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On the Strong Approximation of Pure Jump Processes

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Abstract. This paper constructs strong discrete time approximations for pure jump processes that can be described by stochastic differential equations. Strong approximations based on jump-adapted time discretizations, which produce no discretization bias, are analyzed. The computational complexity of these approximations is proportional to the jump intensity. Furthermore, by exploiting a stochastic expansion for pure jump processes, higher order discrete time approximations, whose computational complexity is not dependent on the jump intensity, are proposed. The strong order of convergence of the resulting schemes is analyzed.

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1 Introduction

As one tries to build more realistic models in economics, finance, biology, social sciences, chemistry, physics and other areas, stochastic effects need to be taken into account. In certain areas, such as finance, the randomness or uncertainty in the dynamics is in fact the essential phenomenon that needs to be modeled. Event driven dynamics become more and more important in most fields of application. In finance one needs to model properly credit events as defaults and credit rating changes, see for instance Jarrow, Lando & Turnbull (1997). Also the short rate, as set by a central bank, jumps after random waiting times typically by a quarter of a percent up or down, see Babbs & Webber (1995). In chemistry and biotechnology the reactions of single molecules or coupled reactions need to be studied in their interaction, see Gillespie (1977, 2001). Simulation methods are in many cases the only practically available numerical methods that allow the study of solutions of certain higher dimensional nonlinear systems of stochastic differential equations (SDEs).

Much progress has been made on what concerns the simulation of continuous solutions of SDEs that are driven by Wiener processes only, see Kloeden & Platen (1999). However, as already mentioned above, advanced modeling needs to capture the effects of single events. Consequently, one has to simulate solutions of SDEs with jumps. The literature in this area is still rather limited, including, for instance, Gillespie (1977), Wright (1980), Platen (1982a), Mikulevicius & Platen (1988), Li (1995), Li & Liu (1997), Protter & Talay (1997), Maghsoodi (1998), Liu & Li (1999, 2000), Kubilius & Platen (2002), Glasserman & Merener (2003), Glasserman (2004), Cont & Tankov (2004), Higham & Kloeden (2004, 2005) and Bruti-Liberati & Platen (2005).

Much attention has been paid to the jump-diffusion case, where Wiener processes and Poisson processes drive the corresponding SDE. This leads, in general, to quite complicated higher order numerical schemes, as one needs to compute multiple stochastic integrals with respect to time, Wiener processes and the Poisson process, see Bruti-Liberati & Platen (2005). Jump-adapted approximations, as introduced by Platen (1982a), severely reduce the complexity of the numerical schemes as they avoid multiple stochastic integrals with respect to the Poisson process. However, for the case of SDEs driven by high intensity jump processes, simulations based on jump-adapted approximations may become unfeasible as their computational complexity is proportional to the intensity level. Therefore, for SDEs driven by high intensity jump processes, to obtain higher order numerical schemes one needs to include multiple stochastic integrals with respect to the Poisson process.

In this paper the focus is on discrete time approximation of SDEs that are pure jump processes. The piecewise constant nature of these solutions simplifies the resulting numerical schemes. Jump-adapted approximations produce in this case no discretization bias. Therefore, in the case of low to medium jump intensity one can construct realistic schemes that show no discretization bias. In the case of high intensity jump processes, jump-adapted schemes are often practically not feasible. However, it is possible to derive higher order discrete time approximations whose complexity turns out to be significantly lower than in the case of jump diffusions.

We consider in this paper piecewise constant discrete time strong approximations with maximum time step size $\Delta \in (0, \Delta_0)$, where $\Delta_0 \in (0, 1)$. A discrete time approximation Y is said to converge with strong order γ if there exist constants C and $\Delta_0 \in (0, 1)$ such that for all $\Delta \in (0, \Delta_0)$ one has

$$\epsilon(\Delta) = \sqrt{E(|X_T - Y_T|^2)} \leq C\,\Delta^\gamma, \tag{1.1}$$

where X_T is the solution of the given pure jump SDE at a terminal time $T \in [0, \infty)$ and Y_T is the corresponding value of the approximation.

The criterion (1.1) allows us to classify different discrete time approximations by their strong order of convergence. These approximations are constructed to approach the solution $X = \{X_t, t \in [0, T]\}$ of the given pure jump SDE in a pathwise sense and are therefore suited for scenario simulation and filtering.

2 Model Dynamics

Let us consider a counting process $N = \{N_t,\ t \in [0, T]\}$, which is right-continuous with left-hand limits and counts the arrival of certain events. For simplicity, we take N to be a *Poisson process* with constant *intensity* $\lambda \in (0, \infty)$ that starts at time $t = 0$ in $N_0 = 0$. It is defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ satisfying the usual conditions.

The counting process N generates an increasing sequence $(\tau_i)_{i\in\{1,2,\ldots,N_T\}}$ of its jump times. One can interpret τ_i as the time of the ith event, $i \in \{1, 2, \ldots, N_T\}$.

For any right-continuous process $Z = \{Z_t,\ t \in [0, T]\}$ we define its *jump size* $\Delta\, Z_t$ at time t as the difference

$$\Delta\, Z_t = Z_t - Z_{t-} \tag{2.1}$$

for $t \in [0, T]$, where Z_{t-} denotes the left-hand limit of Z at time t. Thus, we can, for instance, write

$$N_t = \sum_{s\in(0,t]} \Delta\, N_s \tag{2.2}$$

for $t \in [0, T]$.

For a pure jump process $X = \{X_t,\ t \in [0, T]\}$ that is driven by the counting process N we assume that its value X_t at time t satisfies the SDE

$$dX_t = c(t-, X_{t-})\, dN_t \tag{2.3}$$

for $t \in [0,T]$ with deterministic initial value $X_0 \in \mathbb{R}$. The function $c : [0,T]\times\mathbb{R} \to \mathbb{R}$ denotes the *jump coefficient* and is assumed to be Lipschitz continuous, such that

$$|c(t,x) - c(t,y)| \leq K\,|x-y| \tag{2.4}$$

and satisfies the growth condition

$$|c(t,x)|^2 \leq K\,(1+|x|^2) \tag{2.5}$$

for $t \in [0,T]$ and $x,y \in \mathbb{R}$ with some constant $K \in (0,\infty)$. According to Ikeda & Watanabe (1989) there exists a unique, right-continuous solution of the SDE (2.3).

To provide an example, let us consider the linear SDE

$$dX_t = X_{t-}\,\psi\,dN_t \tag{2.6}$$

for $t \in [0,T]$ with $X_0 > 0$ and constant $\psi \in \mathbb{R}$. Here the jump coefficient has the form $c(t,x) = x\,\psi$. By application of the Itô formula one can show that the solution $X = \{X_t,\ t \in [0,T]\}$ of the SDE (2.6) is a pure jump process with explicit representation

$$X_t = X_0\,\exp\{N_t\,\ln(\psi+1)\} = X_0\,(\psi+1)^{N_t} \tag{2.7}$$

for $t \in [0,T]$.

3 Jump-Adapted Approximations

We consider a *jump-adapted time discretization* $0 = t_0 < t_1 < \ldots < t_{i_T} = T$, where $t_1 < \ldots < t_{i_T-1}$ equal the jump times $\tau_1 < \ldots < \tau_{N_T}$ of the Poisson process N. Here for all $t \in [0,T]$ the index

$$i_t = \max\{i \in \{0,1,\ldots\} : t_i \leq t\} \tag{3.1}$$

denotes the last discretization point before t. On this jump-adapted time grid we construct the *jump-adapted Euler scheme* by the algorithm

$$Y_{t_{i+1}} = Y_{t_i} + c(t_i, Y_{t_i})\,(N_{t_{i+1}} - N_{t_i}), \tag{3.2}$$

for $i \in \{0,1,\ldots,i_T-1\}$ with initial value $Y_0 = X_0$. Between discretization times the right-continuous process Y is set to be piecewise constant.

Since the discretization points are constructed exactly at the jump times of N and the simulation of the increments $N_{t_{i+1}} - N_{t_i} = 1$ of N is exact, the jump-adapted Euler scheme (3.2) produces no discretization bias.

For the implementation of the scheme (3.2) one needs to compute the jump times τ_i, $i \in \{1,2\ldots,N_T\}$, and has then to apply equation (3.2) recursively for every $i \in$

$\{0, 1, 2 \ldots, i_T - 1\}$. One can obtain the jump times via the corresponding waiting times between two consecutive jumps sampling from an exponential distribution with parameter λ. This means that the average of the waiting times is $1/\lambda$.

The computational complexity of algorithm (3.2) is heavily dependent on the intensity λ of the jump process. Indeed, the average number of steps and thus of operations is proportional to the intensity λ.

4 Euler Approximation

As noticed in the previous section, it is possible to derive jump-adapted approximations that produce no discretization bias. However, as their computational complexity is proportional to the intensity level of the jump process, simulations based on jump-adapted approximations may not be feasible when the underlying SDE is driven by a high intensity Poisson process. In the following we develop discrete time strong approximations whose computational complexity is independent of the jump intensity level. To keep our numerical analysis simple we assume for the remainder that the jump coefficient is time homogeneous, that is $c(t, x) = c(x)$ for all $t \in [0, T]$ and $x \in \mathbb{R}$.

We consider a time discretization $0 = t_0 < t_1 < \ldots < t_{i_T} = T$, where i_T is defined in (3.1). For simplicity, we choose an equidistant time discretization with $t_i = i\,\Delta$, for $i \in \{0, 1, \ldots, \frac{T}{\Delta}\}$, where $\Delta \in (0, 1)$ is the time step size.

The simplest strong Taylor approximation $Y = \{Y_t,\ t \in [0, T]\}$ is the *Euler scheme*, which is given by the recursive stochastic difference equation

$$Y_{t_{i+1}} = Y_{t_i} + c(Y_{t_i})\,(N_{t_{i+1}} - N_{t_i}) \tag{4.1}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with initial value $Y_0 = X_0$. Between discretization times the right-continuous process Y is again assumed to be piecewise constant.

By comparing the scheme (4.1) with the algorithm (3.2), we notice that the difference in the schemes consists in the time discretization that is used. We emphasize that the computational complexity of the Euler scheme (4.1) is independent of the jump intensity, if we neglect the additional time needed to sample from a Poisson distribution with higher mean. Therefore, a simulation based on the Euler scheme (4.1) is feasible also in the case of high intensity jump processes. However, while the jump-adapted Euler scheme (3.2) produces no discretization bias, the accuracy of the Euler scheme (4.1) depends on the size of the time step Δ.

For the linear SDE (2.6) the Euler scheme (4.1) has the form

$$Y_{t_{i+1}} = Y_{t_i} + Y_{t_i}\,\psi\,(N_{t_{i+1}} - N_{t_i}) = Y_{t_i}\,(1 + \psi(N_{t_{i+1}} - N_{t_i})) \tag{4.2}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$. Since the equidistant time discretization is not matching the jump times of the underlying Poisson process, we have an approximation error. For what concerns such error there is a substantial similarity to the discrete time approximation of SDEs that are driven purely by Wiener processes, as described, for instance, in Kloeden & Platen (1999).

We will show that, in general, for pure jump SDEs the Euler approximation (4.1) is of strong order $\gamma = 0.5$. This raises the question of constructing higher order discrete time approximations for the case of pure jump SDEs. This problem can be approached by exploiting stochastic Taylor expansions similar to the Wagner-Platen formula, also known as Itô-Taylor formula, which has been described and applied in many ways in Kloeden & Platen (1999) for diffusion SDEs. The stochastic Taylor expansion for pure jump SDEs that we will describe below is a particular case of the stochastic Taylor formula for semimartingales derived already in Platen (1982b).

5 Stochastic Taylor Expansions

Since the use of stochastic Taylor expansions for jump processes is not common in the literature let us at first illustrate the structure of a stochastic Taylor formula for a simple example. For any measurable function $f : \mathbb{R} \to \mathbb{R}$ and a given adapted counting process $N = \{N_t,\, t \in [0,T]\}$ we have the representation

$$f(N_t) = f(N_0) + \sum_{s \in (0,t]} \Delta\, f(N_s) \tag{5.1}$$

for all $t \in [0,T]$. We can formally write the equation (5.1) in the form of an SDE

$$df(N_t) = \left(f(N_{t-}+1) - f(N_{t-})\right) dN_t \tag{5.2}$$

for $t \in [0,T]$. This equation can also be obtained from the Itô formula for semimartingales, see Protter (2004), for the case with jumps.

Obviously, the following difference expression $\widetilde{\Delta}_N f(N_{s-})$ defines a measurable function, as long as

$$\widetilde{\Delta}_N\, f(N) = f(N+1) - f(N) \tag{5.3}$$

is a measurable function of N. By using this function we can rewrite (5.1) in the form

$$f(N_t) = f(N_0) + \int_{(0,t]} \widetilde{\Delta}_N\, f(N_{s-})\, dN_s \tag{5.4}$$

for $t \in [0,T]$. Since $\widetilde{\Delta}_N f(N_{s-})$ is a measurable function we can apply the formula (5.4) to $\widetilde{\Delta}_N f(N_{s-})$ in (5.4), which yields

$$\begin{aligned} f(N_t) &= f(N_0) + \int_{(0,t]} \widetilde{\Delta}_N f(N_0) dN_s + \int_{(0,t]} \int_{(0,s_2)} \left(\widetilde{\Delta}_N\right)^2 f(N_{s_1-}) dN_{s_1} dN_{s_2} \\ &= f(N_0) + \widetilde{\Delta}_N f(N_0) \int_{(0,t]} dN_s + \int_{(0,t]} \int_{(0,s_2)} \left(\widetilde{\Delta}_N\right)^2 f(N_{s_1-}) dN_{s_1} dN_{s_2} \end{aligned} \tag{5.5}$$

for $t \in [0,T]$. Here $\left(\widetilde{\Delta}_N\right)^q$ denotes for integer $q \in \{1,2,\ldots\}$ the q times consecutive application of the function $\widetilde{\Delta}_N$ given in (5.3). Note that a double stochastic integral with respect to the counting process N naturally arises in (5.5). One can now continue in (5.5) to apply the formula (5.4) to the measurable function $(\widetilde{\Delta}_N)^2 f(N_{s_1-})$, which yields

$$f(N_t) = f(N_0) + \widetilde{\Delta}_N f(N_0) \int_{(0,t]} dN_s + \left(\widetilde{\Delta}_N\right)^2 f(N_0) \int_{(0,t]} \int_{(0,s_2)} dN_{s_1} dN_{s_2} + \bar{R}_3(t) \tag{5.6}$$

with remainder term

$$\bar{R}_3(t) = \int_{(0,t]} \int_{(0,s_3)} \int_{(0,s_2)} \left(\widetilde{\Delta}_N\right)^3 f(N_{s_1-})\, dN_{s_1}\, dN_{s_2}\, dN_{s_3} \tag{5.7}$$

for $t \in [0,T]$. In (5.6) we have obtained a double integral in the expansion part. Furthermore, we have a triple integral in the remainder term. We call (5.6) a stochastic Taylor expansion of the function $f(\cdot)$ with respect to the counting process N. Its expansion part only depends on multiple stochastic integrals with respect to the counting process N. These are weighted by some constant coefficient functions with values taken at the expansion point N_0. It is clear how to proceed to obtain higher order Taylor expansion by iterative application of formula (5.4).

Fortunately, the multiple stochastic integrals that arise can be easily computed. It is straightforward to prove by induction, see Engel (1982), that

$$\int_{(0,t]} dN_s = N_t$$

$$\int_{(0,t]} \int_{(0,s_2)} dN_{s_1}\, dN_{s_2} = \frac{1}{2\,!}\, N_t\, (N_t - 1),$$

$$\int_{(0,t]} \int_{(0,s_3)} \int_{(0,s_2)} dN_{s_1}\, dN_{s_2}\, dN_{s_3} = \frac{1}{3\,!}\, N_t\, (N_t - 1)\, (N_t - 2),$$

$$\int_{(0,t]} \int_{(0,s_n)} \cdots \int_{(0,s_2)} dN_{s_1}\, \ldots\, dN_{s_{n-1}}\, dN_{s_n} = \begin{cases} \binom{N_t}{n} & \text{for} \quad N_t \geq n \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

for $t \in [0,T]$. Here we have used the common combinatorial abbreviation $\binom{i}{n}$ for $i \geq n$ with $0\,! = 1$.

With (5.8) we can rewrite the stochastic Taylor expansion (5.6) in the form

$$f(N_t) \;=\; f(N_0) + \widetilde{\Delta}_N\, f(N_0) \binom{N_t}{1} + \left(\widetilde{\Delta}_N\right)^2 f(N_0) \binom{N_t}{2} + \bar{R}_3(t),$$

where

$$\widetilde{\Delta}_N\, f(N_0) = \widetilde{\Delta}_N\, f(0) = f(1) - f(0),$$

$$\left(\widetilde{\Delta}_N\right)^2 f(N_0) = f(2) - 2\,f(1) + f(0).$$

In the given case this leads to the expansion

$$f(N_t) = f(0)+(f(1)-f(0))\,N_t+(f(2)-2\,f(1)+f(0))\,\frac{1}{2}\,N_t\,(N_t-1)+\bar{R}_3(t) \quad (5.9)$$

for $t \in [0,T]$. More generally, by induction it follows the stochastic Taylor expansion

$$f(N_t) = \sum_{k=0}^{n} (\widetilde{\Delta}_N)^k\, f(N_0) \binom{N_t}{k} + \bar{R}_{n+1}(t) \quad (5.10)$$

with

$$\bar{R}_{n+1}(t) = \int_{(0,t]} \cdots \int_{(0,s_2)} (\widetilde{\Delta}_N)^{n+1}\, f(N_{s_1-})\, dN_{s_1} \ldots dN_{s_{n+1}} \quad (5.11)$$

for $t \in [0,T]$ and $n \in \{0,1,\ldots\}$, where $(\widetilde{\Delta}_N)^0\, f(N_0) = f(N_0)$. By neglecting the remainder term in (5.10) one does not consider the occurrence of a higher number of jumps and obtains a useful truncated Taylor approximation of a measurable function f with respect to a counting process N. Note that in (5.10) the truncated expansion is exact if no more than n jumps occur until time t in the realization of N. Consequently, if there is a small probability that more than n jumps occur over the given time period, then the truncated stochastic Taylor expansion can be expected to be quite accurate under any reasonable criterion.

Similar to (5.10) let us now derive a stochastic Taylor expansion for functions of solutions of the pure jump SDE (2.3). We define similarly as above the measurable function $\widetilde{\Delta}_N f(\cdot)$ such that

$$\widetilde{\Delta}_N\, f(X_{t-}) = \Delta\, f(X_t) = f(X_t) - f(X_{t-}) \quad (5.12)$$

for all $t \in [0,T]$. In the same manner as previously shown this leads to the expansion

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_{(0,t]} \widetilde{\Delta}_N\, f(X_{s-})\, dN_s \\
&= f(X_0) + \int_{(0,t]} \left(\widetilde{\Delta}_N\, f(X_0) + \int_{(0,s_2)} \left(\widetilde{\Delta}_N\right)^2 f(X_{s_1-})\, dN_{s_1} \right) dN_{s_2} \\
&= f(X_0) + \sum_{k=1}^{n} \left(\widetilde{\Delta}_N\right)^k f(X_0) \int_{(0,t]} \cdots \int_{(0,s_2)} dN_{s_1} \cdots dN_{s_k} + \tilde{R}^{n+1}_{f,t} \\
&= f(X_0) + \sum_{k=1}^{n} \left(\widetilde{\Delta}_N\right)^k f(X_0) \binom{N_t}{k} + \tilde{R}^{n+1}_{f,t} \quad (5.13)
\end{aligned}$$

with

$$\tilde{R}^{n+1}_{f,t} = \int_{(0,t]} \cdots \int_{(0,s_2)} \left(\widetilde{\Delta}_N\right)^{n+1} f(X_{s_1-})\, dN_{s_1} \cdots dN_{s_{n+1}} \quad (5.14)$$

for $t \in [0,T]$. One notes that (5.13) generalizes (5.10) in a very simple fashion.

Let us give an illustration. For the particular example given by the linear SDE (2.6) we obtain for any measurable function f the function

$$\widetilde{\Delta}_N f(X_{\tau-}) = f(X_{\tau-}(1+\psi)) - f(X_{\tau-}) \tag{5.15}$$

for the jump times $\tau \in [0,T]$ with $\Delta N_\tau = 1$. Therefore, in the case $n=2$, we get from (5.13) and (5.8) the expression

$$\begin{aligned} f(X_t) &= f(X_0) + \big(f(X_0(1+\psi)) - f(X_0)\big)(N_t - N_0) \\ &\quad + \Big(f(X_0(1+\psi)^2) - 2\, f(X_0(1+\psi)) + f(X_0)\Big) \\ &\quad \times \frac{1}{2}(N_t - N_0)\,((N_t - N_0) - 1) + \tilde{R}^3_{f,t} \end{aligned} \tag{5.16}$$

for $t \in [0,T]$. By neglecting the remainder term $\tilde{R}^3_{f,t}$ we obtain for this simple example a truncated Taylor expansion of $f(X_t)$ at X_0.

6 Second Level Taylor Approximation

The Euler scheme (4.1) can be interpreted as being derived from the expansion (5.13) applied to each time step by setting $f(x) = x$, choosing $n = 1$ and neglecting the remainder term. By choosing $n = 2$ in the corresponding truncated Taylor expansion, when applied to each time discretization interval $[t_i, t_{i+1}]$ with $f(x) = x$, we obtain the *second level Taylor approximation*

$$\begin{aligned} Y_{t_{i+1}} &= Y_{t_i} + c(Y_{t_i})\,(N_{t_{i+1}} - N_{t_i}) \\ &\quad + \big(c(Y_{t_i} + c(Y_{t_i})) - c(Y_{t_i})\big)\,\frac{1}{2}\,(N_{t_{i+1}} - N_{t_i})\,((N_{t_{i+1}} - N_{t_i}) - 1) \end{aligned} \tag{6.1}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

In the special case of our linear example (2.6), the second level approximation turns out to be of the form

$$Y_{t_{i+1}} = Y_{t_i}\Big\{1 + \psi\,(N_{t_{i+1}} - N_{t_i}) + \frac{\psi^2}{2}\,(N_{t_{i+1}} - N_{t_i})\,((N_{t_{i+1}} - N_{t_i}) - 1)\Big\} \tag{6.2}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

For the linear SDE (2.6) and a given sample path of the Poisson process, we plot in Figure 6.1 the exact solution (2.7), the Euler approximation (4.2) and the second level Taylor approximation (6.2). We selected a time step size $\Delta = \frac{1}{4}$ and the following parameters: $X_0 = 1$, $T = 1$, $\psi = -0.15$ and $\lambda = 20$. Note in Figure 6.1 that the second level Taylor approximation is at the terminal time

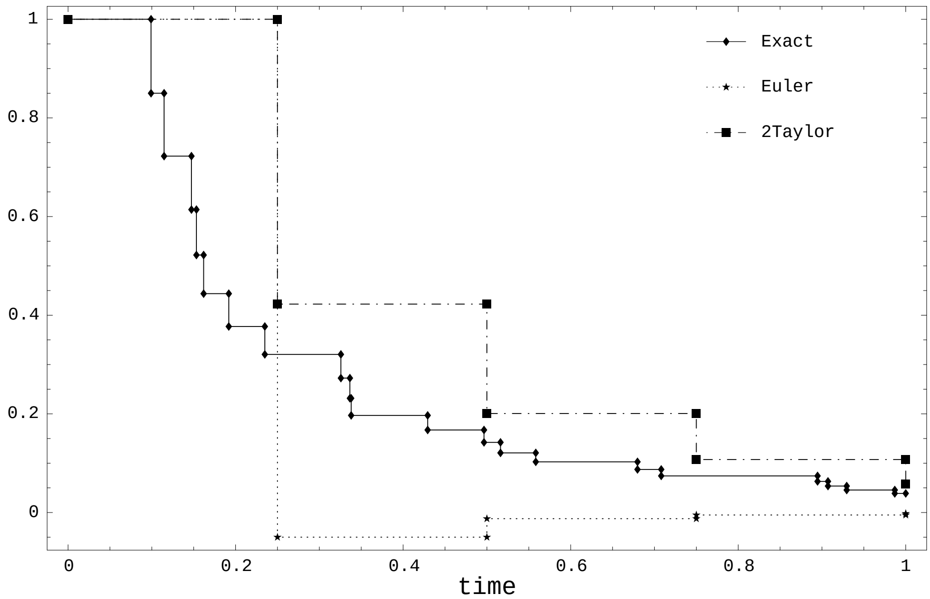

Figure 6.1: Exact solution, Euler and second level Taylor approximations.

$t = 1$ rather close to the exact solution. It appears visually better than the Euler approximation, which is even negative. Below we will present a strong convergence theorem that provides a firm basis for judging the performance of higher order schemes.

7 Third and Fourth Level Taylor Approximations

If we use the truncated stochastic Taylor expansion (5.13) with $n = 3$, when applied to each time interval $[t_i, t_{i+1}]$ with $f(x) = x$, we obtain the *third level Taylor approximation*

$$\begin{aligned} Y_{t_{i+1}} &= Y_{t_i} + c(Y_{t_i})\,(N_{t_{i+1}} - N_{t_i}) + \Big\{c\big(Y_{t_i} + c(Y_{t_i})\big) - c\big(Y_{t_i}\big)\Big\}\binom{N_{t_{i+1}} - N_{t_i}}{2} \\ &\quad + \Big\{c\big(Y_{t_i} + c(Y_{t_i}) + c(Y_{t_i} + c(Y_{t_i}))\big) - 2\,c\big(Y_{t_i} + c(Y_{t_i})\big) + c\big(Y_{t_i}\big)\Big\} \\ &\quad \times \binom{N_{t_{i+1}} - N_{t_i}}{3} \end{aligned} \tag{7.1}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

In the case of our particular example (2.6), the third level approximation is of the form

$$Y_{t_{i+1}} = Y_{t_i}\Big\{1 + \psi\,(N_{t_{i+1}} - N_{t_i}) + \psi^2\binom{N_{t_{i+1}} - N_{t_i}}{2} + \psi^3\binom{N_{t_{i+1}} - N_{t_i}}{3}\Big\} \tag{7.2}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

To construct a fourth level approximation we need to choose $n = 4$ in the truncated expansion (5.13) with $f(x) = x$. Then we obtain the *fourth level Taylor approximation*

$$\begin{aligned} Y_{t_{i+1}} &= Y_{t_i} + c(Y_{t_i})\,(N_{t_{i+1}} - N_{t_i}) + \Big\{c\big(Y_{t_i} + c(Y_{t_i})\big) - c\big(Y_{t_i}\big)\Big\}\binom{N_{t_{i+1}} - N_{t_i}}{2} \\ &+ \Big\{c\big(Y_{t_i} + c(Y_{t_i}) + c(Y_{t_i} + c(Y_{t_i}))\big) - 2\,c\big(Y_{t_i} + c(Y_{t_i})\big) + c\big(Y_{t_i}\big)\Big\} \\ &\times \binom{N_{t_{i+1}} - N_{t_i}}{3} \\ &+ \Big\{c\Big(Y_{t_i} + c\big(Y_{t_i}\big) + c\big(Y_{t_i} + c(Y_{t_i})\big) + c\big(Y_{t_i} + c(Y_{t_i}) + c(Y_{t_i} + c(Y_{t_i}))\big)\Big) \\ &- \;3\,c\Big(Y_{t_i} + c\big(Y_{t_i}\big) + c\big(Y_{t_i} + c(Y_{t_i})\big)\Big) + 3\,c\Big(Y_{t_i} + c\big(Y_{t_i}\big)\Big) - c\Big(Y_{t_i}\Big)\Big\} \\ &\times \binom{N_{t_{i+1}} - N_{t_i}}{4} \end{aligned} \tag{7.3}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

For the linear SDE (2.6) the fourth level approximation is of the form

$$\begin{aligned} Y_{t_{i+1}} &= Y_{t_i}\Big\{1 + \psi\,(N_{t_{i+1}} - N_{t_i}) + \psi^2\binom{N_{t_{i+1}} - N_{t_i}}{2} \\ &\quad + \psi^3\binom{N_{t_{i+1}} - N_{t_i}}{3} + \psi^4\binom{N_{t_{i+1}} - N_{t_i}}{4}\Big\} \end{aligned} \tag{7.4}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$ with $Y_0 = X_0$.

8 General Higher Level Taylor Approximations

It is desirable to be able to construct systematically more and more accurate discrete time approximations for solutions of pure jump SDEs of the form (2.3). For this purpose we use the stochastic Taylor expansion (5.13) to construct the nth level Taylor scheme for pure jump processes.

In Platen (1982a) and in Bruti-Liberati & Platen (2005) convergence theorems for strong Taylor approximations in the more general case of jump diffusions have been presented. When specifying the mentioned theorems to the case of SDEs driven by pure jump processes, it turns out that it is possible to weaken the assumptions on the coefficients of the stochastic Taylor expansions. As we will see below, the Lipschitz and growth conditions (2.4)-(2.5) on the jump coefficient are sufficient to establish the convergence of strong Taylor schemes of any given strong order of convergence $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$.

For a time discretization with maximum step size $\Delta \in (0, 1)$ we define, by using (5.13), the *nth level strong Taylor approximation*

$$Y^{\Delta}_{t_{i+1}} = Y^{\Delta}_{t_i} + \sum_{k=1}^{n} \left(\widetilde{\Delta}_N\right)^k f(Y^{\Delta}_{t_i}) \binom{N_{t_{i+1}} - N_{t_i}}{k} \tag{8.1}$$

for $i \in \{0, 1, \ldots, i_T - 1\}$, with $f(x) = x$, where the operator $\widetilde{\Delta}_N$ is defined in (5.12).

The next three lemmas show that for SDEs driven by pure jump processes the Lipschitz and growth conditions (2.4)-(2.5) imply the conditions on the coefficient functions needed in the convergence theorem for jump diffusions in Bruti-Liberati & Platen (2005).

Lemma 8.1 *Assume that the jump coefficient satisfies the Lipschitz condition*

$$|c(x) - c(y)| \leq K\,|x - y| \tag{8.2}$$

for $x, y \in \mathbb{R}$ with some constant $K \in (0, \infty)$. Then for $k \in \{1, 2, \ldots\}$ the coefficient of the kth level expansion $(\widetilde{\Delta}_N)^k f(x)$, defined in equation (5.12)*, satisfies the Lipschitz condition*

$$|(\widetilde{\Delta}_N)^k f(x) - (\widetilde{\Delta}_N)^k f(y)| \leq C_k\,|x - y| \tag{8.3}$$

for $x, y \in \mathbb{R}$ with $f(x) = x$ and some constant $C_k \in (0, \infty)$, which depends only on the level k of the expansion.

Proof: We will prove the assertion (8.3) by induction on k, where we set $f(x) = x$. For $k = 1$, by the Lipschitz condition (8.2) we obtain

$$\begin{aligned} |(\widetilde{\Delta}_N) f(x) - (\widetilde{\Delta}_N) f(y)| &= \Big| x + c(x) - x - (y + c(y)) + y \Big| \\ &= \Big| c(x) - c(y) \Big| \\ &\leq K\,|x - y|. \end{aligned} \tag{8.4}$$

For $k = n + 1$, by the induction hypothesis, Jensen's inequality and the Lipschitz condition (8.2) we obtain

$$\begin{aligned}
\left|(\widetilde{\Delta}_N)^{n+1} f(x) - (\widetilde{\Delta}_N)^{n+1} f(y)\right| &= \Big|(\widetilde{\Delta}_N)^n f\big(x + c(x)\big) - (\widetilde{\Delta}_N)^n f\big(x\big) \\
&\quad -(\widetilde{\Delta}_N)^n f\big(y + c(y)\big) + (\widetilde{\Delta}_N)^n f\big(y\big)\Big| \\
&\leq C_n\Big|x - y + \big(c(x) - c(y)\big)\Big| + C_n\Big|x - y\Big| \\
&\leq 2\,C_n\Big|x - y\Big| + C_n\,K\Big|x - y\Big| \\
&= C_{n+1}\,\big|x - y\big|, \qquad (8.5)
\end{aligned}$$

which completes the proof of Lemma 8.1. □

Lemma 8.2 *Assume that the jump coefficient satisfies the growth condition*

$$|c(x)|^2 \leq K\,(1 + |x|^2) \qquad (8.6)$$

for $x \in \mathbb{R}$ and some constant $K \in (0, \infty)$. Then for $k \in \{1, 2, \ldots\}$ the coefficient $(\widetilde{\Delta}_N)^k\, f(x)$ of the kth level expansion satisfies the growth condition

$$\Big|(\widetilde{\Delta}_N)^k f(x)\Big|^2 \leq \widetilde{C}_k\,(1 + |x|^2) \qquad (8.7)$$

for $x \in \mathbb{R}$ with $f(x) = x$ and some constant $\widetilde{C}_k \in (0, \infty)$, which depends only on the level k of the expansion.

Proof: We will prove the assertion of Lemma 8.2 by induction on k. For $k = 1$, by applying the growth condition (8.6) we obtain

$$\begin{aligned}
\Big|(\widetilde{\Delta}_N) f(x)\Big|^2 &= \Big|x + c(x) - x\Big|^2 \\
&= \Big|c(x)\Big|^2 \\
&\leq \widetilde{C}_1\,(1 + |x|^2). \qquad (8.8)
\end{aligned}$$

For $k = n + 1$, by the induction hypotheses, Jensen's inequality and the growth condition (8.6) we obtain

$$\begin{aligned}
\Big|(\widetilde{\Delta}_N)^{n+1}\, f(x)\Big|^2 &= \Big|(\widetilde{\Delta}_N)^n f\big(x + c(x)\big) - (\widetilde{\Delta}_N)^n f(x)\Big|^2 \\
&\leq 2\left(\widetilde{C}_n\big(1 + \big|x + c(x)\big|^2\big) + \widetilde{C}_n\big(1 + \big|x\big|^2\big)\right) \\
&\leq 2\Big(\widetilde{C}_n\big(1 + 2(\big|x\big|^2 + \big|c(x)\big|^2)\big) + \widetilde{C}_n\big(1 + \big|x\big|^2\big)\Big) \\
&\leq \widetilde{C}_{n+1}\,(1 + |x|^2), \qquad (8.9)
\end{aligned}$$

which completes the proof of Lemma 8.2. □

Lemma 8.3 *Let us assume that*

$$E(|X_0|^2) < \infty \tag{8.10}$$

and the jump coefficient satisfies the Lipschitz condition

$$|c(x) - c(y)| \leq K\,|x - y| \tag{8.11}$$

and the growth condition

$$|c(x)|^2 \leq K\,(1 + |x|^2) \tag{8.12}$$

for $x, y \in \mathbb{R}$ *with some constant* $K \in (0, \infty)$*. Then for* $k \in \{1, 2, \ldots\}$ *the coefficient* $(\widetilde{\Delta}_N)^k\, f(x)$ *of the kth level expansion satisfies the integrability condition*

$$(\widetilde{\Delta}_N)^k f(\cdot) \in \mathcal{H}_k, \tag{8.13}$$

with $f(x) = x$*, where* $\mathcal{H}_k$ *is the space of predictable stochastic process* $g = \{g(t), t \in [0, T]\}$ *such that*

$$E\left(\int_0^T \int_0^{s_k} \ldots \int_0^{s_2} |g(s, \omega)|^2 d\, s_1 \ldots d\, s_k\right) < \infty. \tag{8.14}$$

Proof: By Lemma 8.2 for $k \in \{1, 2, \ldots\}$ the coefficient of the kth level expansion $(\widetilde{\Delta}_N)^k f(x)$ satisfies the growth condition

$$\left|(\widetilde{\Delta}_N)^k f(x)\right|^2 \leq C_k\,(1 + |x|^2) \tag{8.15}$$

for $x \in \mathbb{R}$ with $f(x) = x$ and some constant $\widetilde{C}_k \in (0, \infty)$. Therefore, for $k \in \{1, 2, \ldots\}$, by condition (8.15) and Fubini's theorem we obtain

$$\begin{aligned} & E\left(\int_0^T \int_0^{s_k} \ldots \int_0^{s_2} |(\widetilde{\Delta}_N)^k f\big(X_{s_1}\big)|^2 d\, s_1 \ldots d\, s_k\right) \\ & \leq E\left(\int_0^T \int_0^{s_k} \ldots \int_0^{s_2} \widetilde{C}_k (1 + |X_{s_1}|^2) d\, s_1 \ldots d\, s_k\right) \\ & = \widetilde{C}_k \frac{T^k}{k!} + \widetilde{C}_k \int_0^T \int_0^{s_k} \ldots \int_0^{s_2} E\left(|X_{s_1}|^2\right) d\, s_1 \ldots d\, s_k \\ & < \infty, \end{aligned} \tag{8.16}$$

where we set $f(x) = x$. The last passage holds, since conditions (8.10), (8.11) and (8.12) ensure that, see Protter (2004),

$$\sup_{t \in [0,T]} E\left(|X_t|^2\right) < \infty \tag{8.17}$$

and this completes the proof of Lemma 8.3. $\square$

Theorem 8.4 *For a given* $\gamma \in \{0.5, 1, 1.5, 2, \ldots\}$*, let* $Y^\Delta = \{Y^\Delta(t), t \in [0,T]\}$ *be the* 2γ *level strong Taylor approximation* (8.1) *corresponding to a time discretization* $(t)_\Delta$ *with* $\Delta \in (0,1)$*. We assume on the jump coefficient* $c(x)$ *the Lipschitz condition* (2.4) *and the growth condition* (2.5)*. Moreover, suppose that*

$$E(|X_0|^2) < \infty \qquad \textit{and} \qquad \sqrt{E(|X_0 - Y_0^\Delta|^2)} \leq K_1 \Delta^\gamma. \tag{8.18}$$

Then the estimate

$$\sqrt{E(\max_{0 \leq i \leq i_T} |X_{t_i} - Y_{t_i}^\Delta|^2)} \leq K\, \Delta^\gamma \tag{8.19}$$

holds, where the constant K *does not depend on* Δ*.*

The proof of Theorem 8.4 is a direct consequence of the convergence theorem for jump diffusions in Bruti-Liberati & Platen (2005). Indeed, by the Lemmas 8.1, 8.2 and 8.3, the coefficients of the nth level approximation satisfy the conditions required by the convergence theorem in Bruti-Liberati & Platen (2005). We emphasize that in the case of SDEs driven by pure jump processes, unlike the more general case of jump diffusions, no extra differentiability conditions on the jump coefficient $c(x)$ are required when deriving higher order approximations.

Theorem 8.4 states that the nth level strong Taylor scheme for pure jump processes achieves a strong order of convergence equal to $\frac{n}{2}$. In fact Theorem 8.4 states that the strong convergence of order $\frac{n}{2}$ is not just at the endpoint T but it is also uniform for all time discretization points. Thus, by including enough terms from the stochastic Taylor expansion (5.13) we are able to construct schemes of any given strong order of convergence $\gamma \in \{0.5, 1, 1.5, \ldots\}$. The multiple stochastic integrals involved are rather simple, as we have seen in (5.8). This makes the above schemes realistically applicable.

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