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### Sharpe Ratio Maximization and Expected Utility when Asset Prices have Jumps

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# Sharpe Ratio Maximization and Expected Utility when Asset Prices have Jumps\*

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## Abstract

We analyze portfolio strategies which are locally optimal, meaning that they maximize the Sharpe ratio in a general continuous time jump-diffusion framework. These portfolios are characterized explicitly and compared to utility based strategies. In the presence of jumps, maximizing the Sharpe ratio is shown to be generally inconsistent with maximizing expected utility, but this is shown to depend strongly on market completeness and whether event risk is priced.

## 1 Introduction

Perhaps the most widely used tool for investment analysis is the so-called *Sharpe ratio*, see e.g. Sharpe (1966, 1975, 1994), derived from the mean-variance based portfolio selection theory originating with Markowitz (1952). In the discrete time case Sharpe ratio based investment and performance measurement have been analyzed extensively and its strengths and weaknesses are reasonably well understood. In particular, maximizing Sharpe ratios in this case has been linked theoretically to a narrow class of return distributions or quadratic utility. Even though the theoretical justification is restrictive, the popularity of this investment performance measure cannot be overlooked. Recently, the continuous time version of this measure, has been studied by, for instance, Nielsen & Vassalou (2002, 2004), Ziemba & Zhao (2003) and Platen (2004). In the case of continuous asset prices it has been shown that Sharpe ratio based investments will in some cases be the choice of all investors maximizing expected utility, see Merton (1971) or the recent more general results by Khanna & Kulldorff (1999) and Nielsen & Vassalou (2002). Most treatments are based on the

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assumption of continuous asset prices. In contrast, empirical studies strongly suggest that jumps are a part of asset price behavior. It is unclear what models with jumps need to satisfy, such that maximizing expected utility coincides with the maximization of Sharpe ratios. Few papers seem to have investigated this issue. Since Sharpe ratio maximization is a generally accepted investment practice, understanding such behavior in a more general framework is an important task. Moreover, recent papers on pricing in incomplete markets based on so called “good deals”, see Cochrane & Saá-Requejo (2000) and Björk & Slinko (2005) assume that investors seek investments with high Sharpe ratios.

We define locally optimal strategies as those which maximize the continuous time analogue of the Sharpe ratio. The analysis is cast in a general incomplete market framework involving jump and diffusion risk, modeled by a multi-dimensional Wiener process and a marked point process. We characterize locally optimal portfolios with and without constraints on the portfolio choice, in particular, in the absence of a locally risk free asset and show that Sharpe ratio based investments always lead to two fund separation, similar to the classical case of mean variance efficient portfolio choice, see Tobin (1958) and Sharpe (1966). Despite the generality of our setting, the results are tractable and derived by using relatively basic vector space techniques.

It is shown that adding jumps to the evolution of asset prices has important consequences. Our specific goal is to determine whether expected utility maximizing investors will maximize the Sharpe ratio of their investments. Since investors may “coincidentally” maximize the Sharpe ratio, we try to determine whether they do so “on purpose” or because they are “forced” to. To see this, we study the choice in a complete, or almost complete model. In complete markets, investors are always capable of reaching their preferred allocation. Often, it may require an infinite number of assets to obtain completeness, because intuitively, completeness requires one asset for each jump size. In this case, approximate completeness is the natural extension, and we extend our results to this case as well. The main result is that if both event risk and diffusion risk are priced and markets are complete or approximately complete, then utility maximizing investors will not choose a locally optimal portfolio, at least not investors with power or logarithmic utility. Although completeness is not common in jump-diffusion models, it is important here, because without completeness investors may in fact maximize the Sharpe ratio “by chance” as will be exemplified. We conclude that there is a significant difference between complete and incomplete markets whenever there is jump and diffusion risk present. We argue that the reason for this effect is the difference in the pricing of event risk and diffusion risk.

## 2 The Jump Diffusion Market Model

### 2.1 The Modeling of Uncertainty

Let there be given a complete filtered probability space,  $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$ , with filtration  $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t \geq 0}$  satisfying the usual conditions, see Protter (2004). The fundamental building block for event driven uncertainty is a marked point process,  $p(\cdot, \cdot)$ , see Jacod & Shiryaev

(1987) or Bremaud (1981). Continuous uncertainty is modelled by an  $m$ -dimensional Wiener process,  $W = \{W(t) = (W^1(t), \dots, W^m(t))^\top, t \in [0, \infty)\}$ , independent of the measure  $p$ . The marked point process is represented by the jump measure  $p(dv, dt)$ , and the corresponding compensated measure is denoted  $q(dv, dt)$ . This compensated measure is a martingale random measure with respect to the filtration  $\underline{\mathcal{F}}$  and the probability measure  $P$ . For simplicity, the mark space  $\mathcal{E}$  is assumed to be some measurable subset of  $\mathbb{R}$  and is equipped with the usual Borel sigma-algebra,  $\mathbb{B}^{\mathcal{E}}$ . In most cases  $\mathcal{E}$  will be either an interval or some finite subset of  $\mathbb{R}$ . It is assumed that the compensated measure,  $q(\cdot, \cdot)$ , admits a time-varying non-negative intensity measure, denoted by  $\phi(\cdot, t)$  at time  $t$  such that  $q(dv, dt) = p(dv, dt) - \phi(dv, t)dt$ . Moreover, it is assumed that  $\phi(dv, t)$  is a finite measure for all  $t \geq 0$ . This means that the total arrival intensity is finite.

## 2.2 The Market

We define a set of primary security accounts as non-negative stochastic processes on the given probability space. The first account is a *savings account* and is assumed to be locally risk free, which means that it is of finite variation and the solution to the differential equation

$$dS^{(0)}(t) = S^{(0)}(t)r(t)dt \quad (1)$$

for  $t \in [0, \infty)$  with  $S^{(0)}(0) = 1$ . It is assumed that the interest rate process  $r = \{r(t), t \in [0, \infty)\}$  is  $\underline{\mathcal{F}}$ -adapted.

The remaining primary security accounts are risky and assumed to be given as the solution to the stochastic differential equation (SDE)

$$dS^{(i)}(t) = S^{(i)}(t-) \left( a^i(t)dt + \sigma^i(t) \cdot dW(t) + \int_{\mathcal{E}} b^i(v, t)q(dv, dt) \right) \quad (2)$$

for  $t \in [0, \infty)$  with  $S^{(i)}(0) > 0$  for  $i \in \{1, 2, \dots, N\}$ . Here  $x \cdot y$  denotes the standard Euclidean inner product in  $\mathbb{R}^m$ . In the SDE (2) it is assumed that the integrands  $a^i, \sigma^{i,j}$  and  $b^i$  are all predictable stochastic processes such that a unique strong solution of the SDE exists, see Protter (2004). In terms of integrability, it is required that for any time  $T > 0$

$$\int_0^T (|r(s)| + |a^i(s)|) ds < \infty \quad (3)$$

and

$$\int_0^T \left( \sigma^{i,j}(s)^2 + \int_{\mathcal{E}} b^i(v, s)^2 \phi(dv, s) \right) ds < \infty \quad (4)$$

almost surely for all  $i \in \{1, 2, \dots, N\}$  and  $j \in \{1, 2, \dots, m\}$ . The assumptions of square integrable volatilities and jump sizes is imposed here, to facilitate the optimization in the following sections. To ensure non-negativity of the primary security accounts, it is assumed that  $b^i(v, t) \geq -1$  for almost every  $(t, v) \in [0, \infty) \times \mathcal{E}$  and all  $i \in \{1, 2, \dots, N\}$ . For simplicity, it can be assumed that  $(\sigma^i(t), b^i(v, t))$ ,  $i \in \{1, 2, \dots, N\}$ , are linearly independent

functions on  $\mathbb{R}^m \times L^2(\phi(dv, t))$  almost surely for all  $t \in [0, \infty)$ . This merely avoids redundant assets and the assumption can be made without loss of generality. Define for each  $N \in \mathbb{N}$  the  $N$ th market

$$\mathbb{S}^N \triangleq \{\mathbb{S}^N(t) = (S^{(0)}(t), S^{(1)}(t), \dots, S^{(N)}(t))^\top, t \in [0, \infty)\}$$

consisting of the first  $N + 1$  primary security accounts. In most parts of this paper,  $N$  will be a fixed number. An exception is Section 4 where the asymptotic properties of the model are considered. Consequently, for a given fixed  $N$  we will use the notation

$$a(t) \triangleq (a^1(t), \dots, a^N(t))^\top, b(t, v) \triangleq (b^1(t, v), \dots, b^N(t, v))^\top, \sigma^i(t) = (\sigma^{i,1}(t), \dots, \sigma^{i,m}(t))^\top$$

and

$$\sigma(t) = (\sigma^1(t), \dots, \sigma^N(t)).$$

### 2.3 Strategies

A strategy,  $\delta$ , in the market  $\mathbb{S}^N$  is defined as a predictable vector process,  $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \dots, \delta^{(N)}(t))^\top, t \in [0, \infty)\}$ , in  $\mathbb{R}^{N+1}$  such that the stochastic integral

$$\int_0^T \delta^{(i)}(t) dS^{(i)}(t)$$

is well-defined for any  $T > 0$  and  $i \in \{0, 1, \dots, N\}$ . The portfolio process,  $S^{(\delta)} = \{S^{(\delta)}(t), t \in [0, \infty)\}$ , is then given as

$$S^{(\delta)}(t) = \sum_{i=0}^N \delta^{(i)}(t) S^{(i)}(t).$$

A strategy,  $\delta$ , in the market  $\mathbb{S}^N$  is called *self-financing* if

$$S^{(\delta)}(t) = S^{(\delta)}(0) + \sum_{i=0}^N \int_0^t \delta^{(i)}(s) dS^{(i)}(s) \quad (5)$$

for all  $t \in [0, \infty)$ . Let  $\underline{\Theta}(\mathbb{S}^N)$  denote the set of non-negative, self-financing portfolios in  $\mathbb{S}^N$ . A portfolio is called *admissible* if it belongs to  $\underline{\Theta}(\mathbb{S}^N)$  for some  $N > 0$ . Unless otherwise stated, portfolios are assumed to be admissible. The requirement that portfolio values should remain non-negative reflects the real life constraint of limited liability for investors. A similar constraint is used in Bielecki, Jin, Pliska & Zhou (2005), which considers a different continuous time mean-variance analysis.

For a strictly positive portfolio,  $S^{(\delta)} \in \underline{\Theta}(\mathbb{S}^N)$ , it is possible to define the corresponding vector of fractions or portfolio weights,

$$\pi_\delta = \{\pi_\delta(t) = (\pi_\delta^1(t), \pi_\delta^2(t), \dots, \pi_\delta^N(t))^\top, t \in [0, \infty)\}$$

with

$$\pi_{\delta}^{(i)}(t) \triangleq \frac{\delta^{(i)}(t)S^{(i)}(t)}{S^{(\delta)}(t)} \quad (6)$$

for  $i \in \{1, \dots, N\}$  and where  $\pi_{\delta}^0(t) \triangleq 1 - \pi_{\delta}(t)^{\top} \mathbf{1}$  is the residual fraction invested in the savings account. We follow the convention of letting  $\mathbf{1}$  denote the  $N$ -dimensional unit vector  $(1, \dots, 1)^{\top}$ . Whenever the meaning is clear from the context, we will write  $\pi$  instead of  $\pi_{\delta}$ . Consequently, the SDE of an admissible portfolio  $S^{(\delta)}$  is given by

$$\begin{aligned} dS^{(\delta)}(t) &= S^{(\delta)}(t-) \left( \left( \pi^{(0)}(t)r(t) + \sum_{i=0}^N \pi^i(t)a^{(i)}(t) \right) dt + \sum_{j=1}^m \sum_{i=1}^N \pi^i(t)\sigma^{i,j}(t)dW^j(t) \right. \\ &\quad \left. + \sum_{i=1}^N \int_{\mathcal{E}} \pi^i(t)b^i(v,t)q(dv, dt) \right) \\ &= S^{(\delta)}(t-) \left( a^{\delta}(t)dt + \sigma^{\delta}(t) \cdot dW(t) + \int_{\mathcal{E}} b^{\delta}(v,t)q(dv, dt) \right), \end{aligned}$$

where

$$a^{\delta}(t) \triangleq \pi^{(0)}(t)r(t) + \pi(t) \cdot a(t), \quad \sigma^{\delta}(t) \triangleq \pi(t)^{\top} \sigma(t) \quad \text{and} \quad b^{\delta}(v,t) \triangleq \pi(t) \cdot b(v,t).$$

For a model to be realistic, it is necessary to ensure that some fundamental form of arbitrage is prohibited. There are different notions of arbitrage in the literature and the one used here is stated formally below.

**Definition 2.1** *An arbitrage is an admissible portfolio  $S^{(\delta)}$ , such that  $S^{(\delta)}(0) = 0$  and  $P(S^{(\delta)}(T) > 0) > 0$  for some  $T > 0$ .*

**Assumption 2.2** *There is no arbitrage in the sense of Definition 2.1.*

Without this assumption, portfolio optimization will lose all meaning, since investors can obtain infinite wealth with no risk and no initial investment. Moreover, the weak requirement of no arbitrage is sufficient to define a market price of risk as done below.

### 3 Local Optimality

For any portfolio,  $S^{(\delta)}$ , define the *local risk premium*,  $p^{\delta}(t)$ , as

$$p^{\delta}(t) \triangleq a^{\delta}(t) - r(t). \quad (7)$$

Recall from Christensen & Platen (2005a), that a *risk premium functional* is a continuous linear functional,

$$\Gamma : \mathbb{R}^m \times L^2(\phi(dv, t)) \rightarrow \mathbb{R}$$

from the space of generalized volatilities to the real numbers, linking uncertainty to the risk premium, that is,

$$p^\delta(t) = \Gamma(\sigma^\delta(t), b^\delta(v, t)). \quad (8)$$

From Assumption 2.2 the existence of a risk premium functional follows, see Christensen & Platen (2005a) or Christensen & Larsen (2004). This is a slight extension of the well-known result that no arbitrage implies the existence of a market price of risk, see for instance Back (1991), Schweizer (1992) and Delbaen & Schachermayer (1995). The risk premium functional is here represented by a vector  $(\theta(t), \psi_\theta(v, t)) \in \mathbb{R}^m \times L^2(\phi(dv, t))$ , such that for any strategy,  $\delta$

$$p^\delta(t) = \sigma^\delta(t) \cdot \theta(t) + \int_{\mathcal{E}} b^\delta(v, t) \psi_\theta(v, t) \phi(dv, t). \quad (9)$$

Any vector,  $(\theta(t), \psi_\theta(v, t))$  satisfying (9), will be termed a *market price of risk representation*. The existence of such a representation is a feature of the Riesz Representation Theorem, see Rudin (1987). Hence (9) is in fact an inner product in the Hilbert space  $\mathbb{R}^m \times L^2(\phi(dv, t))$ , and the risk premium can be written as

$$p^\delta(t) = ((\sigma^\delta(t), b^\delta(v, t)) | (\theta(t), \psi_\theta(v, t))). \quad (10)$$

Here  $(\cdot | \cdot)$  denotes the inner product of two vectors. Obviously, if  $(\theta(t), \psi_\theta(v, t))$  is a market price of risk representation, then for any vector,  $(\theta^\perp(t), \psi_\theta^\perp(v, t))$  orthogonal to the set  $\{(\sigma^i(t), b^i(v, t)) | i \in \{1, \dots, N\}\}$  it holds that  $(\theta^\perp(t), \psi_\theta^\perp(v, t)) + (\theta(t), \psi_\theta(v, t))$  is also a market price of risk representation. The existence of an affine subspace of market price of risk representations is a natural feature of the incompleteness of the market. It will become clear that the particular choice of representation is of no importance. For this reason,  $(\theta(t), \psi_\theta(v, t))$  will just denote *any* market price of risk representation. However, the issue of choosing such a representation will be pursued later. As a matter of integrability, we assume that the representation  $(\theta(t), \psi_\theta(v, t))$  is such that for any  $T > 0$ , one has  $\|\theta(\cdot)\|_{\mathbb{R}^m} \in L^2([0, T])$ .

Let us introduce a measure of local uncertainty,  $V^\delta(t)$ , derived from the conditional quadratic variation, see Protter (2004), where

$$V^\delta(t)^2 = \frac{d(S^{(\delta)})_t}{S^{(\delta)}(t-)^2} = \|\sigma^\delta(t)\|_{\mathbb{R}^m}^2 + \|b^\delta(v, t)\|_{L^2(\phi(dv, t))}^2.$$

To emphasize the vector space structure, the notation

$$V^\delta(t) = \|(\sigma^\delta(t), b^\delta(v, t))\| = (\|\sigma^\delta(t)\|_{\mathbb{R}^m}^2 + \|b^\delta(v, t)\|_{L^2(\phi(dv, t))}^2)^{\frac{1}{2}} \quad (11)$$

will be used, where  $(\sigma^\delta(t), b^\delta(v, t)) \in \mathbb{R}^m \times L^2(\phi(dv, t))$ .

A focus point of this article is the *Sharpe ratio*,  $s^\delta(t)$ , of some portfolio,  $S^{(\delta)}$ , which is defined as

$$s^\delta(t) \triangleq \frac{p^\delta(t)}{V^\delta(t)} \quad (12)$$

for any strategy  $\delta$ , with  $V^\delta(t) \neq 0$ , for  $t \in [0, \infty)$ . This definition provides the continuous time version of the classical Sharpe ratio in the case of jump diffusions.

**Definition 3.1** *A portfolio is said to be locally optimal, if it has a maximal Sharpe ratio. If the maximal Sharpe ratio of all admissible portfolios is zero, then the savings account is defined to be the only locally optimal portfolio.*

Note that the Sharpe ratio is bounded since

$$\begin{aligned} s^\delta(t) &= \frac{\sigma^\delta(t) \cdot \theta(t) + \int_{\mathcal{E}} b^\delta(v, t) \psi_\theta(v, t) \phi(dv, t)}{V^\delta(t)} \\ &\leq \frac{\|(\theta(t), \psi_\theta(v, t))\| \|(\sigma^\delta(t), b^\delta(v, t))\|}{V^\delta(t)} \\ &= \|(\theta(t), \psi_\theta(v, t))\| \end{aligned} \quad (13)$$

by (11) and the Cauchy-Schwartz inequality. This is the so-called *Hansen-Jagannathan* bound, see Hansen & Jagannathan (1991), derived in Björk & Slinko (2005) in a similar set-up. The bound (13) holds for any market price of risk representation,  $(\theta(t), \psi_\theta(v, t))$ . The following lemma characterizes locally optimal portfolios as the solution to a quadratic optimization problem:

**Lemma 3.2** *Suppose there exists some portfolio having a non-zero Sharpe ratio. Then an admissible portfolio,  $S^{(\delta)}$ , is a locally optimal portfolio if and only if it solves the problem*

$$\begin{aligned} p^\delta(t) &= \sup_{S^{(\delta')} \in \mathcal{Q}(\mathbb{S}^N)} p^{\delta'}(t) \\ \text{such that } &V^{\delta'}(t)^2 \leq k(t) \end{aligned} \quad (14)$$

for some non-negative predictable process  $k = \{k(t), t \in [0, \infty)\}$  and all  $t \in [0, \infty)$ .

The proof is given in the Appendix. Equivalently, one might fix the risk premium,  $p^{\delta'}(t)$  and minimize  $V^{\delta'}(t)^2$  in Lemma 3.2.

In any given vector space,  $\mathcal{H}$ , with an inner product,  $(\cdot|\cdot)$ , it is possible to define the projection of a vector,  $x$ , onto a subspace,  $\mathcal{R}$ . From a geometric point of view, a projection is the element in  $\mathcal{R}$  which is closest to  $x$ . The projection of the vector  $x$  onto the subspace  $\mathcal{R}$  will be denoted by  $x|_{\mathcal{R}}$ . It is defined as

$$x|_{\mathcal{R}} \triangleq \arg \min_{y \in \mathcal{R}} \|x - y\|,$$

where  $\|\cdot\| = \sqrt{(x|x)}$ . Let

$$\mathcal{B}_N \triangleq \left\{ x(t) \in \mathbb{R}^m \times L^2(\phi(dv, t)) \mid x(t) = \sum_{i=1}^N \pi^i(t) (\sigma^i(t), b^i(v, t)), \pi(t) \in \mathbb{R}^N, t \in [0, \infty) \right\} \quad (15)$$

denote the subspace containing the set of generalized volatilities. Not all elements of  $\mathcal{B}_N$  will correspond to generalized volatilities of admissible portfolios, since  $b(v, t) \geq -1$  almost everywhere, is required to ensure admissibility. Define the predictable process

$$\Phi \triangleq \{\Phi(t) = (\theta^p(t), \psi_\theta^p(v, t)) \triangleq (\theta(t), \psi_\theta(v, t))|_{\mathcal{B}_N}, t \in [0, \infty)\}. \quad (16)$$

Then  $\Phi$  itself is a market price of risk representation and is independent of the particular market price of risk representation,  $(\theta(t), \psi_\theta(v, t))$ , on the right hand side of equation (16). Moreover, due to the fact that projections are Borel measurable mappings we can apply arguments similar to those in Karatzas & Shreve (1998)[Lemma 1.4.7], to ensure that  $\Phi$  can be chosen in a measurable way.  $\Phi$  will play an important role in the following.

**Theorem 3.3** *Let  $\pi$  denote the fractions of an admissible strategy  $\delta$ . If for almost every  $(\omega, v, t) \in \Omega \times \mathcal{E} \times [0, T]$  it holds that*

$$\pi(t)^\top (\sigma(t), b(v, t)) = \alpha(t) (\theta^p(t), \psi_\theta^p(v, t)) \quad (17)$$

for some non-negative predictable scalar valued process  $\alpha = \{\alpha(t), t \in [0, \infty)\}$ , then  $S^{(\delta)}$  is locally optimal.

Conversely, if  $\psi_\theta(v, t)$  is bounded by some finite valued predictable process almost surely and the portfolio  $S^{(\delta)}$  is locally optimal, then  $\pi^\delta$  satisfies (17), for some non-negative predictable process  $\alpha$ .

Strategies with fractions of the form (17) will consequently be denoted by  $\delta(\alpha)$ . We emphasize that these are only locally optimal if  $\alpha(t) > 0$  for  $t \in [0, \infty)$ . The proof can be found in the Appendix, where it is linked to the proof of Lemma 3.2.

Observe, that fractions of the form (17) in Theorem 3.3 need not correspond to admissible portfolios, since  $\mathcal{B}_N$  does not correspond to the set of generalized volatilities of admissible portfolios as mentioned earlier. However, the fractions corresponding to locally optimal portfolios form a convex subset of the set of solutions. In most cases, it will be possible to choose an admissible locally optimal portfolio, by following a conservative investment strategy, i.e. by choosing  $\alpha(t)$  sufficiently small. Only in the case where

$$-\infty = \inf_{v \in \mathcal{E}} \psi_\theta^p(v, t) \text{ and } \sup_{v \in \mathcal{E}} \psi_\theta^p(v, t) = \infty$$

will it be impossible for the investor to make an admissible investment in a locally optimal portfolio. It is useful to exemplify when such a situation may occur. Intuitively, suppose the events are very bad for the share price of one company and very good for the share price of another company. One could model this situation using  $\mathcal{E} = [-1, \infty)$ ,  $\phi(dv, t) = dv$ , and jump parameters of the form

$$b_1(v, t) = v \text{ and } b_2(v, t) = \frac{1}{\sqrt{v+1}}.$$

By making the risk premium of the first company positive and the risk premium of the second company negative, locally optimal portfolios are long in the shares of the first company

and short in the shares of the second company, creating the situation as described above. In a situation such as this, the martingale measure implied from a Girsanov transformation of  $\Phi$  is a signed measure. In other words, the *minimal martingale measure*, see Schweizer (1995) is a signed probability measure. The assumption that  $\psi_\theta$  is bounded will ensure the existence of some locally optimal strategy. Note that in this example short selling must be allowed. Selling short may not be possible in practice and for this reason, we will briefly consider constraints on portfolio choice in Section 3.2. In the absence of constraints, and supposing that there is an admissible solution to the equations of Theorem 3.3, we can fully characterize the locally optimal portfolio processes.

**Theorem 3.4** *Any locally optimal portfolio,  $S^{(\delta(\alpha))}$ , of the form given in (17) satisfies the SDE*

$$dS^{(\delta(\alpha))}(t) = S^{(\delta(\alpha))}(t-) \left( r(t)dt + \alpha(t) \left( \|\Phi(t)\|^2 dt + \theta^p(t) \cdot dW(t) + \int_{\mathcal{E}} \psi_\theta^p(v, t) q(dv, dt) \right) \right) \quad (18)$$

for some predictable non-negative process  $\alpha = \{\alpha(t), t \in [0, \infty)\}$ .

PROOF: Inserting the optimal fractions and exploiting that  $(\Phi(t)|(\theta(t), \psi_\theta(v, t))) = \|\Phi(t)\|^2$  yields the result. □

One needs to calculate the projection of the market price of risk representations,  $\Phi(t)$ , on the span of generalized volatilities, as indicated in (16). This can be done by solving a set of linear equations as described below. Let  $(\theta^p(t), \psi_\theta^p(v, t)) = \sum_{i=1}^N x^i(t)(\sigma^i(t), b^i(t))$ . We need to determine the coefficients  $x^i$ , but

$$\begin{aligned} p^i(t) &= ((\theta(t), \psi_\theta(t)) | (\sigma^i(t), b^i(t))) = ((\theta^p(t), \psi_\theta^p(t)) | (\sigma^i(t), b^i(t))) \\ &= \sum_{j=1}^N x^j(t) ((\sigma^i(t), b^i(t)) | ((\sigma^j(t), b^j(t))) \\ &\triangleq \sum_{j=1}^N x^j(t) \xi_{ij}(t). \end{aligned}$$

Since we have assumed linear independence of the generalized volatilities, there is exactly one solution and denoting  $\xi(t) = \{\xi_{ij}(t)\}_{i,j \in \{1, \dots, N\}}$ , we have  $x(t) = \xi^{-1}(t)p(t)$  for  $t \in [0, \infty)$ .

To exemplify, assume for the rest of this subsection that there is only an  $m$ -dimensional Wiener process and no jump component. In vector form, the  $N$  risky primary security accounts are then assumed to satisfy the SDE

$$dS(t) = S(t) (a(t)dt + \sigma(t) \cdot dW(t)), \quad (19)$$

where  $W$  is an  $m$ -dimensional standard Wiener process,  $\sigma$  is an  $N \times m$  matrix of rank  $N \leq m$  and both  $a(t)$  and  $S(t)$  are  $N$ -dimensional vectors.

**Corollary 3.5** *If the market is given by equation (19), then an admissible portfolio is locally optimal if and only if its vector of fractions has the form*

$$\pi(t)^\top = \alpha(t)\sigma(t)(\sigma(t)\sigma^\top(t))^{-1}(a(t) - \mathbf{1}r(t)) \quad (20)$$

for  $t \in [0, \infty)$  and some predictable, scalar valued process  $\alpha = \{\alpha(t), t \in [0, \infty)\}$ .

Note that in the case without jump risk,  $\alpha(t) = 1$  for all  $t > 0$  the resulting locally optimal portfolio is the Growth Optimal Portfolio (GOP), see Platen (2004) and Christensen & Platen (2005b). Moreover, any locally optimal portfolio consist of a position in the GOP and the remaining wealth invested in the savings account, a so-called fractional Kelly strategy.

**Remark 3.6** *As a special case, if  $m = N$  and  $\sigma(t)$  is invertible, then the market is complete and there is only one market price of risk,  $\theta(t)$ . This forces  $\pi(t)\sigma(t) = \alpha(t)\theta(t)$  or  $\pi(t) = \alpha(t)\sigma(t)^{-1}\theta(t)$ .*

### 3.1 A Mutual Fund Theorem

For our general set-up, we point out that if investors were assumed to maximize the Sharpe ratio of their investments, then the following *mutual fund theorem* will be a consequence of Theorem 3.3.

**Theorem 3.7** *Any investor, who prefers portfolios with higher Sharpe ratios to portfolios with lower Sharpe ratios will hold a combination of the risk free asset and the mutual fund,  $S^{(mf)}$ , given by the SDE*

$$dS^{(mf)}(t) = S^{(mf)}(t-) \left( (r(t) + \|\Phi(t)\|^2)dt + \theta^p(t) \cdot dW(t) + \int_{\mathcal{E}} \psi_{\theta}^p(v, t)q(dv, dt) \right). \quad (21)$$

Note that the parameter  $\alpha(t)$  introduced in Theorem 3.3 can be interpreted as the fraction held at time  $t$  in the mutual fund. The mutual fund,  $S^{(mf)}$ , corresponds to the choice  $\alpha(t) = 1$  and may involve a position in the risk-free asset. Often, the mutual fund is assumed to consist of risky assets only, but here we define it to be equal to the locally optimal portfolio that corresponds to  $\alpha(t) = 1$ .

From the perspective of expected utility theory,  $\alpha(t)$  is related to the level of risk aversion of the investor, who selects portfolios based on their Sharpe ratios. For instance, in the case of no event risk and deterministic coefficients,  $\alpha(t)$  is the inverse of the Arrow-Pratt measure of relative risk aversion, see Pratt (1964). Since maximizing Sharpe ratios may not coincide with maximizing expected utility  $\alpha(t)$  may not always correspond to the relative risk aversion coefficient in the Arrow-Pratt sense. However, one could still use  $\alpha(t)$  as a convenient and intuitive parametrization of risk aversion in a straightforward manner.

## 3.2 Constraints on Portfolio Selection

The following lemma shows how one might handle restrictions on the choice of portfolios.

**Definition 3.8** *Suppose portfolio constraints are in place such that  $\pi \in \mathcal{J}$  must be satisfied, where  $\mathcal{J}$  is some closed convex set, which may be time dependent. Then there exists an efficient set of strategies maximizing the Sharpe ratios among all portfolio strategies belonging to  $\mathcal{J}$ . These strategies are defined as solutions to the constrained maximization problem*

$$p^\delta(t) = \sup_{\{\delta | \pi_\delta \in \mathcal{J}\}} ((\sigma^\delta(t), b^\delta(v, t)) | (\theta^p(t), \psi_\theta^p(v, t))) \quad (22)$$

such that  $V^\delta(t)^2 \leq k(t)$ ,

almost surely, where  $k = \{k(t), t \in [0, \infty)$  is finite, predictable and non-negative.

We will refer to such portfolios as  $\mathcal{J}$ -locally optimal portfolios. Examples include the set

$$\mathcal{J} \triangleq \{\pi(t) | \pi^i(t) \geq 0, i \in \{0, \dots, N\}, t \in [0, \infty) \text{ and } S^{(\delta^\pi)} \text{ is admissible}\},$$

which corresponds to the situation where no short sales are allowed, whereas the set

$$\mathcal{J} \triangleq \{\pi(t) | \pi^i(t) \geq 0, i \in \{1, \dots, N\}, t \in [0, \infty) \text{ and } S^{(\delta^\pi)} \text{ is admissible}\}$$

allows short positions in the savings account, that is, risk free borrowing is permitted, but no short sale of risky assets. Definition 3.8 still requires one to find the projection of the market price of risk, followed by solving the quadratic optimization program (22) in the vector space  $\mathbb{R}^N \times L^2(\phi(dv, t))$ .

As an example we study in more detail the special case where  $S^{(0)}$  is no longer traded. In the context of Definition 3.8 this corresponds to

$$\mathcal{J} \triangleq \{\pi(t) | \pi^0(t) = 0, t \in [0, \infty), \text{ and } S^{(\delta^\pi)} \text{ is admissible}\}.$$

This has an impact on the set of locally optimal portfolios, which is reduced to one single element, namely the element corresponding to a choice of  $\alpha$  such that  $\sum_{i=1}^N \pi_{\delta(\alpha)}^i(t) = 1$ . Similar to the situation in discrete time, see Markowitz (1952), there is an *efficient frontier* of  $\mathcal{J}$ -locally optimal portfolios. We denote the locally optimal portfolio with a zero net investment in the savings account by  $S^{(\delta_{tan})}$ , the *tangency portfolio*. We now characterize the efficient frontier of  $\mathcal{J}$ -locally optimal portfolios, where  $\pi^0(t) = 0$ .

**Lemma 3.9** *Consider the solution to the problem*

$$S^{(\delta_{mv})}(t) \triangleq \arg \inf_{S^{(\delta')} \in \Theta(\mathbb{S}^N)} V^{\delta'}(t)^2 \quad (23)$$

such that  $\pi^{(0)}(t) = 0$ , for all  $t \in [0, \infty)$ .

*Then  $S^{(\delta_{mv})}(t)$  is the minimum volatility portfolio and is  $\mathcal{J}$ -locally optimal. Two fund separation still holds and any  $\mathcal{J}$ -locally optimal portfolio strategy,  $\pi$ , is at time  $t$  characterized by  $\pi(t) = a(t)\pi_{\delta_{tan}} + (1 - a(t))\pi_{\delta_{mv}}(t)$  for  $t \in [0, \infty)$ , and some non-negative predictable process  $a = \{a(t), t \in [0, \infty)\}$ .*

The proof can be found in the Appendix. Note that the second moment of asset prices have not been assumed to exist. Still, the minimum volatility portfolio, which is well-defined in a pathwise sense, can exist even if it has infinite variance. In the continuous case, if the second moment exists and if the parameters involved are deterministic, then by the Itô isometry the minimum volatility portfolio will have minimal variance among all admissible portfolios. In general, this need not be the case.

There is a striking similarity between the characterization of the minimum variance portfolio known in discrete time and the above continuous time description of the minimum volatility portfolio. Recall the definition of the matrix process

$$\xi = \{\xi(t) = \{\xi_{ij}(t)\}_{i,j \in \{1, \dots, N\}} = \{((\sigma^i(t), b^i(t)) | (\sigma^j(t), b^j(t)))\}_{i,j \in \{1, \dots, N\}}, t \in [0, \infty)\}.$$

The problem (23) can be solved since  $\xi$  is invertible and yields the fractions

$$\pi_{\delta_{mv}}(t) = \frac{\xi^{-1}(t)\mathbf{1}}{\mathbf{1}^\top \xi^{-1}(t)\mathbf{1}}, t \in [0, \infty)$$

for the minimum volatility portfolio. This is similar to the representation of the weights of the minimum variance portfolio in discrete time.

Consequently, although the framework above is held in a general continuous time jump diffusion setting, the simplicity of the typical results of a simple one period model remains intact. The far richer framework presented here will allow a number of interesting and rather general conclusions to be drawn when we start to compare locally optimal strategies to expected utility maximizing strategies.

## 4 Local Optimality versus Expected Utility

It is well-known that, under certain conditions, *any* utility maximizing investor will prefer a locally optimal portfolio when asset prices are continuous. In fact this is the case when the short rate and the total market price of risk are adapted to a certain filtration, as shown by Christensen & Platen (2005b). In this case, investors do not wish to hedge the risk associated with these variables. Similar results have been known in special cases, usually assuming deterministic or even constant coefficients. Knowing that  $r$  and  $\|\theta^p\|$  cannot be arbitrary processes if utility maximization and local optimality are to be related, we make the following simplifying assumption.

**Assumption 4.1** *The short rate,  $r$ , and the total market price of diffusion risk,  $\|\theta^p\|$ , are deterministic processes.*

In the absence of this assumption, local optimality and utility maximization do not coincide in general, even if there are no jumps. Assuming  $r$  and  $\|\theta^p\|$  to be deterministic is more restrictive than necessary, see Christensen & Platen (2005b), however, this allows us to focus on the issues arising from adding jumps to the market dynamics.

We restrict ourselves to a popular class of utility functions, namely that of power utility functions

$$U(x) \triangleq \sup_{\substack{S^{(\delta)} \in \Theta(\mathbb{S}^N) \\ S^{(\delta)}(0)=x}} \mathbb{E} \left[ \frac{(S^{(\delta)}(T))^\gamma - 1}{\gamma} \right], \quad (24)$$

where  $\gamma \in (-\infty, 1)$  and  $1 - \gamma$  is the relative risk aversion of the investor. For  $\gamma = 0$  this is equivalent to log-utility which is thus covered by the analysis. Our aim is to investigate when expected utility maximizing investors will choose a locally optimal portfolio i.e. maximize their Sharpe ratios. Below we show that:

1. In complete markets, if event risk is priced, then neither power nor log-utility investors will choose a locally optimal portfolio.
2. In an incomplete market they may do so, but this is merely an artifact of incompleteness in the sense that when the market approaches completeness, utility maximizing investors will move away from local optimality.
3. In the complete pure jump case utility maximizing strategies can be locally optimal if coefficients are deterministic.

We conclude that the *mixing* of event and diffusion risk is critical, because the pricing of these two types of uncertainty differs fundamentally. Intuitively this arises because the quadratic variation which is used as measure of uncertainty, is symmetric in the sense that uncertainty about upward and downward jumps is “punished” symmetrically. Obviously, for risk-averse investors the downward jumps are the major concern. An illustration is provided by the following example.

**Example 4.2** *Let  $q^1$  and  $q^2$  denote two independent compensated Poisson processes with equal intensity,  $\lambda \in \mathbb{R}_+$ . This means that  $q^i(t) = N^i(t) - \lambda t$  for  $t \in [0, \infty)$  and  $i \in \{1, 2\}$ , where  $N^1, N^2$  are both standard Poisson processes with intensity  $\lambda$ . Consider the price processes*

$$dS^{(i)}(t) = S^{(i)}(t-) \left( (r(t) + \psi_\theta^i b^i) dt + b^i dq^i(t) \right)$$

*where  $t \in [0, \infty)$  and  $i \in \{1, 2\}$ . Suppose  $b^1 = -b^2 = 1$  and  $\psi_\theta^1 = -\psi_\theta^2 = \frac{1}{2}$ . In this case it is seen that  $s^1 = s^2$ , that is, the two securities have identical Sharpe ratios. If an investor were to care only about Sharpe ratios, then she or he would be indifferent between holding either of the two securities. It is easy to see that the second security will default almost surely at some time point. In contrast, the first security will remain strictly positive for all  $t \in [0, \infty)$ . Hence, expected utility maximizing investors with, for instance, log-utility will always prefer the first security to the second security.*

In words, if one considers a portfolio having a ten percent chance of loosing 100% and a given market price of risk and compare it to a portfolio with a 10% chance of doubling and having an equal, but negative, market price of risk. In terms of local optimality the portfolios are similar since they have the same expected return and the same risk,

but no investor having a power or log-utility function will prefer the former. Ranking portfolios according to their Sharpe ratio will then provide a different ranking than by ranking in order of expected utility. The problem is essentially the same in the classical mean-variance framework, since the variance is also symmetric. For diffusion noise, this problem disappears, since investors are assumed to hedge continuously and do not need to fear more than infinitesimal changes in their portfolio over small time-horizons. Even if expected utility and Sharpe ratios provide a different ranking, it may still be the case that in optimum, a utility maximizing agent will choose a locally optimal portfolio.

Section 4.1 below will state the result in a simple case of a finite mark space. Such a model is driven by a finite number of Poisson counting processes. This simplification is made to provide easier access to the results, without having to face the complications arising from, what is in essence, an infinite-dimensional noise source. The assumption of a finite mark space can restrict the models severely, so Section 4.2 will extend the result to the general case, which will be more technical, since completeness is no longer possible when investors are only allowed to trade a finite number of assets.

Let us start by providing a characterization of the solution to the optimal portfolio problem (24) by stating the first order conditions for power and logarithmic utility maximization.

**Theorem 4.3** *Consider an investor with power or log-utility and risk-aversion coefficient  $1 - \gamma$ ,  $\gamma \in (-\infty, 1)$ . Under Assumption 4.1 the first order conditions for optimality are*

$$\sum_{j=1}^m \theta^j(t) \sigma^{i,j}(t) - (1 - \gamma) \sum_{j=1}^m \sigma^{i,j}(t) \left( \sum_{k=1}^N \pi_{\delta_\gamma}^k(t) \sigma^{k,j}(t) \right) + \int_{\mathcal{E}} b^i(v, t) \left( \psi_\theta(v, t) - 1 + \frac{1}{(1 + \sum_{k=1}^N \pi_{\delta_\gamma}^k(t) b^k(v, t))^{1-\gamma}} \right) \phi(dv, t) = 0 \quad (25)$$

for  $t \in [0, \infty)$  and  $i \in \{1, \dots, N\}$ . If there is a predictable process

$$\pi_{\delta_\gamma} = \{ \pi_{\delta_\gamma}(t) = (\pi_{\delta_\gamma}^1(t), \dots, \pi_{\delta_\gamma}^N(t))^\top, t \in [0, \infty) \}$$

satisfying this equation, such that  $\pi_{\delta_\gamma}$  describes the fractions of an admissible portfolio,  $S^{(\delta_\gamma)}$ , and  $\mathbb{E}[U(S^{(\delta_\gamma)}(T))] < \infty$ , then the strategy defined by the fractions  $\pi_{\delta_\gamma}$  is optimal for a power utility investor with relative risk aversion coefficient  $1 - \gamma$ . The first order conditions for log-utility are obtained by putting  $\gamma = 0$  in (25).

The proof is found in, for instance, Kraft & Steffensen (2005)[Appendix A.4] or Christensen & Platen (2005a) for the case of log-utility. A consequence of Assumption 4.1 is that the so-called *Merton-Breeden hedge terms* do not appear in (25), since power investors do not have to hedge uncertainty in the processes  $r$  and  $\|\theta^p\|$ . If  $r$  and  $\|\theta^p\|$  were general processes, then investors may wish to apply a different strategy. For instance, they may take into account that risk free borrowing could be expected to become more expensive and try to hedge this risk, see for instance Merton (1973).

For tractability we add the following assumption.

**Assumption 4.4** For each  $\gamma \in (-\infty, 1)$  the solution to the corresponding investment problem is given by (25).

It should be noted that in a jump-setting, the first order conditions need not define an admissible portfolio because in some cases the process obtained by applying these fractions will become negative. In general, we may have a “corner solution” and Assumption 4.4 rules out this case. If downward jumps do not have a “thin tail” and if the market price of risk is not “too high”, then there will be a solution to (25) which constitutes an admissible strategy and Assumption 4.4 is redundant. We refer to Christensen & Larsen (2004) for a discussion of this in the case of log-utility.

## 4.1 A Simple Case

We assume that the mark space can be represented by a finite set  $\mathcal{E} = \{v_1, \dots, v_{d-m}\}$ . Hence one can think of the market as driven by a  $d - m$  dimensional Poisson process and an  $m$ -dimensional Wiener process. In this setup it is then possible to complete the market by having  $d$  risky primary security accounts apart from the risk free asset. All results of this section are essentially corollaries of those of the next section and all derivations are simpler versions of those applied to the general case.

Before we formulate the main results, let us informally indicate the reason why utility maximization is likely to deviate from local optimality. From the well-known continuous model with Black-Scholes dynamics, see Merton (1971), we conjecture that  $\alpha(t) = \frac{1}{1-\gamma}$  is necessary to maximize power utility among locally optimal strategies in the jump-diffusion case as well. From Theorem 3.3 we know that the relation  $(\sigma^\delta(t), b^\delta(v, t)) = \alpha(t)(\theta^p(t), \psi_\theta^p(v, t))$  characterizes locally optimal portfolios. Now, choose  $\alpha(t) = \frac{1}{1-\gamma}$  and insert this into the first order conditions (25). We see that the first two terms cancel out such that

$$\begin{aligned} & \sum_{j=1}^m (\theta^p)^j(t) \sigma^{i,j}(t) - (1-\gamma) \sum_{j=1}^m \sigma^{i,j}(t) \left( \sum_{k=1}^N \pi_{\delta_\gamma}^k(t) \sigma^{k,j}(t) \right) \\ &= \sum_{j=1}^m (\theta^p)^j(t) \sigma^{i,j}(t) - (1-\gamma) \sum_{j=1}^m \frac{1}{1-\gamma} (\theta^p)^j(t) \sigma^{i,j}(t) = 0. \end{aligned}$$

Hence, to satisfy the first order condition it is necessary that

$$\int_{\mathcal{E}} b^i(v, t) \left( \psi_\theta^p(v, t) - 1 + \frac{1}{(1 + \psi_\theta^p(v, t))^{1-\gamma}} \right) \phi(dv, t) = 0 \quad (26)$$

for all  $i \in \{1, \dots, N\}$  and  $t \in [0, \infty)$ . In terms of inner products

$$\left( b(v, t) | 1 - \psi_\theta^p(v, t) \right)_{L^2(\phi(dv, t))} = \left( b(v, t) | \frac{1}{(\psi_\theta^p(v, t) + 1)^{1-\gamma}} \right)_{L^2(\phi(dv, t))}. \quad (27)$$

If the functions  $b^i$ ,  $i \in \{1, \dots, N\}$ , span the mark space, then we would conclude that

$$1 - \psi_\theta^p(v, t) = \frac{1}{(\psi_\theta^p(v, t) + 1)^{1-\gamma}}$$

for all  $t \in [0, \infty)$  and all  $v \in \mathcal{E}$ . This would then imply that  $\psi_\theta(v, t) = 0$ . The statement below shows that this intuition is not completely misguided.

**Lemma 4.5** *Assume  $m \neq 0$  and  $|\theta^p(t)| \neq 0$ . If the market  $\mathbb{S}^N$  is complete, then a power or log-utility investor will choose a locally optimal portfolio if and only if  $\psi_\theta(v_i, t) = 0$  almost surely for all  $v_i \in \mathcal{E}$  and all  $t \in [0, \infty)$ .*

At this stage it is not clear whether it is only the presence of event risk or the combination of event and diffusion risk, which can create an incompatibility between Sharpe ratio optimization and utility maximization. After all, in the absence of diffusion risk it may still not be possible for locally optimal portfolios to coincide with those chosen by a utility maximizer. To provide a clear answer, we can remove the Wiener terms by putting  $m = 0$ . Doing so provides the following result.

**Lemma 4.6** *Suppose  $m = 0$ , i.e. jumps are the only source of uncertainty. It is sufficient that  $\psi_\theta^p(v, t)$  does not depend on  $v$ , for a power or log-utility investor to choose a locally optimal portfolio. If the market is complete, then the assumption that  $\psi_\theta^p(v, t)$  does not depend on  $v$  is also necessary for a power or log-utility investor to choose a locally optimal portfolio.*

Hence in this case it is possible to make Sharpe ratio maximization consistent with utility maximization. However, it requires that only one price of event driven uncertainty exists. In other words if two types of events affect the stock market, then these events must both carry the same market price of event risk. Such requirement seems rather strong and there is a priori no reason why unrelated systematic risk factors should provide the same expected excess return.

The assumption of completeness cannot be removed from Lemma 4.5 as one might imagine from the preceding discussion. If markets are incomplete it may happen that investors are forced to accept risk they would otherwise prefer to hedge. Consequently, they may have to choose a locally optimal portfolio “by accident”. The following simple example illustrates this point.

**Example 4.7** *Consider a market with one risky asset, whose price process is driven by one Poisson process,  $N$ , and one Wiener process,  $W$ , that is*

$$\begin{aligned} dS(t) &= S(t-) ((r + \sigma\theta + b\psi_\theta)dt + \sigma dW(t) + b(dN(t) - \lambda dt)) \\ &= S(t-) ((r + \sigma\theta + b\psi_\theta)dt + \sigma dW(t) + bdq(t)), \end{aligned}$$

where  $r, \sigma, \theta, b, \psi_\theta, \lambda$  are all positive constants.  $\lambda$  is the intensity of  $N$ , so that the compensated Poisson process  $q(t) = N(t) - \lambda t$  forms a martingale. Assume that  $(\theta, \psi_\theta)$  is

proportional to  $(\sigma, b)$ , that is,  $(\theta, \psi_\theta) = (\theta^p, \psi_\theta^p)$  and we already have the market price of risk representation needed for characterizing locally optimal portfolios. In this case the first order conditions (25) for  $\gamma = 0$  becomes

$$\theta\sigma - \pi\sigma^2 + b(\psi_\theta - 1 + \frac{1}{1 + \pi b}) = 0.$$

For  $\pi$  to represent a locally optimal portfolio, it must hold that  $\pi(\sigma, b) = \alpha(\theta, \psi_\theta)$  for some constant  $\alpha$ . Inserting into the first order conditions we obtain

$$\theta\sigma - \alpha\theta\sigma + b(\psi_\theta - 1 + \frac{1}{1 + \alpha\psi_\theta}) = 0.$$

Rearranging provides a quadratic equation to be solved for  $\alpha$ . Only one of these solutions are positive and hence constitute a locally optimal portfolio. Choosing this solution provides

$$\alpha = \frac{(1 - \psi_\theta)(\theta\sigma + \psi_\theta b) + \sqrt{(1 - \psi_\theta)^2(\theta\sigma + \psi_\theta b)^2 + 4\theta\sigma\psi_\theta(\theta\sigma + \psi_\theta b)}}{2\theta\sigma\psi_\theta}.$$

Since this constitutes an admissible portfolio, the GOP is in fact locally optimal.

The example shows how incompleteness may in some cases “force” investors to maximize the Sharpe ratio. However, they only do so because they have no choice, incompleteness prevents them from attaining the true optimum. Moreover, it is clear even from this simple example that local optimality and utility maximization only coincide “by accident” when event risk is priced. This implies that the use of bounds on the market price of risk to obtain “good deal bounds” is not in general compatible with the assumption that investors maximize utility. At least this is true for power and logarithmic utility functions, when event risk is priced. However, whether investors do in fact maximize some well-specified utility function is an open question. Although it is a conventional assumption in the literature of asset allocation, it has not been answered convincingly so far.

## 4.2 The General Case

Since the uncertainty represented by the marked point process can be of infinite dimension, the  $N$ th market,  $\mathbb{S}^N$ , will in general be incomplete for any  $N$  no matter how large  $N$  is. To deal with this situation, one may study the markets as they become *approximately complete*, see Björk, Di Masi, Kabanov & Runggaldier (1997) or Christensen & Platen (2005a). Consider an increasing sequence of markets,  $\mathbb{S}^N$ , where  $N \in \mathbb{N}$ . It is known that if the set  $\mathcal{B}_N$ , defined by equation (15), is dense in  $\mathbb{R}^m \times L^2(\phi(dv, t))$ , then the sequence of markets is *approximately complete* in the sense that any derivative, satisfying some weak integrability condition, can be approximated by a sequence of admissible portfolios converging in probability. Moreover, if all markets are arbitrage free, then there is a unique market price of risk representation  $(\theta(t), \psi_\theta(v, t))$ .

One may use a simple analogy to understand these features. A continuous function on a discrete space requires its values to be specified at every point to be well-defined. This

refers to the simple case considered earlier. On the other hand, if a continuous function is defined on  $\mathbb{R}$ , then it needs only to be defined on a dense subset, for instance, at every rational number, in order to be determined completely. Similarly, when we move to the general case, the space of generalized volatilities is no longer a finite set. However, the market price of risk functional is a continuous operator and for this reason it needs only to have its values fixed on a dense subset to be well-defined.

The properties needed are summarized in the following theorem. We refer to Christensen & Platen (2005a)[Theorem 2.9] for precise statements and proofs. Björk, Di Masi, Kabanov & Runggaldier (1997)[Proposition 6.10] contains a similar result.

**Theorem 4.8** *Define the subspace  $\mathcal{B} \subseteq \mathbb{R}^m \times L^2(\phi(dv, t))$  as the smallest subspace containing all sets  $\mathcal{B}_N$ ,  $N \in \{1, 2, \dots\}$ . Assume that the closure of  $\mathcal{B}$  is the entire space  $\mathbb{R}^m \times L^2(\phi(dv, t))$ , then:*

1. *If, for some  $T > 0$ ,  $H$  is a bounded  $\mathcal{F}_T$ -measurable random variable, then there exists a sequence of strategies,  $(\delta^N)_{N \in \{1, 2, \dots\}}$ , such that  $S^{(\delta^N)}(T) \rightarrow H$  in probability as  $N \rightarrow \infty$ .*
2. *If there is no arbitrage in any of these markets, then there exists a unique market price of risk representation  $(\theta(t), \psi_\theta(v, t))$ .*

The first property in Theorem 4.8 is referred to as approximate completeness of the sequence of markets. The second property is, under regularity conditions, equivalent to the statement that approximately complete markets have a unique equivalent martingale measure.

**Remark 4.9** *In Christensen & Platen (2005a), denseness is required in terms of the topology induced by the  $L^1$  norm. In this set-up,  $L^1(\mathcal{E}) \subseteq L^2(\mathcal{E})$ , since  $\phi(\mathcal{E}, t)$  is finite and consequently if a subset is dense in  $L^2(\mathcal{E})$  then it is also dense in  $L^1(\mathcal{E})$ .*

For the remainder of this section we make the following assumption, which will ensure that all SDEs that we consider will be well-defined.

**Assumption 4.10** *The set  $\mathcal{B}$  is dense in  $\mathbb{R}^m \times L^2(\phi(dv, t))$  and the unique market price of risk is assumed to satisfy*

$$\int_0^T \|\theta(t)\|^2 dt < \infty \text{ and } |\psi_\theta(v, t)| < 1 - \epsilon \quad (28)$$

*almost surely for all  $T > 0$  and some  $\epsilon > 0$ .*

We call a stochastic process a *generalized portfolio*, if it appears as the limit in probability of a sequence  $(S^{(\delta^N)})_{N \in \{1, 2, \dots\}}$  of traded portfolio processes and the limits

$$\bar{b}(v, t) \triangleq \lim_{N \rightarrow \infty} b^{\delta^N}(v, t) \text{ and } \bar{\sigma}(t) = \lim_{N \rightarrow \infty} \sigma^{\delta^N}(t) \quad (29)$$

exist, where convergence is in  $L^2(\phi(dv, t))$  and the Euclidean norm, respectively. A generalized portfolio having a maximal Sharpe ratio is called a *generalized locally optimal portfolio*. In this setting, a maximal Sharpe ratio may not be attainable by a finite number of assets, due to the incompleteness of the model. Instead, it must be approximated by holding well-diversified portfolios.

**Theorem 4.11** *Assume there is a sequence,  $(\delta^N)_{N \in \{1, 2, \dots\}}$ , of admissible strategies with  $S^{(\delta^N)} \in \mathbb{S}^N$ , such that*

$$\lim_{N \rightarrow \infty} \left( \sum_{i=1}^N \pi_{\delta^N}^i(t) \sigma^i(t), \sum_{i=1}^N \pi_{\delta^N}^i(t) b^i(v, t) \right) = \alpha(t) (\theta(t), \psi_\theta(v, t)) \quad (30)$$

for some predictable process,  $\alpha = \{\alpha(t), t \in [0, \infty)\}$ , where the limit is taken with respect to the standard norm in  $L^2 \times \mathbb{R}^m$ . Then the limit in probability  $S^{(\delta)}(t) \triangleq \lim_{N \rightarrow \infty} S^{(\delta^N)}(t)$  exists for all  $t \in [0, \infty)$  and characterizes a generalized locally optimal portfolio  $S^{(\delta)}$ . Moreover, portfolios satisfying the SDE

$$\begin{aligned} dS^{(\delta)}(t) &= S^{(\delta)}(t-) \left( r(t)dt + \alpha(t) \left( \|(\theta(t), \psi_\theta(v, t))\|^2 dt + \theta(t) \cdot dW(t) \right. \right. \\ &\quad \left. \left. + \int_{\mathcal{E}} \psi_\theta(v, t) q(dv, dt) \right) \right) \end{aligned} \quad (31)$$

are generalized locally optimal portfolios.

**PROOF:** By Theorem 4.8 it follows that  $(\theta(t), \psi_\theta(v, t))$  is unique. Using the Hansen-Jagannathan bound, see equation (13), it follows that any process of the form given by (31) is a candidate for a generalized locally optimal portfolio, since they attain the maximum possible Sharpe ratio. By assumption the processes  $\sigma^{\delta^N}$  and  $b^{\delta^N}$  converge, so in order to prove the remaining part of the theorem, it remains to be shown that the limit

$$S^{(\delta)}(t) = \lim_{N \rightarrow \infty} S^{(\delta^N)}(t)$$

exists in probability for all  $t \in [0, \infty)$  and satisfies the SDE (31). This follows directly from Assumption 4.10 and Christensen & Platen (2005a)[Lemma A.1].

□

A necessary and sufficient condition for the generalized portfolio in (31) to remain positive is that  $\alpha(t)\psi_\theta(v, t) \geq -1$ . On the other hand,  $\psi_\theta(v, t) < 1$  is necessary to avoid arbitrage. Consequently, for  $\alpha(t) > 0$ , if  $S^{(\delta(\alpha))}$  is to be admissible, then  $\psi_\theta$  must be uniformly bounded in  $v$ , see also Theorem 3.3 and the discussion afterwards. This provides a justification for the assumptions on  $\psi_\theta$  in Assumption 4.10.

Since the optimization problem (14) is now of infinite dimension, the set

$$\{\pi_\delta(t) | S^{(\delta)}(t) \in \underline{\Theta}(\mathbb{S}^N), N \in \mathbb{N}, V^\delta(t)^2 \leq k(t), t \in [0, \infty)\}$$

is no longer compact for a given  $t > 0$  and a solution to (14) need not exist. This happens when  $(\theta(t), \psi_\theta(v, t))$  does not belong to the interior of the subspace containing the sets  $\mathcal{B}_N$ ,  $N \in \mathbb{N}$ . In this case, locally optimal portfolios do *not* belong to the set of admissible portfolios. Managers seeking such portfolios would only be able to approximate a locally optimal portfolio, by holding an approximating portfolio. For instance, this can happen if the market contains unsystematic risk, which can be reduced but not eliminated by holding very diversified portfolios.

As with generalized locally optimal portfolios, we need to define a generalized solution to the problem (24), taking into account the fact that we may not be able to buy the portfolio, which provides the true maximum, using only a finite number of assets. The generalization is obtained in the following way.

**Definition 4.12** *A generalized solution to (24), is a generalized portfolio,  $S^{(\delta)}$ , such that for any  $N \in \mathbb{N}$  and any portfolio  $S^{(\delta_N)} \in \underline{\Theta}(\mathbb{S}^N)$  one has*

$$\mathbb{E}[U(S^{(\delta_N)}(T))] \leq \mathbb{E}[U(S^{(\delta)}(T))] < \infty$$

We can now characterize the generalized solutions to the power and log-utility maximization problem.

**Theorem 4.13** *Fix a  $\gamma \in (-\infty, 1)$ . Consider the process  $S^{(\delta_\gamma)}$  given by the SDE*

$$\begin{aligned} dS^{(\delta_\gamma)}(t) = & S^{(\delta_\gamma)}(t-) \left( \left( r(t) + \frac{\|\theta(t)\|^2}{1-\gamma} + \int_{\mathcal{E}} \frac{\psi_\theta(v, t)(1 - (1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}})}{(1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}}} \phi(dv, t) \right) dt \right. \\ & \left. + (1 - \gamma)^{-1} \theta(t) \cdot dW(t) + \int_{\mathcal{E}} \frac{1 - (1 - \psi_\theta(v, t))^{1-\gamma}}{(1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}}} q(dv, dt) \right) \end{aligned} \quad (32)$$

for  $t \in [0, \infty)$ . If  $\mathbb{E}[(S^{(\delta_\gamma)})^\gamma] < \infty$ , then  $S^{(\delta_\gamma)}$  is the unique generalized solution to (24).

The proof can be found in the Appendix.

In the previous section we showed that incompleteness may force the power utility investor to choose a locally optimal portfolio, basically because there is some freedom left when choosing  $\alpha(t)$ . As the market becomes complete, this will no longer hold unless  $\psi_\theta^p(v, t) = 0$ . With the machinery developed here, these conclusions carry over to the approximately complete case and we obtain the following generalization of Lemma 4.5.

**Theorem 4.14** *Assume that markets  $\mathbb{S}^N$  are approximately complete, with  $m \neq 0$  and  $\|\theta^p(t)\| \neq 0$ . Then the generalized solution to a power or log-utility investor is locally optimal if and only if*

$$\psi_\theta(v, t) = 0 \quad (33)$$

almost surely for almost every  $(v, t) \in \mathcal{E} \times [0, \infty)$ .

This observation is important, because the introduction of derivatives in the market imply that one can expect the market to become more complete over time due to the securitization of risk factors. As before, removing the Wiener noise will make local optimality compatible with utility maximization if the market price of event risk is constant across events.

**Theorem 4.15** *Suppose  $m = 0$ . It is sufficient that  $\psi_\theta^p(v, t)$  does not depend on  $v$ , for a power or log-utility investor to choose a locally optimal portfolio. If the market is approximately complete, then the assumption that  $\psi_\theta^p(v, t)$  does not depend on  $v$  is also necessary for a power or log-utility investor to choose a locally optimal portfolio.*

We conjecture that the results of this section using power-utility extend to a wide class of utility functions. We conclude that maximizing Sharpe ratios is quite different from maximizing expected utility, except for special cases, in particular, if the market price of event risk is zero.

## 5 Conclusion

Although it is realized that maximizing Sharpe ratios as an investment goal may conflict with expected utility maximization, it certainly does not appear to limit the large number of academic and practical uses of this performance measure. We showed in a rather general jump-diffusion setting that a striking feature of this form of investment strategy is that it leads to simple, closed form solutions and global two fund separation, making it attractive from an applied perspective. However, it appears that the combination of continuous and event driven uncertainty poses some challenges if one wishes to reconcile Sharpe ratio maximization with expected utility maximization.

## A Proofs

### A.1 A Combined Proof of Lemma 3.2 and Theorem 3.3

PROOF: The proof of Lemma 3.2 and Theorem 3.3 is essentially based on the Projection Theorem of Hilbert spaces, see Rudin (1987). Consider the case where there is some portfolio with a non-zero Sharpe ratio. This is equivalent to the case of non-zero market price of risk representation  $\Phi(t) \neq 0$ , see (16). Note that (14) can be rephrased as

$$\begin{aligned} & \max_{(\sigma^\delta(t), b^\delta(v, t))} ((\sigma^\delta(t), b^\delta(v, t)) | (\theta(t), \psi_\theta(v, t))) \\ & \text{s.t. } \|(\sigma^\delta(t), b^\delta(v, t))\| \leq k(t), \end{aligned} \tag{34}$$

where  $(\sigma^\delta(t), b^\delta(v, t)) \in \mathcal{B}_N$  is obtainable by some admissible strategy. By the Projection Theorem, write  $\mathcal{B}_N = \mathcal{S} \oplus \mathcal{S}^\perp$ , where  $\mathcal{S}$  is the subspace generated by  $\Phi(t) =$

$(\theta^p(t), \psi_\theta^p(v, t))|_{\mathcal{B}_N}$  and  $\mathcal{S}^\perp$  is its orthogonal complement. This means that for any  $(\sigma(t), b(v, t)) \in \mathcal{B}$  there is a number  $\alpha(t)$  and an element  $s^\perp(t) \in \mathcal{S}^\perp$ , such that

$$(\sigma(t), b(v, t)) = \alpha(t)\Phi(t) + s^\perp(t).$$

Now

$$\begin{aligned} ((\sigma(t), b(v, t)) | (\theta(t), \psi_\theta(v, t))) &= (\alpha(t)\Phi(t) + s^\perp(t) | (\theta(t), \psi_\theta(v, t))) \\ &= (\alpha(t)\Phi(t) | \Phi(t)) \\ &= \alpha(t)\|\Phi(t)\|^2. \end{aligned}$$

The second equality holds, since any market price of risk representation  $(\theta(t), \psi_\theta(v, t))$  can also be decomposed as  $\Phi(t) + \hat{s}(t)$ , where  $\hat{s}(t)$  is orthogonal to  $\mathcal{B}_N$ . Since  $\|(\sigma(t), b(v, t))\|^2 = (\alpha(t)\|\Phi(t)\|)^2 + \|s^\perp(t)\|^2$ , it follows directly that optimality is obtained if and only if  $s^\perp(t) = 0$  and  $\alpha(t) = \frac{k(t)}{\|\Phi(t)\|^2}$ . Moreover, it is clear that the optimal fractions are given such that

$$\pi(t) \cdot (\sigma(t), b(v, t)) = \Phi(t) = (\theta(t), \psi_\theta(v, t)).$$

From these observations, Lemma 3.2 and Theorem 3.3 follow directly in the case of non-trivial Sharpe ratios. In the case where  $\Phi(t) = 0$ , which means that the volatilities are orthogonal to the market price of risk, it follows that the Sharpe ratio is identically equal to zero, which means that one invests exclusively in the savings account, according to Definition 3.1.

□

## A.2 Proof of Lemma 3.9

It is relatively straightforward to adapt the proof of the discrete time analogue of this result to the present situation. For completeness, we give the details here.

PROOF: The minimum volatility portfolio  $S^{(\delta_{mv})}$  is unique, by strict concavity of the optimization problem and the assumption of linear independence of volatilities. Consequently, putting  $k(t) = \inf_{S^{(\delta')} \in \mathcal{J}} V^{\delta'}(t)^2$  in Problem (14) this portfolio is  $\mathcal{J}$ -locally optimal. To see that two fund separation holds, assume that  $\delta$  is some admissible strategy. We will show that there is a strategy of the form  $a(t)\pi_{\delta_{tan}}(t) + (1 - a(t))\pi_{\delta_{mv}}(t)$  having the same volatility, but higher expected return. Note that

$$(\sigma^\delta(t), b^\delta(v, t)) = x(t)(\sigma^{\delta_{mv}}(t), b^{\delta_{mv}}(v, t)) + y(t)(\sigma^{\delta_{tan}}(t), b^{\delta_{tan}}(v, t)) + z(t)(\sigma^{\delta^\perp}(t), b^{\delta^\perp}(v, t)),$$

where  $(\sigma^{\delta^\perp}(t), b^{\delta^\perp}(v, t))$  is orthogonal to  $(\sigma^{\delta_{mv}}(t), b^{\delta_{mv}}(v, t))$  and  $(\sigma^{\delta_{tan}}(t), b^{\delta_{tan}}(v, t))$ . Obviously,  $(\sigma^{\delta^\perp}(t), b^{\delta^\perp}(v, t)) \in \mathcal{B}_N$  so this is the generalized volatility of a traded portfolio. By orthogonality, this traded portfolio has zero Sharpe ratio. By switching investments from this portfolio to the tangency portfolio, in a ratio which keeps the generalized volatility fixed, we obtain a higher expected return. This shows that portfolios which are not linear

combinations of the minimum volatility portfolio and the tangency portfolio will not be  $\mathcal{J}$ -locally optimal. To see that such linear combinations are themselves  $\mathcal{J}$ -locally optimal, note that the generalized volatility is strictly increasing in  $\alpha(t)$ , since  $\pi_{\delta_{mv}}$  has the minimal generalized volatility. Hence if there is a portfolio  $S^{(\delta)}$  having a higher expected return and the same generalized volatility as  $a(t)\pi_{\delta_{tan}}(t) + (1 - a(t))\pi_{\delta_{mv}}(t)$ , then such portfolio can not be a different linear combination of the minimal volatility portfolio and the tangency portfolio. However, if this is not the case, then the previous argument can be used to show that the strategy  $a(t)\pi_{\delta_{tan}}(t) + (1 - a(t))\pi_{\delta_{mv}}(t)$  is better than that of  $S^{(\delta)}$  in terms of local optimality. □

### A.3 Proof of Theorem 4.13

PROOF: We need to prove that the process satisfying (32) is a generalized portfolio and that it provides higher utility than any other strategy in any other market  $\mathbb{S}^N$ . To see that  $S^{(\delta)}$  is a generalized portfolio, Assumption 4.10 ensures that  $(b^{\delta\gamma}(v, t), \sigma^{\delta\gamma}(t)) \in L^2 \times \mathbb{R}^m$ . Moreover, Assumption 4.10 which guarantees the existence of a sequence  $(b^{\delta_N}(v, t), \sigma^{\delta_N}(t))$  converging to  $(b^{\delta\gamma}(v, t), \sigma^{\delta\gamma}(t))$  in  $L^2(\mathcal{E}, \mathbb{R}^m)$ . The conclusion follows by applying Christensen & Platen (2005a)[Lemma A.1]. To see that it provides a higher expected utility, consider the market  $\tilde{\mathbb{S}}^N \triangleq (\mathbb{S}^N, S^{\delta\gamma})$  obtained by adding  $S^{(\delta\gamma)}$  to  $\mathbb{S}^N$ . Clearly, in this market the strategy which invests entirely in  $S^{(\delta\gamma)}$  will satisfy the corresponding first-order conditions. Since by assumption the utility of this strategy will be finite, the strategy is optimal proving that  $\mathbb{E}[U(S^{(\delta\gamma)}(T))] \geq \mathbb{E}[U(S^{(\delta)}(T))]$  for any other strategy  $\delta \in \Theta(\mathbb{S}^N)$ . To get uniqueness, notice that the first order conditions (25) can be interpreted as an inner product of the form

$$((\sigma^i(t), b^i(v, t)) | F(\sigma^\delta(t), b^\delta(v, t))) = 0.$$

for some non-linear transformation  $F : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m \times \mathbb{R}$  and all  $(\sigma^i(t), b^i(v, t))$ ,  $i \in \{1, 2, \dots, \}$ . By linearity

$$((\sigma(t), b(v, t)) | F(\sigma^\delta(t), b^\delta(v, t))) = 0.$$

for any  $(\sigma(t), b(v, t)) \in \mathcal{B}$ . Denseness of the set  $\mathcal{B}$  then implies that  $F(\sigma^\delta(t), b^\delta(v, t)) = (0, 0)$ . Solving these equations provides the unique solution

$$(\sigma^\delta(t), b^\delta(v, t)) = (\sigma^{\delta\gamma}(t), b^{\delta\gamma}(v, t))$$

for almost every  $(\omega, v, t)$ . □

## A.4 Proofs of Theorem 4.6 and Theorem 4.15

PROOF: Observe that in order for the first order conditions to be consistent with local optimality, one must find  $\alpha(t)$  such that

$$\left( b^i(v, t) \psi_\theta^p(v, t) - 1 + \frac{1}{(1 + \alpha(t) \psi_\theta^p(v, t))^{1-\gamma}} \right) = 0. \quad (35)$$

Obviously, if  $\psi_\theta^p(v, t) = \psi_\theta^p(t)$  i.e. the market price of jump risk does not depend on  $v$ , then choosing

$$\alpha(t) = \frac{1}{\psi_\theta(t)} \left( \frac{1}{(1 - \psi_\theta^p(t))^{\frac{1}{1-\gamma}}} - 1 \right)$$

solves the equation, proving sufficiency. If markets are complete or approximately complete, then (35) implies that

$$\alpha(t) = \frac{1}{\psi_\theta(v, t)} \left( \frac{1}{(1 - \psi_\theta^p(v, t))^{\frac{1}{1-\gamma}}} - 1 \right)$$

is necessary since the linear span of the set  $\{b^i(v, t)\}_{i \in \{1, 2, \dots\}}$  is dense or equal to  $L^2(\phi(dv, t))$ , in the approximately complete and complete case, respectively. But this requires  $\psi_\theta(v, t)$  to be deterministic and hence we have necessity. □

## A.5 Proofs of Lemma 4.5 and Theorem 4.14

Proving Theorem 4.14 is sufficient since Lemma 4.5 can be viewed as a special case.

PROOF [Theorem 4.14]: Let  $\gamma \in (-\infty, 1]$  be given. If markets are approximately complete it follows that the generalized portfolio providing maximal utility to the power-, or log-utility investor is given by equation (32) and generalized locally optimal portfolios must satisfy (31). Note that by Assumption 4.10 both processes must be locally bounded and hence they are special semimartingales. By uniqueness of the semimartingale decomposition we can match the martingale jump and diffusion terms respectively. Matching the diffusion terms implies that  $\alpha = 1 - \gamma$ . Matching the jump terms imply that

$$(1 - \gamma) \psi_\theta(v, t) = \frac{1 - (1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}}}{(1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}}}$$

for  $\phi(dv, t) \otimes dt$  almost every  $(v, t)$ . Clearly  $\psi_\theta(v, t) = 0$  is sufficient. To see that this is the only solution rewrite the equation as

$$((1 - \gamma) \psi_\theta(v, t) + 1) (1 - \psi_\theta(v, t))^{\frac{1}{\gamma-1}} = 1.$$

By differentiation we see that the left hand side is strictly increasing in  $\psi_\theta$  and consequently  $\psi_\theta = 0$  is the only solution. □

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