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A CONTROL VARIATE METHOD FOR MONTE CARLO SIMULATIONS OF HEATH-JARROW-MORTON MODELS WITH JUMPS

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Abstract. This paper examines the pricing of interest rate derivatives when the interest rate dynamics experience infrequent jump shocks modelled as a Poisson process and within the Markovian HJM framework developed in Chiarella & Nikitopoulos (2003). Closed form solutions for the price of a bond option under deterministic volatility specifications are derived and a control variate numerical method is developed under a more general state dependent volatility structure, a case in which closed form solutions are generally not possible. In doing so, we provide a novel perspective on the control variate methods by going outside a given complex model to a simpler more tractable setting to provide the control variates.

Keywords: HJM model, jump process, bond option prices, control variate, Monte Carlo simulations.
JEL Classification: E43, G33, G13.

1. INTRODUCTION

Interest rate derivatives are securities, the payoffs of which depend in some way on the level of interest rates. The value of an interest-rate option is substantially affected by the presence of skewness and kurtosis in the interest rates. The kurtosis explains the smile effect[1] and results in fat-tailed distributions. The skewness results in asymmetric interest rate distributions that match with the empirically observed distributional profile of the interest rates. Jump-diffusion and stochastic volatility models demonstrate an ability to accommodate these features, providing a modelling setting which explicitly incorporates

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[1] The shape of the implied volatilities (extracted from traded option prices by inverting the Black & Scholes (1973) or Black (1976) option pricing formula, whichever is applicable) for a range of different strikes is called the smile. The smile implies that at-the-money options trade at lower volatilities while the options away from the money trade at higher volatilities.

tail risk to more accurately reflect reality. However, these classes of models come at the expense of an increasing complexity that makes it impossible in most cases to derive closed form or computationally tractable solutions for derivative prices.

Most of the interest rate models under jump-diffusions do not readily admit closed form solutions for derivative prices even when the jump sizes are constant or drawn from well known distributions such as normal and log-normal. Therefore, most of the studies in this area use numerical approximation methods to evaluate interest rate instruments, including those of Ahn & Thompson (1988), Ahn (1988), Mercurio & Runggaldier (1993), Naik & Lee (1995), Baz & Das (1996) and Das (1999). Bates (1996), Duffie & Kan (1996) and Chacko & Das (2002) have considered more advanced models of stochastic volatility with jumps. The work of Jamshidian (1989), Shirakawa (1991), Heston (1993) (stochastic volatility setting), Das & Foresi (1996) and Glasserman & Kou (2003) provides closed form evaluation formulas for bonds and bond options in more specialised cases.

This paper presents two classes of term structure models that incorporate jump behavior of interest rates and more general volatility specifications but also maintain tractability in the pricing of interest rate derivatives. More specifically, we derive closed form solutions for bond options under deterministic volatility specifications and a numerical solution under the more general stochastic volatility case, which is, however, numerically tractable and efficient due to the fact that the term structure model developed admits finite dimensional Markovian representations. The Markovianisation of the jump-diffusion version of the HJM model employed here, even under state dependent volatility specifications, has been achieved by a suitable choice of volatility functions, as explained in Chiarella & Nikitopoulos (2003).

For the deterministic volatility set-up, we consider a parameterisation of the Shirakawa (1991) model of the term structure of interest rates under jump-diffusions. Under an appropriate equivalent probability measure, we consider option pricing within this framework. We use Fourier transform techniques to obtain a representation of the solution. A tractable Black-Scholes type pricing formula is derived under the assumption of a constant jump volatility function.

An extension of the Shirakawa (1991) framework is also considered, in which the volatility evolves stochastically, by means of a volatility function dependent on the state variables of the system. Again under an appropriate equivalent probability measure, we study the pricing of bond options. In this case, however, closed form valuation formulas are not available. Taking the state dependent volatility specifications of the type discussed in Chiarella & Nikitopoulos (2003), the interest rate dynamics become Markovian in a finite dimensional state variable and thus all the quantities involved such as forward rates or bond prices can be expressed in terms of this state variable. Taking advantage of these Markovian representations, we employ these particular Markovian structures to obtain approximate bond option prices by use of the Monte Carlo method. We further improve the efficiency of the Monte Carlo method by using a control variate technique, that makes use of the closed form solutions obtained in the deterministic volatility setting.

This paper is structured as follows. In Section 2 we develop the deterministic volatility model. We solve the bond option pricing equation using Fourier Transform techniques and we obtain closed form solutions for European bond options under constant jump volatility specifications. In Section 3 the state dependent volatility model is considered and the volatility restrictions that lead to Markovian term structures are discussed. Section 4 deals with the numerical implementation of the two models developed. We test the accuracy of the Monte Carlo results in the deterministic volatility model, since closed form solutions are available in this case. In addition, we numerically evaluate bond options under the stochastic volatility model. Finally by combining both models and closed form solutions, we develop a control variate method that significantly reduces computational effort and improves accuracy. Section 5 concludes and provides future directions for research.

2. The Deterministic Volatility Model

We denote as $f(t,T)$ the instantaneous forward interest rate at time t for instantaneous borrowing at time $T(\geqslant t)$. Then the price at time t of a discount zero-coupon bond with maturity T, denoted by $P(t,T)$, is defined as

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \tag{1}$$

so that $P(T,T)=1$. On the filtered probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$,[2] we assume that the dynamics of the instantaneous forward rate $f(t,T)$ are driven by both Gaussian and Poisson risk terms and given by

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n_w}\sigma_i(t,T)dW_i(t) + \sum_{i=1}^{n_p}\beta_i(t,T)[dQ_i(t) - \lambda_i dt], \tag{2}$$

where $\alpha : [0,T] \to R_+$ is the drift function, $W_i(t)$ are standard Wiener processes $(i = 1,2,\ldots,n_w)$, $\sigma_i : [0,T] \to R_+$ are time deterministic volatility functions associated with the Wiener noise processes, $Q_i(t)$ is a Poisson process with constant intensity λ_i $(i=1,2,\ldots,n_p)$, and $\beta_i : [0,T] \to R_+$ are time deterministic volatility functions associated with the Poisson noise processes. The Poisson process Q_i is employed to model the arrival time of the jump events. Thus, the jump feature is modelled by a multivariate point process, allowing for a finite number of jumps.[3]

A number of empirical studies, including Chiarella & Tô (2003), suggest that different types of shocks impact differently across the forward curve. The specification in equation (2) generalises slightly the Shirakawa (1991) volatility structure by allowing the Gaussian noise and Poisson noise to have separate volatility structures.

In stochastic integral equation form, equation (2) may be written as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{n_w}\int_0^t \sigma_i(s,T)dW_i(s) + \sum_{i=1}^{n_p}\int_0^t \beta_i(s,T)[dQ_i(s) - \lambda_i ds]. \tag{3}$$

Thus the stochastic integral equation for the instantaneous spot rate $r(t)$ is expressed as

$$r(t) \equiv f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \sum_{i=1}^{n_w}\int_0^t \sigma_i(s,t)dW_i(s) + \sum_{i=1}^{n_p}\int_0^t \beta_i(s,t)[dQ_i(s) - \lambda_i ds], \tag{4}$$

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[2] In more formal notation we assume that $(\Omega,\ \mathcal{F},\ (\mathcal{F})_{0\leq t\leq T},\ \mathbb{P})$ is the probability space equipped with the natural filtration of a vector of standard Brownian motions $W_i(t)$ $(i=1,2,\ldots,n_w)$ and the Poisson processes $Q_i(t)$ with intensity λ_i $(i=1,2,\ldots,n_p)$, indexed on the time interval $[0,T]$.

[3] Runggaldier (2002) provides a good survey of jump-diffusion models.

and the corresponding stochastic differential equation is

$$dr(t) = \vartheta(t)dt + \sum_{i=1}^{n_w} \sigma_i(t,t)dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t,t)[dQ_i(t) - \lambda_i dt], \tag{5}$$

where $\vartheta(t)$ is defined as

$$\begin{aligned}\vartheta(t) =& \frac{\partial}{\partial t} f(0,t) + \alpha(t,t) + \int_0^t \frac{\partial}{\partial t}\alpha(s,t)ds \\ &+ \sum_{i=1}^{n_w} \int_0^t \frac{\partial}{\partial t}\sigma_i(s,t)dW_i(s) + \sum_{i=1}^{n_p} \int_0^t \frac{\partial}{\partial t}\beta_i(s,t)[dQ_i(s) - \lambda_i ds].\end{aligned} \tag{6}$$

The corresponding dynamics for the bond price are[4]

$$\frac{dP(t,T)}{P(t^-,T)} = [r(t) + H(t,T)]dt - \sum_{i=1}^{n_w} \zeta_i(t,T)dW_i(t) + \sum_{i=1}^{n_p}(e^{-\xi_i(t,T)} - 1)dQ_i(t), \tag{7}$$

where

$$\zeta_i(t,T) \equiv \int_t^T \sigma_i(t,u)du, \tag{8}$$

$$\xi_i(t,T) \equiv \int_t^T \beta_i(t,u)du, \tag{9}$$

$$H(t,T) \equiv -\int_t^T \alpha(t,u)du + \sum_{i=1}^{n_w} \frac{1}{2}\zeta_i^2(t,T) + \sum_{i=1}^{n_p} \lambda_i \xi_i(t,T). \tag{10}$$

Consider a European call option of maturity T_C written on a bond having maturity T ($T > T_C$) and denote by $C = C(r,t,T_C)$ the value of this bond option at time t.

Taking into account that the dynamics of the spot rate are given by (5) and using the jump-diffusion version of Ito's lemma we derive the stochastic differential equation for the bond option price as

$$\begin{aligned}dC =& \left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t,t)\lambda_i\right)\frac{\partial C}{\partial r} + \frac{1}{2}\sum_{i=1}^{n_w} \sigma_i^2(t,t)\frac{\partial^2 C}{\partial r^2}\right)dt \\ &+ \sum_{i=1}^{n_w} \sigma_i(t,t)\frac{\partial C}{\partial r}dW_i(t) + \sum_{i=1}^{n_p}[C(r + \beta_i(t,t), t, T_C) - C(r,t,T_C)]dQ_i(t).\end{aligned} \tag{11}$$

In the next section, we develop the classical hedging portfolio argument in the bond option market, in the spirit of the original Black-Scholes hedging approach, to derive the bond option pricing partial differential-difference equation.

2.1. **Hedging Argument in the Bond Option Market.** In this bond option market, given $n_w + n_p$ sources of risk, n_w due to the Gaussian processes $W_i(t)$ and n_p due to the Poisson processes Q_i, we consider a hedging portfolio containing a bond with maturity T and $n_o = n_w + n_p$ bond options of maturities $T_1, T_2, \cdots, T_{n_o}$. All these options are written on the same bond having maturity T. By taking an appropriate position in bonds and bond options, it is possible to eliminate both Gaussian and Poisson risks. The condition that the riskless hedged portfolio earns the risk-free rate of interest $r(t)$, implies

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[4] See Björk, Kabanov & Runggaldier (1997) for details of the manipulations.

that there must exist[5] a vector $\mathbf{\Phi} = (\phi_1, \ldots, \phi_{n_w})^\top$ and a vector $\mathbf{\Psi} = (\psi_1, \ldots, \psi_{n_p})^\top$ such that for bond options of any maturity T_C it must be the case that[6]

$$\begin{aligned}\frac{\partial C}{\partial t} + (\vartheta(t) - \sum_{i=1}^{n_p} \beta_i(t,t)\lambda_i + \sum_{i=1}^{n_w} \phi_i(t)\sigma_i(t,t))\frac{\partial C}{\partial r} + \frac{1}{2}\sum_{i=1}^{n_w} \sigma_i^2(t,t)\frac{\partial^2 C}{\partial r^2} - rC \\ + \sum_{i=1}^{n_p} \psi_i(t)[C(r + \beta_i(t,t), t) - C(r,t)] = 0.\end{aligned} \tag{12}$$

Also as a result of the no-riskless arbitrage condition, the following drift restriction holds

$$\alpha(t,T) = \sum_{i=1}^{n_w} \sigma_i(t,T)(-\phi_i(t) + \zeta_i(t,T)) - \sum_{i=1}^{n_p} \beta_i(t,T)(\psi_i(t)e^{-\xi_i(t,T)} - \lambda_i). \tag{13}$$

Equation (12) is the partial differential-difference equation for the bond option price that is solved over $0 \leq t \leq T_c$ and under boundary conditions appropriate to the type of option being evaluated. The boundary conditions in the case of a call bond option price with exercise price E are

$$C(r(T_C), T_C, T_C) = (P(r(T_C), T_C, T) - E)^+, \tag{14}$$

and

$$C(\infty, t, T_C) = 0,$$

as a result of the condition on the bond price that $P(\infty, t, T) = 0$.

Note that the gist of the argument is establishing no-arbitrage consistency among a set of instruments sufficient to complete the market. Thus we may derive the conditions (12) and (13) by using alternative portfolios, for instance a portfolio consisting of a bond option and n_o bonds.

In deriving the martingale representation of the bond option price, the money market account has been initially used as the numeraire. By changing the numeraire, the bond option pricing equation can be formulated within a framework similar to that used by Merton (1976) to evaluate stock options involving Gaussian-Poisson risk. Further the price of the zero-coupon bond with maturity T_C will be employed as the numeraire for bond option pricing.

For every fixed finite time horizon T, we can obtain a unique equivalent probability measure $\widetilde{\mathbb{P}}$, under which the $\widetilde{W}_i(t) = -\int_0^t \phi_i(s)ds + W_i(t)$ are standard Wiener processes and the Q_i are Poisson processes with intensity $\psi_i(t)$. Thus imposing the drift restriction (13) on equation (7), the dynamics for $P(t, T_C)$, the zero coupon bond maturing at bond option maturity, under $\widetilde{\mathbb{P}}$, are given by

$$\frac{dP(t,T_C)}{P(t^-,T_C)} = r(t)dt - \sum_{i=1}^{n_w} \zeta_i(t,T_C)d\widetilde{W}_i(t) - \sum_{i=1}^{n_p}(1 - e^{-\xi_i(t,T_C)})[dQ_i(t) - \psi_i(t)dt]. \tag{15}$$

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[5] Note that the underlying Gaussian and jump risks (dW_i, dQ_i) driving the option price dynamics are the same as those driving the bond price dynamics and the instantaneous spot rate dynamics, thus the market price of these risks will be the same as those in the bond hedging portfolio.

[6] See Appendix 1 for details on the development of the continuous hedging argument.

Using equations (11) and (12), the dynamics for $C(r,t,T_C)$ under $\widetilde{\mathbb{P}}$ are given by

$$\begin{aligned}\frac{dC}{C} =& r(t)dt + \frac{1}{C}\frac{\partial C}{\partial r}\sum_{i=1}^{n_w}\sigma_i(t,t)d\widetilde{W}_i(t)\\ &+\frac{1}{C}\sum_{i=1}^{n_p}[C(r+\beta_i(t,t),t,T_C)-C(r,t,T_C)][dQ_i(t)-\psi_i(t)dt].\end{aligned} \tag{16}$$

Define the relative option and bond prices (with respect to the price of a zero-coupon bond that has the same maturity T_C as the option) as

$$Y(t)=\frac{C(r,t,T_C)}{P(r,t,T_C)}, \tag{17}$$

and

$$X(t)=\frac{P(r,t,T)}{P(r,t,T_C)}, \tag{18}$$

respectively. An application of the jump-diffusion version of the Ito's lemma gives the dynamics for Y[7] as

$$\begin{aligned}\frac{dY}{Y} =& \sum_{i=1}^{n_w}\left(\zeta_i(t,T_C)+\frac{\sigma_{0i}}{C}\frac{\partial C}{\partial r}\right)[d\widetilde{W}_i(t)+\zeta_i(t,T_C)dt]\\ &+\sum_{i=1}^{n_p}\left(\frac{C(r+\beta_{0i},t,T_C)}{C(r,t,T_C)}e^{\xi_i(t,T_C)}-1\right)[dQ_i(t)-\psi_i(t)e^{-\xi_i(t,T_C)}dt],\end{aligned} \tag{19}$$

and the dynamics for X as

$$\begin{aligned}\frac{dX}{X} =& \sum_{i=1}^{n_w}(\zeta_i(t,T_C)-\zeta_i(t,T))[d\widetilde{W}_i(t)+\zeta_i(t,T_C)dt]\\ &+\sum_{i=1}^{n_p}(\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}-1)[dQ_i(t)-\psi_i(t)e^{-\xi_i(t,T_C)}dt].\end{aligned} \tag{20}$$

By an application of Girsanov's theorem, a new measure $\mathbb{P}^*$ may be found under which the new processes (specified here in increment form)

$$dW_i^*(t)=d\widetilde{W}_i(t)+\zeta_i(t,T_C)dt, \tag{21}$$

are standard Gaussian processes, and

$$dQ_i^*(t)=dQ_i(t)-\psi_i(t)e^{-\xi_i(t,T_C)}dt, \tag{22}$$

are Poisson processes associated with the intensity vector $\mathbf{\Psi}^* = (\psi_1(t)e^{-\xi_1(t,T_C)}, \psi_2(t)e^{-\xi_2(t,T_C)}, \dots \psi_n(t)e^{-\xi_n(t,T_C)})^\top$.

It follows from (19) and (20) that the relative option price Y and the relative bond price X are martingales under $\mathbb{P}^*$ and using the expectation operator $\mathbb{E}^*$ under this new measure, we may write

$$Y(t)=\mathbb{E}^*[Y(T_C)\mid\mathcal{F}_t], \tag{23}$$

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[7] See Appendix 2 for details.

where the Wiener processes $W_i^*(t)$ and the Poisson process $Q_i(t)^*$ with intensity $\mathbf{\Psi}^*$ generate the $\mathbb{P}$-augmentation of the filtration $\mathcal{F}_t$. By using the definition of $Y(t)$,[8] equation (23) expands to

$$\begin{aligned}\frac{C(r,t,T_C)}{P(r,t,T_C)} &= \mathbb{E}^*\left[\frac{C(r,T_C,T_C)}{P(r,T_C,T_C)} \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[(P(r,T_C,T)-E)^+ \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[(X(T_C)-E)^+ \mid \mathcal{F}_t\right].\end{aligned} \tag{24}$$

Therefore, the relative option price Y can be expressed as a function of the relative bond price X, i.e.,

$$Y(X,t) = \mathbb{E}^*\left[(X(T_C)-E)^+ \mid \mathcal{F}_t\right]. \tag{25}$$

The value of the adjusted option[9] $Y(X,t)$ is driven by the dynamics for X, which are given by equation (20). Given the assumption on the volatility function, this process reduces to a form that puts us essentially in the framework used by Merton to price stock options under a geometric jump-diffusion process, the only difference being that the coefficients of the stochastic differential equation are time dependent. Application of the Feynman-Kac Theorem for processes with jumps to equation (25) leads to the partial differential-difference equation

$$\begin{aligned}\frac{\partial Y}{\partial t} &+ \sum_{i=1}^{n_p}\left(e^{-\xi_i(t,T_C)} - e^{-\xi_i(t,T)}\right)\psi_i(t) X \frac{\partial Y}{\partial X} \\ &+ \frac{1}{2}\sum_{i=1}^{n_w}\left(\zeta_i(t,T_C)-\zeta_i(t,T)\right)^2 X^2 \frac{\partial^2 Y}{\partial X^2} \\ &+ \sum_{i=1}^{n_p}\psi_i(t) e^{-\xi_i(t,T_C)}\left(Y(X(t)\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}) - Y(X(t))\right) = 0,\end{aligned} \tag{26}$$

subject to the boundary condition

$$\lim_{t\to T_C} Y(X,t) = (X(T_C)-E)^+. \tag{27}$$

In the next section, a technique to solve the partial differential-difference equation (26) is proposed, by employing Fourier Transform methods, that will lead to a pricing formula for the bond option. The Fourier Transform provides a quite general framework for solving partial differential equations of financial economics, since it handles a variety of pricing frameworks such as the jump-diffusion setting or the American option problem.[10]

2.2. **Solution to the Option Pricing Equation by Fourier Transform Techniques.**

By changing the variable X to the logarithmic variable $Z = \ln X$ and defining the new function

$$\Upsilon(Z,t) = Y(e^Z,t)$$

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[8] Recall that $C(r,t,T_C)$ is the value of a European option written on a bond with maturity T thus $C(r,T_C,T_C) = (P(r,T_C,T)-E)^+$ and $P(r,T_C,T_C)=1$.

[9] The value of the option under the new T_C-forward measure is given by

$$C(t,T_C) = P(t,T_C)\mathbb{E}^*\left[(P(T_C,T)-E)^+ \mid \mathcal{F}_t\right].$$

[10] See for example Carr & Madan (1999) and Chiarella, Kucera & Ziogas (1999).

the partial differential-difference equation (26) becomes

$$\begin{aligned}
&\frac{\partial \Upsilon}{\partial t} + \left[\sum_{i=1}^{n_p} \psi_i(t)\left(e^{-\xi_i(t,T_C)} - e^{-\xi_i(t,T)}\right) - \frac{1}{2}\sum_{i=1}^{n_w}\left(\zeta_i(t,T_C) - \zeta_i(t,T)\right)^2\right]\frac{\partial \Upsilon}{\partial Z} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n_w}\left(\zeta_i(t,T_C) - \zeta_i(t,T)\right)^2 \frac{\partial^2 \Upsilon}{\partial Z^2} \\
&\quad + \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t,T_C)}\left(\Upsilon\left(Z(t) + \ln\left[\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}\right]\right) - \Upsilon(Z(t))\right) = 0,
\end{aligned} \tag{28}$$

subject to the boundary condition

$$\lim_{t \to T_C} \Upsilon(Z,t) = (e^{Z(T_C)} - E)^+. \tag{29}$$

Define the Fourier transform of the solution $\Upsilon = \Upsilon(Z,t)$ to the partial differential-difference equation (28) by

$$\overline{\Upsilon}(\omega,t) = \int_{-\infty}^{\infty} \Upsilon(Z,t) e^{-\mathbf{i}\omega Z} dZ, \tag{30}$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary number. By employing Fourier transform techniques, as Appendix 3 shows, the function $\overline{\Upsilon}(\omega,t)$ satisfies an ordinary differential equation with complex coefficients having solution

$$\overline{\Upsilon}(\omega,t) = \overline{\Upsilon}(\omega,T_C)\exp\left\{(T_C - t)\left(-\overline{c}(t,T_C) + \mathbf{i}\omega[\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)] - \frac{\omega^2}{2}\overline{\sigma}^2(t,T_C) + \overline{\xi}(\omega,t,T_C)\right)\right\}, \tag{31}$$

where

$$\overline{c}(t,T_C) = \frac{1}{T_C - t}\sum_{i=1}^{n_p}\int_t^{T_C} \psi_i(s) e^{-\xi_i(s,T_C)} ds, \tag{32}$$

$$\overline{v}(t,T_C) = \frac{1}{T_C - t}\sum_{i=1}^{n_p}\int_t^{T_C} \psi_i(s)\left(e^{-\xi_i(s,T_C)} - e^{-\xi_i(s,T)}\right) ds, \tag{33}$$

$$\overline{\sigma}^2(t,T_C) = \frac{1}{T_C - t}\int_t^{T_C}\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C) - \zeta_i(s,T)\right)^2 ds, \tag{34}$$

$$\overline{\xi}(\omega,t,T_C) = \frac{1}{T_C - t}\sum_{i=1}^{n_p}\int_t^{T_C} \psi_i(s) e^{-\xi_i(s,T_C)}\left(\frac{e^{-\xi_i(s,T)}}{e^{-\xi_i(s,T_C)}}\right)^{\mathbf{i}\omega} ds. \tag{35}$$

By the Fourier inversion theorem, we have that

$$\Upsilon(Z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Upsilon}(\omega,t) e^{\mathbf{i}\omega Z} d\omega. \tag{36}$$

Thus, by substituting (31) into (36) we obtain

$$\begin{aligned}
&\Upsilon(Z,t) \\
&= \frac{e^{-(T_C-t)\overline{c}(t,T_C)}}{2\pi}\int_{-\infty}^{\infty} \overline{\Upsilon}(\omega,T_C) e^{\mathbf{i}\omega([\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+Z) - \frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t) + \overline{\xi}(\omega,t,T_C)(T_C-t)} d\omega,
\end{aligned}$$

and by changing the variable Z back to the variable X (recall that $Z = \ln X$), we obtain

$$Y(X,t) = e^{-\overline{c}(t,T_C)(T_C-t)} \int_{-\infty}^{\infty} Y(e^Z, T_C)\mathcal{K}(Z,X,t)dZ, \tag{37}$$

where the kernel $\mathcal{K}$ is defined by

$$\mathcal{K}(Z,X,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathbf{i}\omega([\overline{\upsilon}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X - Z)-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)+\overline{\xi}(\omega,t,T_C)(T_C-t)}d\omega. \tag{38}$$

Thus, the value of a call bond option can be expressed as

$$C(r,t,T_C) = e^{-\overline{c}(t,T_C)(T_C-t)}P(r,t,T_C)\int_{\ln E}^{\infty}(e^Z - E)\mathcal{K}(Z,X,t)dZ. \tag{39}$$

Unlike the corresponding result in Merton's jump-diffusion stock option model, it does not seem possible to proceed further with (39) and obtain a closed form solution under the more general volatility specifications. This is apparently due to the term $\left(\frac{e^{-\xi_i(s,T)}}{e^{-\xi_i(s,T_C)}}\right)^{\mathbf{i}\omega}$ in the expression for $\overline{\xi}(\omega,t,T_C)$ in equation (38) for the kernel $\mathcal{K}$. Since in Merton's analysis the coefficients of his integro partial differential equation were not time varying this term reduces to 1 allow him to obtain closed form solutions.

2.2.1. *Constant Jump Volatility Case.* We will show in this section that restricting the model to constant jump volatilities will provide closed form solutions for the option price.

Assumption 2.1. *For $i = 1,\dots,n_p$, the deterministic Wiener volatility structure is of the form*

$$\sigma_i(s,t) = \sigma_{0i}(s)e^{-\int_s^t \kappa_{\sigma i}(u)du}, \tag{40}$$

and for $i = 1,\dots,n_w$, the Poisson volatility functions are of the form

$$\beta_i(s,t) = \beta_{0i}, \tag{41}$$

where β_{0i} are constant.

Under Assumption 2.1 the quantity $\overline{\xi}(\omega,t,T_C)$ in (35) simplifies to

$$\overline{\xi}(\omega,t,T_C) = \frac{1}{T_C - t}\sum_{i=1}^{n_p}\frac{\psi_i e^{-\beta_{0i}(T-T_C)\mathbf{i}\omega}}{\beta_{0i}}\left(1 - e^{-\beta_{0i}(T_C-t)}\right), \tag{42}$$

which simplifies further the term $e^{\overline{\xi}(\omega,t,T_C)(T_C-t)}$ and allows us to complete the inversion of the Fourier Transform.

Proposition 2.1. *Under the volatility specifications of Assumption 2.1, the value of the bond option is given by*

$$\begin{aligned}C(r,t,T_C) = e^{-\overline{c}(t,T_C)(T_C-t)}&\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\cdots\sum_{p_{n_p}=0}^{\infty}\frac{\varsigma_1^{p_1}}{p_1!}\frac{\varsigma_2^{p_2}}{p_2!}\cdots\frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!}\\ &\left[P(r,t,T)e^{\overline{\upsilon}(t,T_C)(T_C-t)+(\sum_{i=1}^{n_p}p_i\mu_i)}\Phi(d_1(\boldsymbol{p})) - EP(r,t,T_C)\Phi(d_2(\boldsymbol{p}))\right],\end{aligned} \tag{43}$$

where

$$d_1(\boldsymbol{p}) = \frac{\ln\frac{X}{E} + (p_1\mu_1 + p_2\mu_2 + \ldots + p_{n_p}\mu_{n_p}) + [\overline{\upsilon}(T_C,t) + \frac{1}{2}\overline{\sigma}^2(T_C,t)](T_C - t)}{\overline{\sigma}(T_C,t)\sqrt{T_C - t}}, \tag{44}$$

$$d_2(\boldsymbol{p}) = d_1(\boldsymbol{p}) - \overline{\sigma}(T_C, t)\sqrt{T_C - t}, \tag{45}$$

and the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt.$$

Proof. By an application of the Taylor expansion of $e^x = \sum_{p=0}^{\infty} \frac{x^p}{p!}$ we write

$$\begin{aligned} e^{\overline{\xi}(\omega,t,T_C)(T_C-t)} &= \prod_{i=1}^{n_p} \exp\left[\frac{\psi_i \left(1 - e^{-\beta_{0i}(T_C-t)}\right)}{\beta_{0i}} e^{-\beta_{0i}(T-T_C)\mathbf{i}\omega} \right] \\ &= \prod_{i=1}^{n_p} \sum_{p=0}^{\infty} \frac{\varsigma_i^p}{p!} e^{p\mu_i \mathbf{i}\omega} \\ &= \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\varsigma_1^{p_1}}{p_1!} \frac{\varsigma_2^{p_2}}{p_2!} \cdots \frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!} e^{(p_1\mu_1 + p_2\mu_2 + \ldots + p_{n_p}\mu_{n_p})\mathbf{i}\omega} \end{aligned} \tag{46}$$

where $\varsigma_i = \frac{\psi_i\left(1-e^{-\beta_{0i}(T_C-t)}\right)}{\beta_{0i}}$ and $\mu_i = -\beta_{0i}(T - T_C)$.
Substituting the expression (46) into the kernel function (38) and simplifying, we obtain

$$\begin{aligned} \mathcal{K}(Z, X, t) = \frac{1}{2\pi} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\varsigma_1^{p_1}}{p_1!} \frac{\varsigma_2^{p_2}}{p_2!} \cdots \frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ \int_{-\infty}^{\infty} e^{\mathbf{i}\omega([\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \ldots + p_{n_p}\mu_{n_p})) - \frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)} d\omega. \end{aligned} \tag{47}$$

By using the result

$$\int_{-\infty}^{\infty} e^{-q\omega - \varrho\omega^2} d\omega = \sqrt{\frac{\pi}{\varrho}}\, e^{\frac{q^2}{4\varrho}}, \tag{48}$$

we are able to evaluate the integral expression in (47), to derive

$$\begin{aligned} \mathcal{K}(Z, X, t) = \frac{1}{\sqrt{2\pi(T_C - t)}\overline{\sigma}(t, T_C)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\varsigma_1^{p_1}}{p_1!} \frac{\varsigma_2^{p_2}}{p_2!} \cdots \frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ \exp\left\{ -\frac{([\overline{v}(t, T_C) - \frac{1}{2}\overline{\sigma}^2(t, T_C)](T_C - t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \ldots + p_{n_p}\mu_{n_p}))^2}{2\overline{\sigma}^2(t, T_C)(T_C - t)} \right\}. \end{aligned} \tag{49}$$

Thus equation (37) reduces to

$$\begin{aligned} Y(X, t) = e^{-\overline{c}(t,T_C)(T_C-t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\varsigma_1^{p_1}}{p_1!} \frac{\varsigma_2^{p_2}}{p_2!} \cdots \frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ \int_{\ln E}^{\infty} \frac{e^Z - E}{\sqrt{2\pi(T_C - t)}\overline{\sigma}(t, T_C)} e^{-\frac{([\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t) + \ln X - Z + (p_1\mu_1 + p_2\mu_2 + \ldots + p_{n_p}\mu_{n_p}))^2}{2\overline{\sigma}^2(t,T_C)(T_C-t)}} dZ. \end{aligned}$$

Further by evaluating separately the integrals[11] in the last expression and using the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt,$$

we obtain

$$\begin{aligned} Y(X,t) = e^{-\overline{c}(t,T_C)(T_C-t)} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \cdots \sum_{p_{n_p}=0}^{\infty} \frac{\varsigma_1^{p_1}}{p_1!} \frac{\varsigma_2^{p_2}}{p_2!} \cdots \frac{\varsigma_{n_p}^{p_{n_p}}}{p_{n_p}!} \\ [X e^{\overline{v}(t,T_C)(T_C-t)+(p_1\mu_1+p_2\mu_2+\ldots+p_{n_p}\mu_{n_p})} \Phi(d_1(\mathbf{p})) - E\Phi(d_2(\mathbf{p}))], \end{aligned} \tag{50}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_{n_p})$. By recalling the definitions (17) and (18) of Y and X, we derive the result (43). □

The closed form bond option pricing formula derived is in the spirit of Shirakawa's (1991) closed form bond option evaluation results, in which the Poisson risk was assumed to be binomial, however here we provide a pricing formula allowing for multi-factor Poisson risk.

2.3. **Markovian Spot Rate Dynamics under a Deterministic Volatility Structure.**

Within the jump-diffusion framework and under particular volatility specifications, the above term structure model admits a finite dimensional Markovian representation. These results are obtained and presented in Nikitopoulos (2005), however in this section, we summarise the main results obtained under deterministic Wiener volatilities and constant jump volatilities, which are the volatility specifications that lead to a closed form solution for the bond option price as we have seen in the previous section.

Substitution of the condition (13) into (4), leads to the spot rate dynamics under the risk neutral measure, which are of the form

$$\begin{aligned} r(t) = f(0,t) + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s,t)\zeta_i(s,t) ds + \sum_{i=1}^{n_p} \int_0^t \psi_i(s)\beta_i(s,t)[1 - e^{-\xi_i(s,t)}] ds \\ + \sum_{i=1}^{n_w} \int_0^t \sigma_i(s,t) d\widetilde{W}_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s,t)[dQ_i(s) - \psi_i(s)ds]. \end{aligned} \tag{51}$$

Assume the case of deterministic volatility specifications with constant jump volatilities described in Assumption 2.1. Using results from Nikitopoulos (2005) and Chiarella & Nikitopoulos (2003) Markovian spot rate dynamics can be obtained under these volatility specifications. For volatility specifications satisfying Assumption 2.1, the dynamics for the spot rate (51) can be expressed in terms of a number of Markovian stochastic quantities, (See Proposition 2.3.1. of Nikitopoulos (2005) for details) as

$$\begin{aligned} dr(t) = \left[D(t) - \sum_{i=2}^{n_w} \hat{\kappa}_{\sigma i}(t)\mathcal{D}_{\sigma i}(t) + \sum_{i=1}^{n_p} \kappa_{\sigma 1}(t)\mathcal{D}_{\beta i}(t) - \kappa_{\sigma 1}(t) r(t) \right] dt \\ + \sum_{i=1}^{n_w} \sigma_{0i}(t) d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i} \left[dQ_i(t) - \psi_i(t)dt \right], \end{aligned} \tag{52}$$

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[11] See Appendix 4 for detailed evaluation of the integrals.

where

$$D(t) = \kappa_{\sigma 1}(t) f(0,t) + \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta i}(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma i}(t), \tag{53}$$

$$\hat{\kappa}_{\sigma i}(t) = \kappa_{\sigma i}(t) - \kappa_{\sigma 1}(t), \tag{54}$$

and

$$\mathcal{E}_{\sigma i}(t) = \int_0^t \sigma_i^2(s,t) ds, \tag{55}$$

$$\mathcal{E}_{\beta i}(t) = \int_0^t \psi_i(s) \beta_{0i}^2 e^{-\beta_{0i}(t-s)} ds, \tag{56}$$

$$\mathcal{D}_{\sigma i}(t) = \int_0^t \sigma_i(s,t) \zeta_i(s,t) ds + \int_0^t \sigma_i(s,t) d\widetilde{W}_i(s), \tag{57}$$

$$\mathcal{D}_{\beta i}(t) = \int_0^t \psi_i(s) \beta_{0i} [1 - e^{-\beta_{0i}(t-s)}] ds + \int_0^t \beta_{0i} (dQ_i(s) - \psi_i(s) ds). \tag{58}$$

As the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ display Markovian dynamics, the instantaneous spot rate dynamics (52) are Markovian under the forward rate volatility specifications of Assumption 2.1. The corresponding multi-factor bond price formula in terms of $r(t)$ and the stochastic quantities $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ assumes the multi-factor exponential affine representation (See Proposition 2.3.3. of Nikitopoulos (2005) for details)

$$\begin{aligned} P(t,T) = & \frac{P(0,T)}{P(0,t)} \exp \Bigg\{ \mathcal{M}(t,T) - \mathcal{N}_{\sigma 1}(t,T) r(t) - \sum_{i=2}^{n_w} (\mathcal{N}_{\sigma i}(t,T) - \mathcal{N}_{\sigma 1}(t,T)) \mathcal{D}_{\sigma i}(t) \\ & - \sum_{i=1}^{n_p} ((T-t) - \mathcal{N}_{\sigma 1}(t,T)) \mathcal{D}_{\beta i}(t) \Bigg\}, \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathcal{M}(t,T) = & \mathcal{N}_{\sigma 1}(t,T) f(0,t) - \frac{1}{2} \sum_{i=1}^{n_w} \mathcal{N}_{\sigma i}^2(t,T) \mathcal{E}_{\sigma i}(t) \\ & - \sum_{i=1}^{n_p} \int_0^t \int_t^T \psi_i(s) \beta_{0i} [1 - e^{-\beta_{0i}(y-s)}] dy ds + \sum_{i=1}^{n_p} (T-t) \int_0^t \psi_i(s) \beta_{0i} [1 - e^{-\beta_{0i}(t-s)}] ds, \end{aligned} \tag{60}$$

and

$$\mathcal{N}_{\sigma_i}(t,T) \equiv \int_t^T e^{-\int_t^y \kappa_{\sigma_i}(u) du} dy. \tag{61}$$

These Markovian representations of the jump-diffusion version of the Hull & White (1990) type model developed in this section, will be used in Section 4 where the simulated bond option prices are compared to the closed form solution (43) for the bond option price obtained in Section 2.2.1.

3. The State Dependent Volatility Model

The state dependent forward rate volatility structure is specified by allowing the Wiener volatilities to depend on $\bar{f}(t) = (r(t), f(t,T_1), f(t,T_2), \dots, f(t,T_{\bar{n}_s}))^\top$, a vector of state dependent variables including the instantaneous spot rate and instantaneous forward rates of different fixed maturities. This specific volatility structure is determined by the following assumption

Assumption 3.1. *For $i = 1, \dots, n_p$, the state dependent Wiener volatility structure is of the form*

$$\sigma_i(s,t,\bar{f}(s)) = \sigma_{0i}(s,\bar{f}(s))e^{-\int_s^t \kappa_{\sigma i}(u)du}, \tag{62}$$

and for $i = 1, \dots, n_w$, the time dependent Poisson volatility functions continue to be of the form,

$$\beta_i(s,t) = \beta_{0i}(s)e^{-\int_s^t \kappa_{\beta i}(u)du},$$

where $\kappa_{\sigma i}(t)$, $\kappa_{\beta i}(t)$ and $\beta_{0i}(t)$ are deterministic functions of time and $\sigma_{0i}(t,\bar{f}(t))$ are time and state dependent functions.

It has been shown in Nikitopoulos (2005) that the above volatility specifications allow for Markovian representations of the term structure model.

Consider a European call option of maturity T_C written on a bond having maturity T ($T > T_C$) and denote by $C = C(\bar{f}(t), t, T_C)$ the value of this bond option at time t. Now however the underlying variables include not only $r(t)$ but also a number of forward rates of fixed maturities.

For notational convenience set $f_i = f(t,T_i)$, with $i = 1, 2, \dots, \bar{n}_s$. Thus using the jump-diffusion version of Ito's lemma to derive the stochastic differential equation for the bond option price, we have to take into account the dynamics of all the underlying factors, namely (2) for the $f(t,T_i)$, ($i = 1, 2, \dots, \bar{n}_s$) and (5) for the $r(t)$. However the Wiener volatilities now depend on the vector $\bar{f}(t)$. The multi-dimensional version of Ito's lemma is applied to derive

$$dC = \left(\frac{\partial C}{\partial t} + \mathrm{K}C\right)dt + \sum_{i=1}^{n_w} \mathrm{D}_i C dW_i(t) + \sum_{i=1}^{n_p} \mathrm{J}_i C dQ_i(t), \tag{63}$$

where

$$\begin{aligned}
\mathrm{K}C \equiv{}& \left(\vartheta(t) - \sum_{i=1}^{n_p}\beta_i(t,t)\lambda_i\right)\frac{\partial C}{\partial r} + \sum_{j=1}^{\bar{n}_s}(\alpha(t,T_j) - \sum_{i=1}^{n_p}\beta_i(t,T_j)\lambda_i)\frac{\partial C}{\partial f_j} \\
&+ \frac{1}{2}\frac{\partial^2 C}{\partial r^2}\sum_{i=1}^{n_w}\sigma_i^2(t,t,\bar{f}(t)) + \frac{1}{2}\sum_{j=1}^{\bar{n}_s}\frac{\partial^2 C}{\partial r \partial f_j}\sum_{i=1}^{n_w}\sigma_i(t,t,\bar{f}(t))\sigma_i(t,T_j,\bar{f}(t)) \\
&+ \frac{1}{2}\sum_{k=1}^{\bar{n}_s}\sum_{j=1}^{\bar{n}_s}\frac{\partial^2 C}{\partial f_k \partial f_j}\sum_{i=1}^{n_w}\sigma_i(t,T_k,\bar{f}(t))\sigma_i(t,T_j,\bar{f}(t)),
\end{aligned} \tag{64}$$

$$\mathrm{D}_i C \equiv \sigma_i(t,t,\bar{f}(t))\frac{\partial C}{\partial r} + \sum_{j=1}^{\bar{n}_s}\sigma_i(t,T_j,\bar{f}(t))\frac{\partial C}{\partial f_j}, \tag{65}$$

for $i = 1, 2, \ldots, n_w$, and,

$$\mathrm{J}_i C = C(r + \beta_i(t,t), f_1 + \beta_i(t,T_1), \ldots, f_{\bar{n}_s} + \beta_i(t, T_{\bar{n}_s}), t, T_C) - C(\bar{f}(t), t, T_C), \qquad (66)$$

for $i = 1, 2, \ldots, n_p$.

Developing the continuous hedging argument in a bond option market and imposing the condition that the riskless hedge portfolio earns the risk-free rate of interest $r(t)$, it follows that there exist a vector $\mathbf{\Phi} = (\phi_1, \ldots, \phi_{n_w})^\top$ and a vector $\mathbf{\Psi} = (\psi_1, \ldots, \psi_{n_p})^\top$ such that for bond options of any maturity T_C it must be the case that

$$\frac{\partial C}{\partial t} + \mathrm{K}C + \sum_{i=1}^{n_w} \phi_i(t)\mathrm{D}_i C + \sum_{i=1}^{n_p} \psi_i(t)\mathrm{J}_i C - rC = 0. \qquad (67)$$

Furthermore in the current context, the no-arbitrage forward rate drift restriction becomes

$$\alpha(t,T) = \sum_{i=1}^{n_w} \sigma_i(t,T,\bar{f}(t))(-\phi_i(t) + \zeta_i(t,T,\bar{f}(t))) - \sum_{i=1}^{n_p} \beta_i(t,T)(\psi_i(t)e^{-\xi_i(t,T)} - \lambda_i). \qquad (68)$$

As in Section 2, the partial differential equation (67) would need to be solved over $0 \leq t \leq T_c$ and under boundary conditions appropriate to the type of option being evaluated.

3.1. **Markovian Spot Rate Dynamics under a State Dependent Volatility Structure.**

Similarly as in Section 2.3, the spot rate dynamics under the risk neutral measure, are of the form

$$\begin{aligned} r(t) = f(0,t) &+ \sum_{i=1}^{n_w} \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,,\bar{f}(s))ds + \sum_{i=1}^{n_p} \int_0^t \psi_i(s)\beta_i(s,t)[1 - e^{-\xi_i(s,t)}]ds \\ &+ \sum_{i=1}^{n_w} \int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s) + \sum_{i=1}^{n_p} \int_0^t \beta_i(s,t)[dQ_i(s) - \psi_i(s)ds]. \qquad (69) \end{aligned}$$

Given the volatility structure consisting of state dependent Wiener volatility functions and deterministic Poisson volatility functions as expressed in Assumption 3.1, the spot rate dynamics (69) can be expressed in Markovian form, as stated in the following proposition.

Proposition 3.1. *Let $\sigma_i(s,t,\bar{f}(s))$ and $\beta_i(s,t)$, satisfy Assumption 3.1. Then the dynamics for the spot rate (69) can be expressed as*

$$\begin{aligned} dr(t) = &\left[\bar{D}(t) + \sum_{i=1}^{n_w} \mathcal{E}_{\sigma i}(t) - \sum_{i=2}^{n_w} \hat{\kappa}_{\sigma i}(t)\mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n_p} \hat{\kappa}_{\beta i}(t)\mathcal{D}_{\beta i}(t)) - k_{\sigma 1}(t)r(t)\right] dt \\ &+ \sum_{i=1}^{n_w} \sigma_{0i}(t,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{i=1}^{n_p} \beta_{0i}(t)\left[dQ_i(t) - \psi_i(t)dt\right], \qquad (70) \end{aligned}$$

where

$$\bar{D}(t) = \kappa_{\sigma 1}(t) f(0,t) + \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n_p} \mathcal{E}_{\beta i}(t), \tag{71}$$

$$\hat{\kappa}_{\sigma i}(t) = \kappa_{\sigma i}(t) - \kappa_{\sigma 1}(t), \tag{72}$$

$$\hat{\kappa}_{\beta i}(t) = \kappa_{\beta i}(t) - \kappa_{\sigma 1}(t), \tag{73}$$

and

$$\mathcal{E}_{\sigma i}(t) = \int_0^t \sigma_i^2(s,t,\bar{f}(s))ds, \tag{74}$$

$$\mathcal{E}_{\beta i}(t) = \int_0^t \psi_i(s)\beta_i^2(s,t)e^{-\xi_i(s,t)}ds, \tag{75}$$

$$\mathcal{D}_{\sigma i}(t) = \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds + \int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s), \tag{76}$$

$$\mathcal{D}_{\beta i}(t) = \int_0^t \psi_i(s)\beta_i(s,t)[1-e^{-\xi_i(s,t)}]ds + \int_0^t \beta_i(s,t)(dQ_i(s) - \psi_i(s)ds). \tag{77}$$

Proof. See Proposition 2.3.4 of Nikitopoulos (2005) for details. □

The corresponding multi-factor bond price formula in terms of $r(t)$ and the stochastic quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$ is given by

$$\begin{aligned} P(t,T) =& \frac{P(0,T)}{P(0,t)} \exp\Bigg\{ \bar{\mathcal{M}}(t,T) - \mathcal{N}_{\sigma_1}(t,T)r(t) - \frac{1}{2}\sum_{i=1}^{n_w} \mathcal{N}_{\sigma_i}^2(t,T)\mathcal{E}_{\sigma_i}(t) \\ & - \sum_{i=2}^{n_w}(\mathcal{N}_{\sigma i}(t,T) - \mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\sigma_i}(t) - \sum_{i=1}^{n_p}(\mathcal{N}_{\beta i}(t,T) - \mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\beta i}(t) \Bigg\}, \end{aligned} \tag{78}$$

where,

$$\begin{aligned} \bar{\mathcal{M}}(t,T) =& \mathcal{N}_{\sigma_1}(t,T)f(0,t) - \sum_{i=1}^{n_p}\int_0^t\int_t^T \psi_i(s)\beta_i(s,y)[1-e^{-\xi_i(s,y)}]dyds \\ & + \sum_{i=1}^{n_p}\mathcal{N}_{\beta i}(t,T)\int_0^t \psi_i(s)\beta_i(s,t)[1-e^{-\xi_i(s,t)}]ds, \end{aligned} \tag{79}$$

and

$$\mathcal{N}_x(t,T) \equiv \int_t^T e^{-\int_t^y \kappa_x(u)du}dy, \quad x \in \{\sigma_i, \beta_i\}. \tag{80}$$

Using the exponential affine term structure of interest rates (78), we can express the instantaneous forward rate in terms of $r(t)$ and the stochastic quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$, as

$$\begin{aligned} f(t,T) = f(0,T) - \frac{\partial\bar{\mathcal{M}}(t,T)}{\partial T} + \frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T} + \sum_{i=1}^{n_w}\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}\mathcal{N}_{\sigma i}(t,T)\,\mathcal{E}_{\sigma i}(t) \\ + \sum_{i=2}^{n_w}(\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T} - \frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T})\,\mathcal{D}_{\sigma i}(t) + \sum_{i=1}^{n_p}\left(\frac{\partial\mathcal{N}_{\beta i}(t,T)}{\partial T} - \frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\mathcal{D}_{\beta i}(t). \end{aligned} \tag{81}$$

The relationship (81) can be used to express the benchmark forward rates $f(t, T_j)$, with $j = 1, 2, \dots, \bar{n}_s$ of the state dependent volatility functions in terms of the stochastic state variables $r(t)$, $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta i}(t)$. In addition, by taking the number $\bar{n}_s$ of the fixed forward rate maturities used in the state dependent volatility structure $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), \dots, f(t, T_{\bar{n}_s}))^\top$, equal to the number of stochastic quantities $(2n_w + n_p)$, we have a fully specified system with the forward rates of any maturity evaluated by (81). See Section 2.4 of Nikitopoulos (2005) for more discussion on how finite dimensional affine realisations in terms of forward rates may be obtained. These Markovian representations of the jump-diffusion version of the state dependent term structure model developed here, will be used in Section 4 where the above Markovian term structure of interest rates is simulated in order to obtain bond option prices. In particular, we use the Markovian representation in terms of the stochastic quantities as in Proposition 3.1, rather than in terms of a set of benchmark forward rates.

4. Monte Carlo Simulations

Monte Carlo simulation for derivative pricing, when the underlying asset follows a multivariate state dependent volatility jump-diffusion process, is extremely intensive computationally, as the variance of the sampled variable is usually large and for N sample paths the standard errors of the Monte Carlo simulations decreases only as $1/\sqrt{N}$. To improve the Monte Carlo efficiency, one should employ some sort of variance reduction methodology, namely antithetic variable, control variates, stratified sampling and importance sampling and/or use low discrepancy sequences. A control variate technique was developed by Chiarella, Clewlow & Musti (2003) for a state dependent volatility HJM model when the forward rate dynamics are driven by diffusion processes. We extend this to accommodate our jump-diffusion setting, also taking advantage of the Markovian representations that have been obtained under the particular volatility specifications.

For the one Wiener/two Poisson case we examine two classes of models. The first one is the deterministic volatility (DV) model with volatilities

$$\sigma(t, T) = \sigma_0 e^{-\kappa_\sigma (T-t)}, \tag{82}$$

and

$$\beta_i(t, T) = \beta_{0i}, \quad \text{with } i = 1, 2. \tag{83}$$

This model yields closed form solutions of the form (43) for bond option prices.[12]

The second model is the state dependent volatility (SV) model,

$$\sigma(t, T, \bar{f}(t)) = \sigma_0(t, \bar{f}(t)) e^{-\kappa_\sigma (T-t)},$$

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[12] For the volatility specifications (82) and (83), the quantities (32), (33) and (34) simplify

$$\overline{c}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \frac{\psi_i}{\beta_{0i}} \left(1 - e^{-\beta_{0i}(T_C - t)}\right), \tag{84}$$

$$\overline{\upsilon}(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_p} \frac{\psi_i}{\beta_{0i}} \left(1 - e^{-\beta_{0i}(T_C - t)} + e^{-\beta_{0i}(T - t)} - e^{-\beta_{0i}(T - T_C)}\right), \tag{85}$$

$$\overline{\sigma}^2(t, T_C) = \frac{1}{T_C - t} \sum_{i=1}^{n_w} \frac{\sigma_0^2}{2\kappa_i^3} \left(e^{\kappa_i T_c} - e^{\kappa_i T}\right)^2 \left(e^{-2\kappa_i T} - e^{2\kappa_i t - 2\kappa_i T - 2\kappa_i T_c}\right). \tag{86}$$

where

$$\sigma_0(t,\bar{f}(t)) = \begin{cases} 0.05\,\sigma_0(t), & L_f(t) < 0.005; \\ \sigma_0(t)[(L_f(t)-0.005)^\gamma + 0.05], & L_f(t) \geq 0.005; \end{cases} \tag{87}$$

with $L_f(t) = c_0 r(t) + \sum_{h=1}^{3} c_h f(s,T_h)$ and $\gamma = \frac{1}{2}$. Also we consider $\beta_i(t,T) = \beta_{0i} e^{-k_{\beta i}(T-t)}$ and constant ψ_i. Recall that in the current setup $\bar{f}(t) = (r(t), f(t,T_1), f(t,T_2), f(t,T_3))^\top$. Also note that, since these Markovian structures may drive the forward rate to negative values, the state dependent volatility functions (87) have been selected so as to be well-defined in such a case (see Appendix 5).

4.1. **Simulation Scheme.** Let t be time, T be maturity, and $\mathbb{T}$ be the time horizon where $0 \leq t \leq T \leq \mathbb{T}$. The time horizon $(0,\mathbb{T})$ is subdivided into N intervals of length $\Delta t = \frac{\mathbb{T}}{N}$ so that $t = n\Delta t$ and $T = t + m\Delta t$. This scheme requires the knowledge of the initial forward curve $f(0,T)$. The initial forward rate curve considered here has the functional form $f(0,t) = (a_0 + a_1 t + a_2 t^2)\, e^{-vt}$ with parameters being estimated as $a_0 = 0.033287$, $a_1 = 0.014488$, $a_2 = -0.000117$, and $v = 0.0925$, which gives a reasonable fit to the US zero yields on July 20, 2001, with maturities up to 10 years and including the overnight rate.

4.2. **The initial bond price.** We recall that the initial bond price $P(0,T)$ is given by relationship

$$P(0,T) = \exp\left(-\int_0^T f(0,s)ds\right). \tag{88}$$

The bond price $P(t,T)$ can be also expressed in terms of a risk neutral expectation as

$$P(t,T) = \tilde{\mathbb{E}}\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right], \tag{89}$$

so that the initial bond price $P(0,T)$ may also be expressed as

$$P(0,T) = \tilde{\mathbb{E}}\left[\exp\left(-\int_0^T r(s)ds\right) \mid \mathcal{F}_0\right]. \tag{90}$$

The bond price estimated by performing simulations over Π paths, should coincide closely with the input initial bond price

$$P(0,T) \simeq \frac{1}{\Pi}\sum_{i=0}^{\Pi} \exp\left(-\sum_{j=0}^{N} r_i(j\Delta t)\Delta t\right). \tag{91}$$

σ_0	0.015
κ_σ	0.18
β_{01}	0.02
β_{02}	-0.03
ψ_1	1
ψ_2	1.5

Table 1. Parameter Values.

Equation (91) can be used to provide a check on the accuracy of our simulation schemes, in particular giving an indication of the size of the discretisation bias. Table 2 provides the simulated initial bond prices for the deterministic volatility models when the parameter values are set as in Table 1.

The discretised spot rate dynamics used in the simulation scheme are the Markovian dynamics presented in Section 2.3, recall equation (52).

$P(0,1) = \exp\left(-\int_0^1 f(0,s)ds\right) = 0.9381583$				
N	Π	P(0,1)	St. Dev.	St. Err.
200	5,000	0.938044	*0.000336*	0.34033
	50,000	0.938009	*0.000108*	1.37195
	500,000	0.937833	*0.000034*	8.81883
400	5,000	0.938262	*0.000341*	-0.30571
	50,000	0.938035	*0.000108*	1.13973
	500,000	0.937984	*0.000034*	5.10233
800	5,000	0.938087	*0.000336*	0.21052
	50,000	0.938086	*0.000108*	0.66752
	500,000	0.938094	*0.000034*	1.86966

TABLE 2. Initial Bond Prices - DV models.

Table 3 presents the simulated initial bond prices, of a bond maturing in 1 year, for the state dependent volatility models, and when the parameter values are set as in Table 1. In addition, we set $\kappa_{\beta 1} = 0.31$, $\kappa_{\beta 2} = 0.17$, $c_0 = 1$, $c_1 = 2$, $c_2 = 1$, $c_3 = 2$. The discretised spot rate dynamics are the Markovian dynamics described in Section 3.1, see in particular equation (70), with the benchmark forward rates expressed in terms of the stochastic factors of the system, by using equation (81).

$P(0,1) = \exp\left(-\int_0^1 f(0,s)ds\right) = 0.9381583$				
N	Π	P(0,1)	St. Dev.	St. Err.
200	5,000	0.938080	*0.000312*	0.25150
	50,000	0.937930	*0.000098*	2.32676
	500,000	0.937856	*0.000031*	9.28062
400	5,000	0.938430	*0.000310*	-0.87725
	50,000	0.938091	*0.000098*	0.68753
	500,000	0.937999	*0.000033*	4.86024
800	5,000	0.938161	*0.000311*	-0.00876
	50,000	0.938026	*0.000098*	1.34415
	500,000	0.938108	*0.000031*	1.62363

TABLE 3. Initial Bond Prices - SV models.

We consider discretisation error[13] to be evident when the distance between the true and simulated bond prices exceeds two standard deviations, i.e. when the true price lies outside the 95% confidence interval around the Monte Carlo estimate. In Table 2, this is the case for 500,000 paths or more, with the error appearing to be somewhat reduced when Δt is reduced as would be expected. In particular, discretisation error is no longer

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[13] Standard error is defined as the difference between exact price and simulated price divided by the standard deviation of the simulated prices.

evident at the current number of paths when the discretisation level is increased to 800. It is important to keep the magnitude of the error from this source in mind when interpreting the results from the simulations in the subsequent sections.

The initial bond price results obtained by the simulations for both models (deterministic volatility (DV) and stochastic volatility (SV) model) are consistent, to four decimal place accuracy, (especially when we reduce the discerisation bias by setting the discretisation level to $N = 800$) with the value obtained from the analytical bond price (88), providing evidence of the effectiveness of this numerical scheme.

4.3. **Bond Option Price Evaluation.** Denote with $C(t, T_c, \mathbb{T})$ the time t-value of a European call option maturing at T_c on the zero-coupon bond with maturity $\mathbb{T}$, where $0 \leq t \leq T_c \leq \mathbb{T}$. The current value of a European call option $C(0, T_c, \mathbb{T})$ can be evaluated, under the risk neutral measure as the expected discounted payoff of the option at the option's maturity

$$C(0, T_c, \mathbb{T}) = \widetilde{\mathbb{E}}\left[\exp\left\{-\int_0^{T_c} r(s)ds\right\}(P(T_c, \mathbb{T}) - E)^+ | \mathcal{F}_0\right], \tag{92}$$

or, alternatively, under the T_c-forward measure, as

$$C(0, T_c, \mathbb{T}) = P(0, \mathbb{T})\mathbb{E}^*\left[(P(T_c, \mathbb{T}) - E)^+ | \mathcal{F}_0\right]. \tag{93}$$

For simulation based approaches to bond option pricing, we have found that the use of one or the other probability measure does not seem to provide any significant advantage. Here we report the simulations under the risk neutral probability measure.

Given the Markovian spot rate dynamics under the risk neutral measure, which are equation (52) for the deterministic volatility models and equation (70) for the state dependent volatility models, the bond option price is evaluated using formula (92), by using the Euler-Maruyama scheme for the integration, as

$$C(0, T_c, \mathbb{T}) = \frac{1}{\mathrm{K}}\sum_{k=1}^{\mathrm{K}} \exp\left\{-\sum_{i=1}^{N} r_k(i\Delta t)\Delta t\right\}(P_k(T_c, \mathbb{T}) - E)^+. \tag{94}$$

The bond price $P_k(T_c, \mathbb{T})$ is computed by the exponential affine term structure (59) for the deterministic volatility model and (78) for the state dependent volatility model.

Table 4 shows the simulated bond option prices for the deterministic volatility model under the parameter values given in Table 1. The exercise price is set $E = 0.95$ and the exact value for the bond option price is evaluated from (43).

Exact Option Price		$C^{DV}_{exact}(0, 0.5, 1) = 0.018181443925$		
N	Π	$C^{DV}_{MC}(0, 0.5, 1)$	St. Dev.	St. Err.
200	5,000	0.018285	*0.000200*	-0.51624
	50,000	0.018121	*0.000063*	0.96650
	500,000	0.018157	*0.000020*	1.24335
400	5,000	0.018245	*0.000200*	-0.31898
	50,000	0.018179	*0.000063*	0.03544
	500,000	0.018180	*0.000020*	0.05745

TABLE 4. Call Bond Option Prices - DV models.

The bond option prices obtained by the Monte Carlo simulations for the deterministic volatility model are consistent with the value obtained from the analytical bond option price (43) with accuracy reaching three significant figures for the 50,000 simulated paths and over. This provides evidence that the numerical scheme employed here is effective.

Table 5 shows the simulated bond option prices for the stochastic volatility type of models under the parameter values shown in Table 1, and for $\kappa_{\beta 1} = 0.31$ and $\kappa_{\beta 2} = 0.17$. Recall that the Wiener state dependent volatilities have the functional form (87), with $c_0 = 1$, $c_1 = 2$, $c_2 = 1$ and $c_3 = 2$. The exercise price is set to be $E = 0.95$.

N	Π	$C^{SV}_{MC}(0, 0.5, 1)$	St. Dev.
200	5,000	0.022210	*0.000182*
	50,000	0.022240	*0.000058*
	500,000	0.022303	*0.000018*
400	5,000	0.022107	*0.000181*
	50,000	0.022414	*0.000058*
	500,000	0.022280	*0.000018*

TABLE 5. Call Bond Option Prices - SV models.

The bond option prices obtained by the Monte Carlo simulations for the stochastic volatility model are consistent to at least two significant figures, however in the next section we will attempt to improve convergence of the stochastic volatility numerical scheme by an application of a control variate method.

4.4. **Control Variate Method.** The application of Monte Carlo simulations to evaluate bond option prices under the HJM framework comes at the expense of significant computational effort. To improve convergence we propose to use a control variate method.

The DV model under deterministic Wiener volatilities and constant jump sizes accommodates a closed form option pricing formula and thus we can compute the option price C^{DV}_{exact}. The SV model (state dependent Wiener volatilities and deterministic jump volatilities) can only be evaluated numerically. Running simulations of these two models, the option prices C^{DV}_{MC} under the deterministic volatility model and the option prices C^{SV}_{MC} under the stochastic volatility model are estimated. The control variate adjustment proposes that the approximated option value of the stochastic volatility model is evaluated by

$$C^{SV} = C^{SV}_{MC} - C^{DV}_{MC} + C^{DV}_{exact}. \tag{95}$$

The rationale of the control variate method is that the known error imposed by the Monte Carlo simulations in the case of the deterministic volatility model

$$C^{DV}_{MC} - C^{DV}_{exact},$$

is assumed to be close to the error of the Monte Carlo estimation for the case of stochastic volatility model, namely

$$C^{SV}_{MC} - C^{SV}.$$

Evaluating (95) can be time consuming since it requires the results of two simulations. However, use of the Markovian representations of the models considered have considerably simplified and sped up the calculation. As Table 6 shows the standard errors of the option values estimated by the control variate method are of the order of approximately one seventh with respect to the values obtained by the standard Monte Carlo simulation of the Markovian stochastic volatility term structure model.[14] This reduction is uniform across the order of discretisation and the number of simulated paths. The accuracy on the bond option price has increased to three significant figures by the application of the control variate scheme compared to the two significant figures accuracy obtained by the application of the standard Monte Carlo simulation.

N	Π	$C^{SV}(0,0.5,1)$	St. Dev.	$C^{CV}(0,0.5,1)$	St. Dev.
200	5,000	0.022484	*0.000186*	0.022340	*0.000026*
	50,000	0.022220	*0.000058*	0.022347	*0.000008*
	500,000	0.022313	*0.000019*	0.022323	*0.000003*
400	5,000	0.022325	*0.000186*	0.022315	*0.000026*
	50,000	0.022299	*0.000059*	0.022325	*0.000008*
	500,000	0.022298	*0.000019*	0.022326	*0.000003*

Table 6. Call Bond Option Prices - SV models; Control Variate Method.

To justify the efficiency of the control variate method, we ensure firstly that

$$\mathbb{E}[C_{MC}^{DV} - C_{exact}^{DV}] = 0. \tag{96}$$

From Table 4, we conclude that condition (96) holds since insignificant discretisation error exists in particular when we increase the order of the discretisation to 400.

The control variate method is employed here to price the same product - bond options - under two different models. This is a somewhat unorthodox and - to our knowledge - new perspective on control variate methods in pricing derivatives. Typically a control variate method is applied to another (closely related) instrument priced in the same model, whereas here the control variate is the same instrument priced in a closely related model. Under these model specifications the state variables evolve differently. However, the state variables can be seen as simply two different sets of functions of the driving Wiener and Poisson processes, which are the same in both models. Therefore, the control variate method will be correctly used if the state variables of the two models considered are highly correlated. Table 7 presents the correlation coefficients of the state variables, which are common in these two classes of models, and these are the $r(t)$, $\mathcal{D}_{\beta 1}(t)$ and $\mathcal{D}_{\beta 2}(t)$. The state variables are clearly highly positively correlated, thus one can reasonably expect the control variate method to be effective in this context.

Thus, the combination of these Markovian structures with a control variate method provides an efficient numerical scheme that may yield good results in Monte Carlo simulation even with a relatively small number of simulated paths. Given that the control variate method improves the standard error by seven times relative to the standard error obtained from the standard Monte Carlo simulation on the stochastic volatility model, one may obtain the same order of accuracy with forty-nine times less the number of simulated paths. An important contribution to the efficiency of this numerical scheme must

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[14] Thus a back-of-an-envelope calculation indicates that computational efficiency can be gained by using the method proposed here: Control variates at most double the computational effort compared to the same number of simulations without control variates, whereas the number of simulations without control variates would have to be increased by a factor of $7^2 = 49$ in order to achieve the same accuracy as a given number of simulations with control variates.

N	Π	r(t)	$\mathcal{D}_{\beta 1}(t)$	$\mathcal{D}_{\beta 2}(t)$
200	5,000	0.995985	0.998992	0.999704
	50,000	0.995856	0.999007	0.999700
	500,000	0.995885	0.999002	0.999699
400	5,000	0.995788	0.998948	0.999712
	50,000	0.995895	0.998994	0.999705
	500,000	0.995902	0.998998	0.999699

Table 7. Correlation Coefficients.

be attributed to the fact that the models developed here possess Markovian dynamics. All the parameters used in the simulations such as bond prices, benchmark forward rates used in the volatility structure could be expressed in terms of the state variables of the Markovian system. Discretisation has only been applied to the dynamics of the state variables of the system, therefore a lot of numerical evaluations have been avoided.

5. Conclusions

This paper develops two models to price bond options when interest rates are subject to jumps. In the first model, both Wiener and Poisson volatilities are time dependent, and working within the Shirakawa general HJM model, we have derived the partial differential-difference equation for the pricing of bond options. In addition, by employing Fourier transform techniques, bond option prices have been evaluated and an easily tractable Black-Scholes type bond option pricing formula under the assumption of constant jump volatility has been derived. In the second model, the volatility structure is more general, by allowing for state dependent Wiener volatilities and time dependent Poisson volatilities. In this second model, it is difficult to explicitly solve the bond option pricing problem, therefore Monte Carlo simulation techniques are used to evaluate bond options. However, under appropriate volatility functions, the term structures obtained for both models display Markovian dynamics. These Markovian representations contribute to increase the efficiency and accuracy of the application of the Monte Carlo simulations. Additionally, taking advantage of the closed form solutions obtained under the deterministic volatility setting, we employ a control variate method that significantly improves the efficiency of the numerical procedure.

The important characteristic of the solutions for bond option prices proposed in this paper is that they incorporate the complexity of a stochastic volatility and/or jump-diffusion model although they enjoy computational tractability due to the Markovian structures used.

A worthwhile extension of this work would be to fit to empirical information to calibrate the model parameters as well as the volatility smile.

Appendix 1. The No-Arbitrage Condition in the Bond Option Market

Recall the stochastic differential equation for the spot rate

$$dr(t) = \vartheta(t)dt + \sum_{i=1}^{n_w} \sigma_i(t,t)dW_i(t) + \sum_{i=1}^{n_p} \beta_i(t,t)[dQ_i(t) - \lambda_i dt], \tag{97}$$

where $\vartheta(t)$ is defined as

$$\begin{aligned}\vartheta(t) =& \frac{\partial}{\partial t}f(0,t) + \alpha(t,t) + \int_0^t \frac{\partial}{\partial t}\alpha(s,t)ds \\ &+ \sum_{i=1}^{n_w}\int_0^t \frac{\partial}{\partial t}\sigma_i(s,t)dW_i(s) + \sum_{i=1}^{n_p}\int_0^t \frac{\partial}{\partial t}\beta_i(s,t)[dQ_i(s) - \lambda_i ds].\end{aligned} \tag{98}$$

Using the jump-diffusion version of Ito's lemma we derive the stochastic differential equation for the bond option price

$$\begin{aligned}dC =& \left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p}\beta_i(t,t)\lambda_i\right)\frac{\partial C}{\partial r} + \frac{1}{2}\sum_{i=1}^{n_w}\sigma_i^2(t,t)\frac{\partial^2 C}{\partial r^2}\right)dt \\ &+ \sum_{i=1}^{n_w}\sigma_i(t,t)\frac{\partial C}{\partial r}dW_i(t) + \sum_{i=1}^{n_p}[C(r+\beta_i(t,t),t,T_C) - C(r,t,T_C)]dQ_i(t),\end{aligned} \tag{99}$$

We consider a hedging portfolio containing a bond with maturity T and $n_o = n_w + n_p$ bond options of maturities $T_1,\ T_2,\ \cdots,\ T_{n_o}$ in proportions $w_1,\ w_2,\ \cdots,\ w_{n_o}$ with $w_1 + w_2 + \cdots + w_{n_o+1} = 1$, where w_{n_o+1} is the proportion corresponding to the bond. All these options are written on the bond having maturity T. If we denote with $C_i(t) = C(t,T_i)$ $(i = 1,2,\ldots,(n_o))$ the value of the i^{th} bond option, we may write the stochastic differential equation for $C_i(t)$ in the general form

$$\frac{dC_i(t)}{C_i(t)} = \mu_{C_i}(t)dt + \sum_{j=1}^{n_w}\nu_{C_{i,j}}(t)dW(t) + \sum_{j=1}^{n_p}\chi_{C_{i,j}}(t)dQ_j(t),$$

where

$$\begin{aligned}\mu_{C_i}(t) &= \frac{1}{C_i}\left(\frac{\partial C_i}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p}\beta_i(t,t)\lambda_i\right)\frac{\partial C_i}{\partial r} + \frac{1}{2}\frac{\partial^2 C_i}{\partial r^2}\sum_{j=1}^{n_w}\sigma_{ij}^2(t,t)\right), \\ \nu_{C_{i,j}}(t) &= \frac{1}{C_i}\frac{\partial C_i}{\partial r}\sum_{j=1}^{n_w}\sigma_{ij}(t,t), \\ \chi_{C_{i,j}}(t) &= \frac{1}{C_i}\sum_{j=1}^{n_p}[C_i(r+\beta_{ij}(t,t),t) - C_i(r,t)].\end{aligned}$$

Also recall the stochastic differential equation for the bond price P

$$\frac{dP(t)}{P(t)} = \mu_P(t)dt + \sum_{i=1}^{n_w}\nu_{P_i}(t)dW_i(t) + \sum_{i=1}^{n_p}\chi_{P_i}(t)dQ_i(t),$$

where

$$\mu_P(t) = r(t) + H(t,T),\ \nu_{P_i}(t) = -\zeta_i(t,T), \text{ and } \chi_{P_j}(t) = \eta_j(t,T) - 1.$$

Let V be the value of the hedging portfolio then the return on the portfolio is given by

$$\begin{aligned}\frac{dV}{V} &= w_1\frac{dC_1}{C_1} + w_2\frac{dC_2}{C_2} + \cdots + w_{n_o}\frac{dC_{n_o}}{C_{n_o}} + w_{n_o+1}\frac{dP}{P} \\ &= \sum_{i=1}^{n_o} w_i\mu_{C_i}dt + w_{n_o+1}\mu_P dt + \sum_{i=1}^{n_o} w_i \sum_{j=1}^{n_w} \nu_{C_{ij}} dW_j(t) + w_{n_o+1}\sum_{j=1}^{n_w}\nu_{P_j} dW_j(t) \\ &\quad + \sum_{i=1}^{n_o} w_i \sum_{j=1}^{n_p} \chi_{C_{i,j}} dQ_j(t)) + w_{n_o+1}\sum_{j=1}^{n_p}\chi_{P_j} dQ_j(t).\end{aligned}$$

In order to eliminate both Gausian and Poisson risks we need to choose w_1, w_2, $\cdots$, w_{n_o+1} so that

$$\sum_{i=1}^{n_o} w_i\nu_{C_{i,j}} + w_{n_o+1}\nu_{P_j} = 0, \text{ when } j = 1, 2, \ldots, n_w \tag{100}$$

$$\sum_{i=1}^{n_o} w_i\chi_{C_{i,j}} + w_{n_o+1}\chi_{P_j} = 0, \text{ when } j = 1, 2, \ldots, n_p. \tag{101}$$

The hedging portfolio then becomes riskless, thus, it should earn the risk-free rate of interest $r(t)$, of the Gaussian bond market, i.e.,

$$\frac{dV}{V} = \sum_{i=1}^{n_o} w_i\mu_{C_i}dt + w_{n_o+1}\mu_P dt = r(t)dt,$$

which can be simplified to

$$\sum_{i=1}^{n_o} w_i(\mu_{C_i} - r(t)) + w_{n_o+1}(\mu_P - r(t)) = 0, \tag{102}$$

using also the fact that $w_1 + w_2 + \cdots + w_{n_o+1} = 1$. Equations (100), (101) and (102) form a system of $(n_o + 1)$ equations with $(n_o + 1)$ unknowns w_1, w_2, $\cdots$, w_{n_o+1}. This system can only have a non-zero solution if the determinant

$$\begin{vmatrix} \nu_{C_{1,1}}(t) & \nu_{C_{2,1}}(t) & \cdots & \nu_{C_{n_o,1}}(t) & \nu_{P_1}(t) \\ \vdots & \vdots & \vdots & \vdots & \\ \nu_{C_{1,n_w}}(t) & \nu_{C_{2,n_w}}(t) & \cdots & \nu_{C_{n_o,n_w}}(t) & \nu_{P_{n_w}}(t) \\ \chi_{C_{1,1}}(t) & \chi_{C_{2,1}}(t) & \cdots & \chi_{C_{n_o,1}}(t) & \chi_{P_1}(t) \\ \vdots & \vdots & \vdots & \vdots & \\ \chi_{C_{1,n_p}}(t) & \chi_{C_{2,n_p}}(t) & \cdots & \chi_{C_{n_o,n_p}}(t) & \chi_{P_{n_p}}(t) \\ \mu_{C_1}(t) - r(t) & \mu_{C_2}(t) - r(t) & \cdots & \mu_{C_{n_o}}(t) - r(t) & \mu_P(t) - r(t) \end{vmatrix}$$

is equal to zero. The above equation implies that for $h = 1, 2, \ldots, n_o$ there exist $\phi_1(t)$, $\phi_2(t)$,..., $\psi_{n_w}(t)$ and $\psi_1(t)$, $\psi_2(t)$, ..., $\psi_{n_p}(t)$ such that

$$\mu_{C_i}(t) - r(t) = -\sum_{j=1}^{n_w}\phi_j(t)\nu_{C_{i,j}}(t) - \sum_{j=1}^{n_p}\psi_j(t)\chi_{C_{i,j}}(t), \tag{103}$$

and

$$\mu_P(t) - r(t) = -\sum_{i=1}^{n_w}\phi_i(t)\nu_{P_i}(t) - \sum_{i=1}^{n_p}\psi_i(t)\chi_{P_i}(t). \tag{104}$$

Thus using equations (103) for bond options of any maturity T_C we must have that

$$\mu_C(t) - r(t) = -\sum_{i=1}^{n_w} \phi_i(t)\nu_{C_i}(t) - \sum_{i=1}^{n_p} \psi_i(t)\chi_{C_i}(t),$$

and substituting the expressions for $\mu_C(t)$, $\nu_{C_i}(t)$ and $\chi_{C_i}(t)$, we have that

$$\begin{aligned}\frac{1}{C}\left(\frac{\partial C}{\partial t} + \left(\vartheta(t) - \sum_{i=1}^{n_p}\beta_i(t,t)\lambda_i\right)\frac{\partial C}{\partial r} + \frac{1}{2}\sum_{i=1}^{n_w}\sigma_i^2(t,t)\frac{\partial^2 C}{\partial r^2}\right) - r(t)\\ = -\frac{1}{C}\frac{\partial C}{\partial r}\sum_{i=1}^{n_w}\phi_i(t)\sigma_i(t,t) - \sum_{i=1}^{n_p}\psi_i(t)\frac{1}{C}[C(r+\beta_i(t,t),t) - C(r,t)],\end{aligned} \tag{105}$$

or after further manipulations

$$\begin{aligned}\frac{\partial C}{\partial t} + (\vartheta(t) - \sum_{i=1}^{n_p}\beta_i(t,t)\lambda_i + \sum_{i=1}^{n_w}\phi_i(t)\sigma_i(t,t))\frac{\partial C}{\partial r} + \frac{1}{2}\sum_{i=1}^{n_w}\sigma_i^2(t,t)\frac{\partial^2 C}{\partial r^2} - rC\\ + \sum_{i=1}^{n_p}\psi_i[C(r+\beta_i(t,t),t) - C(r,t)] = 0.\end{aligned} \tag{106}$$

By substituting the expressions for $\mu_P(t)$, $\nu_{P_i}(t)$ and $\chi_{P_i}(t)$ in equation (104), we derive the drift restriction

$$\alpha(t,T) = \sum_{i=1}^{n_w}\sigma_i(t,T)(-\phi_i(t) + \zeta_i(t,T)) - \sum_{i=1}^{n_p}\beta_i(t,T)(\psi_i(t)e^{-\xi_i(t,T)} - \lambda_i). \tag{107}$$

Appendix 2. Application of Ito's lemma on Y

The dynamics for $P(t,T_C)$ are given by

$$\begin{aligned}dP(t,T_C) = r(t)P(t,T_C)dt - \sum_{i=1}^{n_w}\zeta_i(t,T_C)P(t,T_C)d\widetilde{W}_i(t)\\ + P(t,T_C)\sum_{i=1}^{n_p}(e^{-\xi_i(t,T_C)} - 1)[dQ_i(t) - \psi_i dt],\end{aligned} \tag{108}$$

and the dynamics for $C(r,t,T)$ are

$$\begin{aligned}dC =& r(t)Cdt + \frac{\partial C}{\partial r}\sum_{i=1}^{n_w}\sigma_{0i}(t)d\widetilde{W}_i(t)\\ &+ \sum_{i=1}^{n_p}[C(r+\beta_{0i},t,T) - C(r,t,T)][dQ_i(t) - \psi_i dt].\end{aligned} \tag{109}$$

Define the new quantity

$$Y(C,P) = \frac{C(r,t,T)}{P(r,t,T_C)} \tag{110}$$

then

$$\begin{aligned} \frac{\partial Y}{\partial C} =& \frac{1}{P}, \qquad \frac{\partial Y}{\partial P} = -\frac{C}{P^2} \\ \frac{\partial^2 Y}{\partial C^2} =& 0, \qquad \frac{\partial^2 Y}{\partial P^2} = \frac{2C}{P^3}, \qquad \frac{\partial^2 Y}{\partial P \partial C} = -\frac{1}{P^2}. \end{aligned} \tag{111}$$

Application of the multi-dimensional jump-diffusion version of Ito's Lemma leads to

$$\begin{aligned} dY = & \Bigg[\frac{\partial Y}{\partial t} + (rP - \sum_{i=1}^{n_p} (e^{-\xi_i(t,T_C)} - 1)\psi_i P) \frac{\partial Y}{\partial P} + (rC - \sum_{i=1}^{n_p} [C(r+\beta_{0i}) - C(r)]\psi_i) \frac{\partial Y}{\partial C} \\ & + \frac{1}{2} \sum_{i=1}^{n_w} \left(\zeta_i^2(t,T_C) P^2 \frac{\partial^2 Y}{\partial P^2} - 2\zeta_i(t,T_C) P \sigma_{0i} \frac{\partial C}{\partial r} \frac{\partial^2 Y}{\partial P \partial C} + \sigma_{0i}^2 (\frac{\partial C}{\partial r})^2 \frac{\partial^2 Y}{\partial C^2} \right) \Bigg] dt \\ & + \sum_{i=1}^{n_w} \left(-\zeta_i(t,T_C) P \frac{\partial Y}{\partial P} + \sigma_{0i} \frac{\partial C}{\partial r} \frac{\partial Y}{\partial C} \right) d\widetilde{W}_i(t) \\ & + \sum_{i=1}^{n_p} [\{ Y(C(r) + C(r+\beta_{0i}) - C(r), P + P(e^{-\xi_i(t,T_C)} - 1)) - Y(C(r), P) \}] dQ_i(t), \end{aligned} \tag{112}$$

and after substitution of the partial derivatives it is simplified to

$$\begin{aligned} dY = & \left[-rY + rY + \frac{1}{2} \sum_{i=1}^{n_w} \left(2\zeta_i^2(t,T_C) Y + 2\zeta_i(t,T_C) \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} Y + 0 \right) \right] dt \\ & + \sum_{i=1}^{n_p} \left((e^{-\xi_i(t,T_C)} - \frac{C(r+\beta_{0i})}{C} \right) \psi_i Y dt \\ & + \sum_{i=1}^{n_w} \left(\zeta_i(t,T_C) Y + \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} Y \right) d\widetilde{W}_i(t) \\ & + \sum_{i=1}^{n_p} Y \left(\frac{C(r+\beta_{0i})}{e^{-\xi_i(t,T_C)} C(r)} - 1 \right) dQ_i(t), \end{aligned} \tag{113}$$

and further to

$$\begin{aligned} \frac{dY}{Y} = & \sum_{i=1}^{n_w} \left(\zeta_i(t,T_C) + \frac{\sigma_{0i}}{C} \frac{\partial C}{\partial r} \right) [d\widetilde{W}_i(t) + \zeta_i(t,T_C) dt] \\ & + \sum_{i=1}^{n} \left(\frac{C(r+\beta_{0i})}{C(r)} e^{\xi_i(t,T_C)} - 1 \right) [dQ_i(t) - \psi_i e^{-\xi_i(t,T_C)} dt]. \end{aligned} \tag{114}$$

Appendix 3. Fourier Transform Technique

Define the Fourier transform of the solution $\Upsilon = \Upsilon(Z,t)$ to the partial differential equation (28) by

$$\overline{\Upsilon}(\omega,t) = \int_{-\infty}^{\infty} \Upsilon(Z,t) e^{-\mathrm{i}\omega Z} dZ, \tag{115}$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit. Then

$$\int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial t} e^{-\mathbf{i}\omega Z} dZ = \frac{\partial \overline{\Upsilon}(\omega, t)}{\partial t}, \tag{116}$$

while[15]

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} dZ &= \Upsilon e^{-\mathbf{i}\omega Z}|_{-\infty}^{\infty} + \mathbf{i}\omega \int_{-\infty}^{\infty} \Upsilon e^{-\mathbf{i}\omega Z} dZ \\ &= \mathbf{i}\omega \overline{\Upsilon}(\omega, t), \end{aligned} \tag{117}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial^2 \Upsilon}{\partial Z^2} e^{-\mathbf{i}\omega Z} dZ &= \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z}|_{-\infty}^{\infty} + \mathbf{i}\omega \int_{-\infty}^{\infty} \frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z} dZ \\ &= \Upsilon e^{-\mathbf{i}\omega Z}|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} \Upsilon e^{-\mathbf{i}\omega X} dZ \\ &= -\omega^2 \overline{\Upsilon}(\omega, t). \end{aligned} \tag{118}$$

Also note that

$$\begin{aligned} &\int_{-\infty}^{\infty} \Upsilon(Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}) e^{-\mathbf{i}\omega Z} dZ \\ &\quad = e^{\mathbf{i}\omega \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}} \int_{-\infty}^{\infty} \Upsilon(Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}}) e^{-\mathbf{i}\omega (Z(t) + \ln \frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}})} dZ \\ &\quad = \left(\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}} \right)^{\mathbf{i}\omega} \overline{\Upsilon}(\omega, t), \end{aligned} \tag{119}$$

Using the results (115)-(119), the partial differential equation (28) for $\Upsilon(Z, t)$ becomes an ordinary differential equation with complex coefficients for $\overline{\Upsilon}(\omega, t)$, i.e.,

$$\begin{aligned} \frac{\partial \overline{\Upsilon}(\omega, t)}{\partial t} = &\Bigg(-\mathbf{i}\omega \left[\sum_{i=1}^{n_p} \psi_i(t) \left(e^{-\xi_i(t,T_C)} - e^{-\xi_i(t,T)} \right) - \frac{1}{2} \sum_{i=1}^{n_w} \left(\zeta_i(t, T_C) - \zeta_i(t, T) \right)^2 \right] \\ &+ \frac{\omega^2}{2} \sum_{i=1}^{n_w} \left(\zeta_i(t, T_C) - \zeta_i(t, T) \right)^2 \\ &+ \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t,T_C)} - \sum_{i=1}^{n_p} \psi_i(t) e^{-\xi_i(t,T_C)} \left(\frac{e^{-\xi_i(t,T)}}{e^{-\xi_i(t,T_C)}} \right)^{\mathbf{i}\omega} \Bigg) \overline{\Upsilon}(\omega, t). \end{aligned} \tag{120}$$

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[15] Note that we assume $\Upsilon e^{-\mathbf{i}\omega Z}|_{-\infty}^{\infty} = 0$ and $\frac{\partial \Upsilon}{\partial Z} e^{-\mathbf{i}\omega Z}|_{-\infty}^{\infty} = 0$. It is necessary later to verify that the solution obtained based on these assumptions satisfies the partial differential equation (28). The assumption is then justified on the basis of uniqueness of the solution.

Equation (120) may be expressed as

$$\begin{aligned}\frac{\partial}{\partial t}\Bigg[\overline{\Upsilon}(\omega,t)\exp\Bigg\{&-\sum_{i=1}^{n_p}\int_0^t \psi_i(s)e^{-\xi_i(s,T_C)}ds\\&+\mathbf{i}\omega\left[\sum_{i=1}^{n_p}\int_0^t \psi_i(s)\left(e^{-\xi_i(s,T_C)}-e^{-\xi_i(s,T)}\right)ds-\frac{1}{2}\int_0^t\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C)-\zeta_i(s,T)\right)^2 ds\right]\\&-\frac{\omega^2}{2}\int_0^t\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C)-\zeta_i(s,T)\right)^2 ds\\&+\sum_{i=1}^{n_p}\int_0^t \psi_i(s)e^{-\xi_i(s,T_C)}\left(\frac{e^{-\xi_i(s,T)}}{e^{-\xi_i(s,T_C)}}\right)^{\mathbf{i}\omega}ds\Bigg\}\Bigg]=0.\end{aligned}\tag{121}$$

Integrating from t to T_C

$$\begin{aligned}\overline{\Upsilon}(\omega,t)=&\overline{\Upsilon}(\omega,T_C)\exp\Bigg\{-\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)e^{-\xi_i(s,T_C)}ds\\&+\mathbf{i}\omega\left[\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)\left(e^{-\xi_i(s,T_C)}-e^{-\xi_i(s,T)}\right)ds-\frac{1}{2}\int_t^{T_C}\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C)-\zeta_i(s,T)\right)^2 ds\right]\\&-\frac{\omega^2}{2}\int_t^{T_C}\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C)-\zeta_i(s,T)\right)^2 ds\\&+\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)e^{-\xi_i(s,T_C)}\left(\frac{e^{-\xi_i(s,T)}}{e^{-\xi_i(s,T_C)}}\right)^{\mathbf{i}\omega}ds\Bigg\}.\end{aligned}\tag{122}$$

Let

$$\overline{c}(t,T_C)=\frac{1}{T_C-t}\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)e^{-\xi_i(s,T_C)}ds,\tag{123}$$

$$\overline{v}(t,T_C)=\frac{1}{T_C-t}\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)\left(e^{-\xi_i(s,T_C)}-e^{-\xi_i(s,T)}\right)ds,\tag{124}$$

$$\overline{\sigma}^2(t,T_C)=\frac{1}{T_C-t}\int_t^{T_C}\sum_{i=1}^{n_w}\left(\zeta_i(s,T_C)-\zeta_i(s,T)\right)^2 ds,\tag{125}$$

$$\overline{\xi}(\omega,t,T_C)=\frac{1}{T_C-t}\sum_{i=1}^{n_p}\int_t^{T_C}\psi_i(s)e^{-\xi_i(s,T_C)}\left(\frac{e^{-\xi_i(s,T)}}{e^{-\xi_i(s,T_C)}}\right)^{\mathbf{i}\omega}ds,\tag{126}$$

then equation (122) is simplified to

$$\begin{aligned}&\overline{\Upsilon}(\omega,t)=\\&\overline{\Upsilon}(\omega,T_C)e^{(T_C-t)\left(-\overline{c}(t,T_C)+\mathbf{i}\omega[\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)]-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)+\overline{\xi}(\omega,t,T_C)\right)}.\end{aligned}\tag{127}$$

By the Fourier inversion theorem, we have that

$$\Upsilon(Z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\Upsilon}(\omega,t)e^{\mathbf{i}\omega Z}d\omega.\tag{128}$$

Thus, by substituting (31) into (128) we obtain

$$\begin{aligned}\Upsilon(Z,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Upsilon}(\omega,T_C) \\ &\exp\left\{(T_C-t)\left(-\overline{c}(t,T_C)+\mathbf{i}\omega[\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)]-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)+\overline{\xi}(\omega,t,T_C)\right)+\mathbf{i}\omega Z\right\}d\omega \\ &= \frac{e^{-(T_C-t)\overline{c}(t,T_C)}}{2\pi}\int_{-\infty}^{\infty}\overline{\Upsilon}(\omega,T_C)e^{\mathbf{i}\omega([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+Z)-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)+\overline{\xi}(\omega,t,T_C)(T_C-t)}d\omega.\end{aligned}$$

Note that by changing the variable Z back to the variable X (recall that $Z=\ln X$), we obtain

$$\begin{aligned}\overline{\Upsilon}(\omega,T_C) &= \int_{-\infty}^{\infty}\Upsilon(Z,T_C)e^{-\mathbf{i}\omega Z}dZ \\ &= \int_{-\infty}^{\infty}Y(e^Z,T_C)e^{-\mathbf{i}\omega Z}dZ, \end{aligned} \tag{129}$$

and so

$$\begin{aligned}&Y(X,t) \\ &= \frac{e^{-\overline{c}(t,T_C)(T_C-t)}}{2\pi} \\ &\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}Y(e^Z,T_C)e^{-\mathbf{i}\omega Z}dZ\right)e^{\mathbf{i}\omega([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X)-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)+\overline{\xi}(\omega,T_C,t)(T_C-t)}d\omega \\ &= \frac{e^{-\overline{c}(t,T_C)(T_C-t)}}{2\pi} \\ &\int_{-\infty}^{\infty}Y(e^Z,T_C)\left(\int_{-\infty}^{\infty}e^{\mathbf{i}\omega([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X-Z)-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)+\overline{\xi}(\omega,t,T_C)(T_C-t)}d\omega\right)dZ \\ &= e^{-\overline{c}(t,T_C)(T_C-t)}\int_{-\infty}^{\infty}Y(e^Z,T_C)\mathcal{K}(Z,X,t)dZ, \end{aligned} \tag{130}$$

where the kernel $\mathcal{K}$ is defined by

$$\mathcal{K}(Z,X,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mathbf{i}\omega([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X-Z)-\frac{\omega^2}{2}\overline{\sigma}^2(t,T_C)(T_C-t)+\overline{\xi}(\omega,t,T_C)(T_C-t)}d\omega. \tag{131}$$

APPENDIX 4. DERIVATION OF BLACK-SCHOLES TYPE INTEGRAL

We set as I the integral

$$I = \int_{\ln E}^{\infty}(e^Z-E)\,e^{-\frac{([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X-Z+\sum_{i=1}^{np}p_i\mu_i)^2}{2\overline{\sigma}^2(t,T_C)(T_C-t)}}dZ, \tag{132}$$

then by performing further manipulations

$$\begin{aligned}I &= \int_{\ln E}^{\infty}e^Z\,e^{-\frac{([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X-Z+\sum_{i=1}^{np}p_i\mu_i)^2}{2\overline{\sigma}^2(t,T_C)(T_C-t)}}dZ \\ &\quad - E\int_{\ln E}^{\infty}e^{-\frac{([\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X-Z+\sum_{i=1}^{np}p_i\mu_i)^2}{2\overline{\sigma}^2(t,T_C)(T_C-t)}}dZ.\end{aligned} \tag{133}$$

The change of the variable

$$u = \frac{[\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C - t) + \ln X - Z + \sum_{i=1}^{n_p} p_i\mu_i}{\overline{\sigma}(t,T_C)\sqrt{T_C - t}},$$

and the setting of

$$d_2 = \frac{[\overline{v}(t,T_C) - \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C - t) + \ln \frac{X}{E} + \sum_{i=1}^{n_p} p_i\mu_i}{\overline{\sigma}(t,T_C)\sqrt{T_C - t}},$$

leads to

$$\begin{aligned}
I &= \int_{d_2}^{-\infty} e^{Z}\, e^{-\frac{u^2}{2}}(-\overline{\sigma}(t,T_C)\sqrt{T_C - t})du - E\int_{d_2}^{-\infty} e^{-\frac{u^2}{2}}(-\overline{\sigma}(t,T_C)\sqrt{T_C - t})du \\
&= \overline{\sigma}(t,T_C)\sqrt{T_C - t}\left(\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}+[\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X+\sum_{i=1}^{n_p} p_i\mu_i - u\overline{\sigma}(t,T_C)\sqrt{T_C-t}}du - E\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}}du\right) \\
&= \overline{\sigma}(t,T_C)\sqrt{T_C - t}\left(e^{[\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X+\sum_{i=1}^{n_p} p_i\mu_i}\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}-u\overline{\sigma}(t,T_C)\sqrt{T_C-t}}du - E\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}}du\right) \\
&= \overline{\sigma}(t,T_C)\sqrt{T_C - t}\Big(e^{[\overline{v}(t,T_C)-\frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C-t)+\ln X+\sum_{i=1}^{n_p} p_i\mu_i+\frac{1}{2}\overline{\sigma}^2(t,T_C)(T_C-t)} \\
&\qquad\qquad \int_{-\infty}^{d_2} e^{-\frac{1}{2}(u+\overline{\sigma}(t,T_C)\sqrt{T_C-t})^2}du - E\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}}du\Big).
\end{aligned} \tag{134}$$

By changing the variable to $U = u + \overline{\sigma}(t,T_C)\sqrt{T_C - t}$, defining d_1 as

$$d_1 = d_2 + \overline{\sigma}(t,T_C)\sqrt{T_C - t} = \frac{[\overline{v}(t,T_C) + \frac{1}{2}\overline{\sigma}^2(t,T_C)](T_C - t) + \ln \frac{X}{E} + \sum_{i=1}^{n_p} p_i\mu_i}{\overline{\sigma}(t,T_C)\sqrt{T_C - t}},$$

and using the standard normal cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}dt,$$

we obtain

$$\begin{aligned}
I &= \overline{\sigma}(t,T_C)\sqrt{T_C - t}\left(e^{\overline{v}(t,T_C)(T_C-t)+\ln X+\sum_{i=1}^{n_p} p_i\mu_i}\int_{-\infty}^{d_1} e^{-\frac{1}{2}U^2}dU - E\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}}du\right) \\
&= \sqrt{2\pi}\,\overline{\sigma}(t,T_C)\sqrt{T_C - t}\left(e^{\overline{v}(t,T_C)(T_C-t)+\ln X+\sum_{i=1}^{n_p} p_i\mu_i}\Phi(d_1) - E\Phi(d_2)\right).
\end{aligned} \tag{135}$$

Appendix 5. Model Limitations

The Markovian term structure model developed here does not guarantee positivity of the interest rates, a feature that we must handle with caution given the state dependent nature of the model's volatility functions. In fact, there is a positive probability that this type of dynamics may drive interest rates to negative values. This is due to the functional properties of the jump adjusted drift coefficient. To understand this effect, the functional behavior of the drift coefficient of the forward rate is examined in detail. The risk neutral forward rate dynamics under the volatility specifications of Assumption 3.1

are expressed as

$$\begin{aligned} df(t,T) = & \sum_{i=1}^{n_w} \frac{1}{k_{\sigma i}} {\sigma_{0i}}^2(t, \bar{f}(t)) e^{-\kappa_{\sigma i}(T-t)} \left(1 - e^{-\kappa_{\sigma i}(T-t)}\right) dt \\ & - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}(T-t)} [e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}(T-t)}-1)} - 1] dt - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}(T-t)} dt \\ & + \sigma_0(t, \bar{f}(t)) e^{-\kappa_\sigma (T-t)} \, d\widetilde{W}(t) + \sum_{i=1}^{2} \beta_{0i} e^{-k_{\beta i}(T-t)} \, dQ_i(t). \end{aligned} \tag{136}$$

The drift function of the forward rate dynamics is bounded by the function $\mathbb{D}(\tau)$ (set $\tau = T - t$)

$$\begin{aligned} \mathbb{D}(\tau) &= \sum_{i=1}^{n_w} \frac{{\sigma_{0i}}^2}{k_{\sigma i}} c_0 \, e^{-\kappa_{\sigma i}\tau} \left(1 - e^{-\kappa_{\sigma i}\tau}\right) - \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}\tau} e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}\tau}-1)}, \\ &= \mathbb{D}_G(\tau) + \mathbb{D}_J(\tau) \end{aligned} \tag{137}$$

where $\mathbb{D}_G(\tau)$ is the Gaussian drift contribution and $\mathbb{D}_J(\tau)$ is the contribution of the jump component to the drift. The derivative of $\mathbb{D}(\tau)$ is

$$\begin{aligned} \frac{d\mathbb{D}(\tau)}{d\tau} = & \sum_{i=1}^{n_w} {\sigma_{0i}}^2 c_0 \, e^{-\kappa_{\sigma i}\tau} \left(2e^{-\kappa_{\sigma i}\tau} - 1\right) \\ & + \sum_{i=1}^{n_p} \psi_i \beta_{0i} e^{-k_{\beta i}\tau} e^{\frac{\beta_{0i}}{k_{\beta i}}(e^{-k_{\beta i}\tau}-1)} \left(k_{\beta i} + \beta_{0i} e^{-k_{\beta i}\tau}\right). \end{aligned}$$

First by assuming only positive jump sizes, the drift function is originally negative for some time close to the maturity as the following arguments shows. The $\mathbb{D}_J(\tau)$ is an increasing function in τ with negative minimum $-\sum_{i=1}^{n_p} \psi_i \beta_{0i}$ at $\tau = 0$. As $\tau \to \infty$, $\mathbb{D}_J(\tau)$ converges to 0. The $\mathbb{D}_G(\tau)$ has a minimum of 0 at $\tau = 0$. As $\tau \to \infty$, $\mathbb{D}_G(\tau)$ converges to 0. Thus $\mathbb{D}$ has always a negative minimum of $-\sum_{i=1}^{n_p} \psi_i \beta_{0i}$ at $\tau = 0$.

By assuming only negative jump sizes, then the drift function is always positive. However, the negative jump noise terms may drive interest rates to negative values. Consequently, we cannot avoid the situation of interest rates becoming negative. Thus the volatility functions must be well defined for negative values. As an illustrative example, let us assume that the state dependence is modelled as a linear combination of benchmark forward rates and the instantaneous spot rate of the state vector $\bar{f}(t)$, as

$$L_f(t) = c_0 r(t) + \sum_{h=1}^{n_d} c_h f(t, T_h).$$

When $L_f(s)$ becomes very small or negative then the model may behave as a deterministic volatility Hull-White type of model. Thus essentially extending the Cox, Ingersoll & Ross (1985) model to jump diffusions, a suggested volatility function may be

$$\sigma_0(t, \bar{f}(t)) = \begin{cases} c_f \sigma_0, & L_f(t) < 0.005; \\ \sigma_0[(L_f(t) - 0.005)^\gamma + c_f], & L_f(t) \geq 0.005; \end{cases} \tag{138}$$

with $\gamma = \frac{1}{2}$ and $c_f > 0$.

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