\eta\,\bar{S}^{\delta_*}_0}{\alpha\,(\exp\{\eta\,t\} - 1)} \right\} \tag{22}$$

of the candidate risk neutral measure as a function of time $t \in [0,T]$. The explicit expression in (22) is obtained by integrating against the analytically available transition density of the squared Bessel process of dimension four, see Platen (2002).

It can be demonstrated that the MMM implies the existence of "free lunches with vanishing risk", in the sense of Delbaen & Schachermayer (1994, 1998). The situation is not desperate, however, since the MMM does not allow the weak form of arbitrage discussed earlier, see Platen (2002). A non-replicable payoff can be consistently priced via the real world pricing formula (11) when using Theorem 3.2.

Pricing and Hedging of Long Dated Zero Coupon Bonds

The pricing and hedging of long dated derivative contracts has been a challenging problem in finance and insurance. Since we derived a model for the dynamics of the MP we can now study its applicability for pricing and hedging long dated contracts on such a stock index.

In the case of a nonnegative payoff H with maturity date T we obtain its price according to Theorem 3.2 by the real world pricing formula (11). If H is independent of the value $S_T^{\delta_*}$ of the GOP at maturity T, then the *actuarial pricing formula*

$$P_H(t,T) = S_t^{\delta_*}\, E\left(\frac{H}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t\right) = P_1(t,T)\, E\left(H \,\big|\, \mathcal{A}_t\right) \tag{23}$$

follows directly from (11) for $t \in [0,T]$. Here $P_1(t,T)$ denotes the zero coupon bond price at time t with payoff $H = 1$ at maturity T. We emphasize that the actuarial pricing formula (23) is of central importance in insurance. Its key ingredients are the expectation of the independent payoff H under the real world probability and the zero coupon bond price.

Now, using (22) we obtain under the above stylized MMM with deterministic short rate for the zero coupon bond price the explicit formula

$$P_1(t,T) \;=\; \exp\left\{-\int_t^T r_s\, ds\right\}\left(1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_{\tau_t}^{\delta_*}}{\alpha\,(\exp\{\eta\, T\} - \exp\{\eta\, t\})}\right\}\right) \tag{24}$$

for $t \in [0,T]$. For illustration, let us interpret the historically observed US short rate as being deterministic. This allows us to plot in Figure 4 the evolution of the price of the zero coupon bond at time t which matures at T in March 2006 using the stylized MMM with the above estimated parameters. In the same figure we display also the *savings bond* $P^*(t,T) = \exp\left\{-\int_t^T r_s\, ds\right\}$ as a function of $t \in [0,T]$. The zero coupon bond has an initial value of $P_1(0,T) = 0.0042$ which represents only about 8.5% of the initial savings bond value $P^*(0,T) = 0.0495$. The key property of the benchmarked zero coupon bond is that it is a martingale, whereas the benchmarked savings bond is a strict supermartingale. Since both replicate the same payoff at time T, the martingale determines the lowest price for the payoff $H = 1$, which forms a fair price process. Note that under the benchmark approach different self-financing portfolios can replicate the same payoff. Only for fair portfolios one has the law of one price.

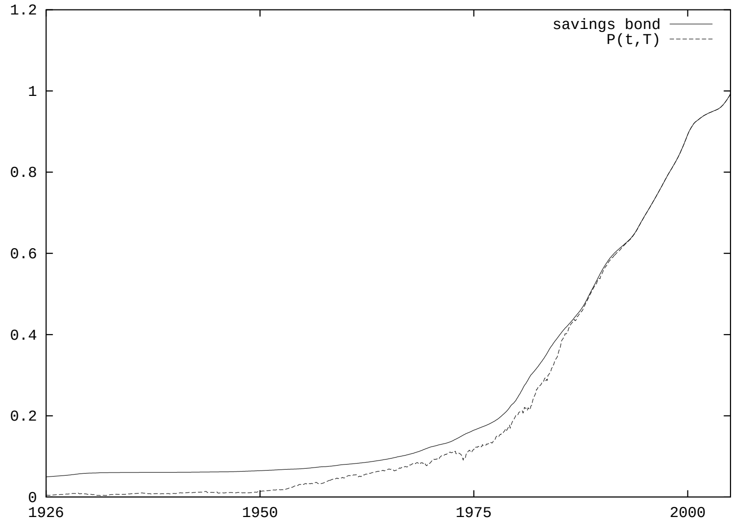

Figure 4: Zero coupon bond and savings bond.

Theoretically, the payoff $H = 1$ can be replicated at time T by investing at time $t \in [0, T]$

$$\begin{aligned}\delta_*(t) &= \frac{\partial P_1(t,T)}{\partial S_t^{\delta_*}} \\ &= P^*(t,T)\,\exp\left\{\frac{-2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}\frac{2\,\eta}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})} \qquad (25)\end{aligned}$$

units in the MP and the remainder in the savings account with continuous reallocation of the wealth.

For illustration we perform a hedge simulation, where we calculate the self-financing portfolio that starts at the theoretical initial zero coupon bond price $P_1(0,T)$ and keep during each following month the number of units invested as determined by (25). It turns out that the hedge portfolio almost perfectly replicates with monthly rehedging the payoff of the zero coupon bond at maturity T. We plot in Figure 5 the resulting benchmarked profit and loss of the hedge portfolio, which remains very close to zero.

The resulting zero coupon bond has an initial value, which is far less than that of the savings bond. This phenomenon is difficult to explain under the risk neutral approach even if one considers stochastic interest rates. However, under the benchmark approach it can be easily explained as a consequence of the strict supermartingale property of the benchmarked savings account. These findings raise serious concerns about the use of risk neutral pricing and hedging for long dated derivative contracts in finance and insurance. Forthcoming work will demonstrate for long dated index and currency derivatives similar explanatory power of the above stylized MMM.

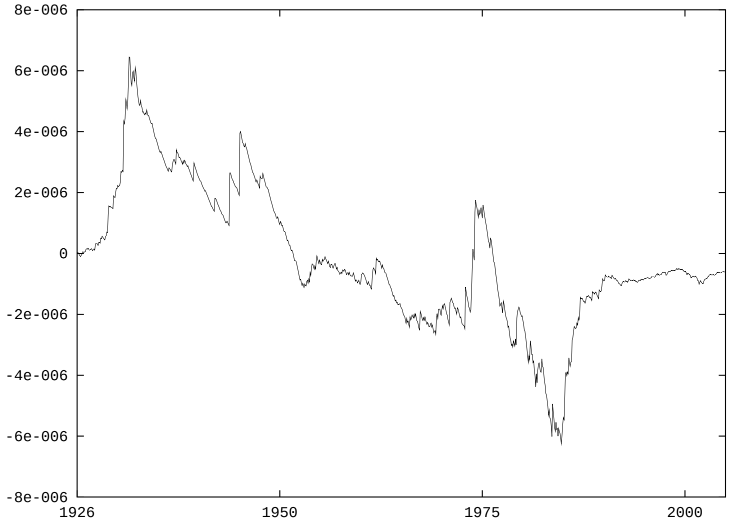

Figure 5: Benchmarked profit and loss.

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Appendix A: Proof of Theorem 3.1

We express the constrained optimization problem (12), which maximizes over discounted, fair portfolios, by introducing a Lagrange multiplier $\lambda \in (0,\infty)$ in the functional

$$v^\delta = E\left(U\left(\bar S^\delta_{\tau_T}\right)\,\Big|\,\mathcal{A}_0\right) - \lambda\left(E\left(\frac{\bar S^\delta_{\tau_T}}{\bar S^{\delta_*}_{\tau_T}}\,\Big|\,\mathcal{A}_0\right) - \frac{x}{\bar S^{\delta_*}_0}\right) = E\left(F\left(\bar S^\delta_{\tau_T}\right)\,\Big|\,\mathcal{A}_0\right) \quad (26)$$

that has to be maximized for $\bar S^\delta \in \bar{\mathcal{V}}^+_x$. When maximizing the function $F(\bar S^\delta_{\tau_T}) = U(\bar S^\delta_{\tau_T}) - \lambda\left(\frac{\bar S^\delta_{\tau_T}}{\bar S^{\delta_*}_{\tau_T}} - \frac{x}{\bar S^{\delta_*}_0}\right)$ with respect to $\bar S^\delta_{\tau_T}$ one obtains the first order condition $U'(\bar S^\delta_{\tau_T}) - \frac{\lambda}{\bar S^{\delta_*}_{\tau_T}} = 0$. Since U is concave, $U'(0) = \infty$ and $U'(\infty) = 0$, this characterizes a maximum. By applying the inverse function U'^{-1} of U' on both sides of the first order condition it follows for the candidate optimal portfolio that $\bar S^{\tilde\delta}_{\tau_T} = U'^{-1}\left(\frac{\lambda}{\bar S^{\delta_*}_{\tau_T}}\right)$. Since $S^{\tilde\delta}$ is assumed to be a fair portfolio one needs to choose $\lambda \in (0,\infty)$ such that

$$\frac{x}{\bar S^{\delta_*}_0} = \frac{\bar S^{\tilde\delta}_0}{\bar S^{\delta_*}_0} = E\left(\frac{\bar S^{\tilde\delta}_{\tau_T}}{\bar S^{\delta_*}_{\tau_T}}\,\Big|\,\mathcal{A}_0\right) = E\left(U'^{-1}\left(\frac{\lambda}{\bar S^{\delta_*}_{\tau_T}}\right)\frac{1}{\bar S^{\delta_*}_{\tau_T}}\,\Big|\,\mathcal{A}_0\right). \quad (27)$$

Since $\hat{S}^0_\tau = (\bar{S}^{\delta_*}_\tau)^{-1}$ is a scalar diffusion process in market time τ there exists by the Feynman-Kac formula a function $\hat{u}(\cdot,\cdot)$ such that

$$\hat{S}^{\tilde{\delta}}_{\tau_t} = \hat{u}(\tau_t, \hat{S}^0_{\tau_t}) = E\left(U'^{-1}\left(\frac{\lambda}{\bar{S}^{\delta_*}_{\tau_T}}\right)\frac{1}{\bar{S}^{\delta_*}_{\tau_T}}\,\Big|\,\mathcal{A}_t\right) = E\left(U'^{-1}\left(\lambda\,\hat{S}^0_{\tau_T}\right)\hat{S}^0_{\tau_T}\,\Big|\,\mathcal{A}_t\right) \quad (28)$$

forms an $(\underline{\mathcal{A}}, P)$-martingale for $t \in [0,T]$. It is straightforward to verify that the resulting strategy $\tilde{\delta}$, with the fraction (14) invested in the GOP, maximizes the expected utility. □

Appendix B: Proof of Theorem 3.2

The following expected utility difference can be derived from (15) by the Taylor expansion in the form

$$\frac{1}{\varepsilon}\left|v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V}\right| \leq \frac{1}{\varepsilon}\left|E\left(U\left(\bar{S}^{\tilde{\delta}}_{\tau_T}\right) + U'\left(\bar{S}^{\tilde{\delta}}_{\tau_T}\right)\varepsilon\left(\bar{H} - V\frac{\bar{S}^{\tilde{\delta}}_{\tau_T}}{x}\right)\Big|\,\mathcal{A}_0\right) - v^{\tilde{\delta}}_{0,V}\right| + \frac{\varepsilon}{2}K, \quad (29)$$

see (17). We have from (13) $\bar{S}^{\tilde{\delta}}_{\tau_T} = U'^{-1}\left(\frac{\lambda}{\bar{S}^{\delta_*}_{\tau_T}}\right)$, which yields due to (29) and (17) for the utility indifference price V the relation

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V}\right) = E\left(\frac{\lambda}{\bar{S}^{\delta_*}_{\tau_T}}\left(\bar{H} - V\frac{\bar{S}^{\tilde{\delta}}_{\tau_T}}{x}\right)\Big|\,\mathcal{A}_0\right) \\ &= \lambda\left(E\left(\frac{H}{S^{\delta_*}_{\tau_t}}\,\Big|\,\mathcal{A}_0\right) - \frac{V}{x}E\left(\hat{S}^{\tilde{\delta}}_{\tau_T}\,\big|\,\mathcal{A}_0\right)\right). \end{aligned} \quad (30)$$

This provides for the payoff H the real world pricing formula (11) when exploiting the martingale property of $\hat{S}^{\tilde{\delta}}$, where $E\left(\hat{S}^{\tilde{\delta}}_{\tau_T}\,\big|\,\mathcal{A}_0\right) = \hat{S}^{\tilde{\delta}}_0 = \frac{x}{\bar{S}^{\delta_*}_0}$. □

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