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THE MULTIFACTOR NATURE OF THE VOLATILITY OF THE EURODOLLAR FUTURES MARKET

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ABSTRACT. This paper seeks to estimate a multifactor volatility model so as to describe the dynamics of interest rate markets, using data from the highly liquid but short term futures markets. The difficult problem of estimating such multifactor models is resolved by using a genetic algorithm to carry out the optimization procedure. The ability to successfully estimate a multifactor volatility model also eliminates the need to include a jump component, the existence of which would create difficulties in the practical use of interest rate models, such as pricing options or producing forecasts.

Key words: Term structure; Volatility; Mutlifactor; Jump; Eurodollar futures; Genetic algorithm;

JEL classifications: C51, C61, E43

1. INTRODUCTION

The volatility of interest rate markets has long been of interest to researchers. The market volatility can be thought of as the result of the incessant arrival of information into the market. A smooth change in the market is expected with the release of routine information, whereas bursts in information (e.g. unexpected "good" or "bad"

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economic news) can cause abrupt jumps (Merton (1976)). For simplicity, we shall refer to these two processes as smooth volatility and jump volatility.

A great number of research papers have analyzed these volatility components in considerable detail. The smooth volatility component is generally found to be determined by the level effect, the slope effect and the curvature effect, giving a nice intuitive understanding on how volatility is linked to the characteristics of the yield curve. Some notable research papers in this area include Litterman and Scheinkman (1991), Chen and Scott (1993), Knez, Litterman and Scheinkman (1994), Singh (1995). The jump volatility component has been analyzed in Babbs and Webber (1995), Naik and Lee (1995), El-Jahel, Lindberg and Perraudin (1997), Das (2002) and Piazzesi (2005), albeit the specification of the jump models are different in these various studies.

The research so far has focused on the spot rate of interest, and analyzed the bond market in particular. In the very efficient futures markets, where instruments with much shorter maturities are traded, it is harder to distinguish and identify different volatility components. As far as the smooth volatility process is concerned, Amin and Morton (1994) argue that there is usually insufficient variation in the term structure across different maturities to separate the effect of different uncertainty sources.[1] However, in a later piece of research, Jegadeesh and Pennacchi (1996) are successful in applying Kalman filter to Eurodollar futures contracts to estimate a linear two-factor interest rate model.

It may be conjectured that different methods of estimation have different abilities in identifying the volatility components in this liquid and short term market. The Kalman filter is a powerful tool, however, it is not readily applicable when a jump component is present. Chiarella and Tô (2003) find that in addition to a single smooth volatility factor, there exists a significant jump component in the futures markets. It is the aim of this paper to empirically analyze different interest rate markets using a model that has a multifactor smooth volatility component as well as a jump component, via a simple yet capable estimation method. We seek in particular to determine whether the inclusion of a sufficient number of smooth volatility factors obviates the need to include a jump volatility component to the stochastic differential equation system describing interest rate dynamics. This would be a desirable outcome as the inclusion of jump volatility

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[1] In the bond market, Duffie and Singleton (1999), Dai and Singleton (2000) have found that allowance should be made for at least two factors driving the term structure.

components makes application of the interest rate modelling framework to areas such as bond and option pricing far more difficult.

The method we use is still the very popular maximum likelihood method that has nice asymptotic properties. The difficulty of the estimation lies on the irregular likelihood function (e.g. a multi-modal versus a smooth and convex function), therefore it is difficult for a standard gradient-based optimization routine to obtain convergence. Moreover, if the optimization converges to an inferior local maximum, the estimates are usually not significant. We, in contrast, use a search-based genetic algorithm to carry out the optimization procedure (see Davis (1991), Goldberg (1989, 2002) and Michalewicz (1999)). We find that the algorithm does give satisfactory results.

The method is used to empirically analyze the volatility structure of the Eurodollar futures contracts traded in the United States, the United Kingdom and the Australian markets. These are all very liquid markets and represent different economic areas in the world economy. The empirical analysis seeks to answer two questions, (1) does the futures market exhibit multi-factor smooth volatility, and (2) if yes, is the jump volatility component still important when this multi-factor nature is taken into account.

2. MODEL FRAMEWORK

2.1. **The framework.**

Among various interest rate models, the no-arbitrage framework of Heath-Jarrow-Morton (1992) (hereafter HJM) offers both parsimony and flexibility to the modelling of the market dynamics. It matches the initial yield curve by construction and ensures the nonexistence of arbitrage opportunities.

The original HJM models only contain smooth volatility noise, which is represented by a Wiener increment in the stochastic differential equation for the evolution of interest rates, namely

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where $f(t,T)$ is the instantaneous T-maturity forward rate at time t, and $W(t)$ is a standard Wiener process. Here $\alpha(t,T)$ and $\sigma(t,T)$ are respectively the maturity dependent instantaneous drift and volatility of $f(t,T)$. The first attempt to introduce a jump volatility component into the HJM class of models perhaps was that of Shirakawa (1991), who replaced the Wiener noise increment $dW(t)$ by the maturity independent

Wiener-Poisson noise increment

$$dW(t) + \sum_{m=1}^{M} \gamma_m \big(dN_m(t) - \lambda_m dt\big),$$

where $N_1, N_2, \ldots, N_M$ are M Poisson processes respectively associated with intensities $\lambda_1(t), \lambda_2(t), \ldots, \lambda_M(t)$. Chiarella and Tô (2003) further allowed the "generalized noise term" to be maturity dependent. However, their empirical analysis only considers a single Wiener noise. We will use their framework here with a multiple Wiener noise to investigate our hypothesis that it is possible to distinguish different volatility components in the very liquid but short term futures markets by using a more powerful optimization procedure than is normally used in similar econometric studies.

With the addition of multifactor Wiener and Poisson processes the evolution for the forward rate now becomes

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{I} \sigma_i(t,T)dW_i(t) + \sum_{m=1}^{M} \delta_m(t,T)dN_m(t), \tag{2.1}$$

where the jump volatility $\delta_m(t,T)$ are assumed to be maturity dependent. Moreover, note that for simplicity of exposition we have included the compensator terms of the jump processes in the $\alpha(t,T)$ term. Different market behaviour can be captured by choosing appropriate smooth volatility functions σ_i and jump volatility function δ_m.

Björk, Kabanov and Runggaldier (1997) (Proposition 3.13, p222) show that to eliminate arbitrage opportunities, the relation

$$\begin{aligned} \alpha(t,T) &= \sum_{i=1}^{I} \sigma_i(t,T) \left(\int_t^T \sigma_i(t,s)ds - \phi_i \right) - \sum_{m=1}^{M} \delta_m(t,T)\gamma_m \, \mathrm{e}^{D_m(t,T)} \, \psi_m(t) \\ &\equiv \tilde{\alpha}(t,T) \end{aligned} \tag{2.2}$$

must hold, ie. the drift is determined by the volatility functions and the market prices of Wiener risks ϕ_i as well as the market prices of Poisson jump risks ψ_i. We will assume constant market prices of risk for simplicity, taking into account the efficiency and liquidity of the futures markets.

Let $F(t, T_F, T_B)$ be the price at time t of a futures contract maturing at time $T_F(> t)$. The contract is written on a pure discount instrument which has a face value of \$1 and matures at time $T_B(> T_F)$. The dynamics of the futures price F is governed by the stochastic differential equation (see Chiarella and Tô (2003))

$$\begin{aligned}\frac{dF(t,T_F,T_B)}{F(t^-,T_F,T_B)} = \left(\alpha^{(F)}(t,\cdot) - \sum_{i=1}^{I} \phi_i \sigma_i^{(F)}(t,\cdot)\right) dt + \sum_{i=1}^{I} \sigma_i^{(F)}(t,\cdot) d\widetilde{W}(t) \\ + \sum_{m=1}^{M} \delta_m^{(F)}(t,\cdot) dN_m(t),\end{aligned} \tag{2.3}$$

where

$$\sigma_i^{(F)}(t,\cdot) \equiv -\int_{T_F}^{T_B} \sigma_i(t,s)ds,$$

$$\delta_m^{(F)}(t,\cdot) \equiv \exp\left(-\int_{T_F}^{T_B} \delta_m(t,s)ds\right) - 1,$$

$$\alpha^{(F)}(t,\cdot) \equiv -\sum_{m=1}^{M} \psi_m \delta_m^{(F)}.$$

2.2. **The model and estimation method.**

We propose a model with a 3-factor smooth volatility component and a jump component. The specification for each volatility function in (2.1) is as follows

$$\begin{aligned}
\sigma_1(t,T) &= \gamma_1, \\
\sigma_2(t,T) &= \gamma_2 \exp(-\kappa_2(T-t)), \\
\sigma_3(t,T) &= \gamma_3 (T-t) \exp(-\kappa_3(T-t)), \\
\delta(t,T) &= \gamma \exp(-\kappa(T-t)), \quad (I=3, M=1).
\end{aligned}$$

The first volatility function σ_1 can be thought of as the long run volatility value, the volatility of a forward rate with infinite maturity. The second volatility function σ_2 (with $\kappa_2 > 0$) allows the normal shock in the market to have higher impact at the short end of the curve than at the longer end of the curve. The volatility function σ_3 (with $\kappa_3 > 0$) creates a hump in the volatility curve, which is a feature some previous research have found (Kahn (1991), Amin and Morton (1994), Moraleda and Vorst (1997a, 1997b), Ritchken and Chuang (1999), Chiarella and Tô (2003)), although the specification for the hump might be slightly different. The jump volatility δ is specified as a simple exponential function since (for $\kappa > 0$) the arrival of some “surprise” in information is expected to have more impact on rates maturing in the near-future.

With this specification, it is a relatively routine task to write out the evolution (2.3) of the observed futures prices. A maximum likelihood estimator using discrete data can be derived using the method of Lo (1988), details can be found in Chiarella and Tô (2003). The basic idea is to find the density of F via the transformation $X = \ln(F)$. As X follows a Poisson mixture of normal distributions, a Bernoulli approximation can be used for computational purposes, see Appendix A for a brief summary.

Even though it is straightforward to write out the likelihood function, it is not such a simple matter to maximize this function. Normal hill-climbing optimization routines often fail to locate the maximum point, and the estimates may be insignificant. To overcome these difficulties we use a genetic algorithm[2], which is known for its ability to solve difficult optimization problems which have complex fitness landscapes. The algorithm is designed to move the set of parameters away from local minima where a traditional hill-climbing algorithm may get stuck. A review of the technique is presented in the next session.

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[2] The Fortran computer code of Dr Hing Hung is gratefully acknowledged.

3. OPTIMIZATION METHOD

The genetic algorithm (GA) is an evolutionary algorithm which follows the "survival of fittest" strategy. The algorithm starts the search of the parameter domain from a completely random generation, then evolves from generation to generation. In each generation, the individuals are modified by mutation and crossover (ie. recombination), and the fittest individuals are selected for the next generation.

There is a wide range of variants in the actual implementation of the GA since the premier work of Holland (1975), Davis (1991), Goldberg (1989, 2002) and Michalewicz (1999). Any variant requires the determination of 5 fundamental issues: solution representation, creation of initial population, reproduction operators, selection function and termination criteria. Our implementation is as follows;

3.1. **Solution representation.** Each individual in the population of interest is presented by a chromosome. Our chromosome is made up of a sequence of genes from an alphabet containing only binary digits (0 and 1).

3.2. **Reproduction operators.** There are two basic genetic operators: crossover (or recombination) and mutation. Crossover produces two new individuals using two existing ones, whereas mutation produces one new individual by altering one existing one.

We use two types of crossover. A uniform crossover interchanges every bit of two chromosomes with a certain probability. A double point crossover randomly select two points of the chromosomes to perform the interchanging task. The mutation is a binary mutation which flips every bit of a chromosome with a certain probability.

3.3. **Selection function.** The GA selects individuals for successive generations using probabilistic approach. The selection is based on each individual's fitness value, so that better individuals have higher chance of being selected. We use the standard roulette wheel selection method of Holland (1975). The probability p_i of individual i being chosen for the next generation is

$$p_i = \frac{f_i}{\sum_{j=1}^{N} f_j},$$

where f_i is the fitness value of individual i and N is the population size.

Based on this roulette wheel selection, we further have 2 methods to select the parents and offsprings, which we name "common method" and "élite method". The common method is a plain-vanilla method of selecting parents. The élite method takes the three fittest individuals, chooses their partners among the rest of the population and the number of offsprings from each (élite) individual based on the roulette wheel principle. If this method is used, we let individuals have finite ages. Under both methods, we introduce new immigration into the population to keep it diverse.

3.4. **Initial population and termination criteria.** The initial population is randomly generated. Depending on the method of selecting parents, different termination criterion is used. If the common method is used, the GA stops when the best solution fails to improve over a specified number of generations. If the élite method is used, the GA stops when a specified maximum number of generations is reached.[3]

4. EMPIRICAL ANALYSIS

4.1. **Data.**

The data used in this research is the Eurodollar futures contract traded on the Chicago Mercantile Exchange (CME). Daily data is downloaded from Datastream^{TM}, from the first trading day of 1988 to the 30th June 2004. Each day multiple contracts are used in the estimation. Even though the same set of contracts is used for each year, due to their relatively short lives, a new set of contracts is rolled over each year.

4.2. **Empirical results.**

4.2.1. *Models with no jump component.*

We first estimated the 3-factor smooth volatility model without the jump volatility component. Different from previous research where it was claimed that different factors could not be separated using futures data, we found significant estimates for all of the factors in all three markets. The estimates and their standard errors can be found in Table 1.

The first volatility factor, which affects the whole yield curve equally, is measured by γ_1. This permanent shock is quite small, respectively being 7.3, 7.9 and 9.5 basis points in the U.S, U.K and Australian market. The second volatility factor affects

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[3] In our empirical work, we use both methods as a double check on each other.

TABLE 1. Estimation for 3-factor model

	U.S.		U.K.		Australia	
Parameter	Estimate	Std.err.	Estimate	Std.err.	Estimate	Std.err.
γ_1	0.00073	0.00006	0.00080	0.00002	0.00095	0.00003
γ_2	0.0195	0.0003	0.0649	0.0016	0.0434	0.0010
γ_3	1.8650	0.0156	1.2118	0.0338	1.6202	0.0421
κ_2	11.3150	0.1454	17.163	0.5198	15.935	0.3747
κ_3	8.2519	0.0650	19.028	0.5036	19.284	0.3773
ϕ_1	-8.6466	1.3527	38.677	1.6962	44.020	1.4100
ϕ_2	27.145	0.3621	27.559	0.6756	2.7420	0.2090
ϕ_3	-3.5924	0.1952	5.1674	0.5454	49.641	1.7779
σ_e	0.00031	0.000003	0.00020	0.00000	0.00023	0.00000

the short end of the curve more than the long end. This volatility component for the instantaneous spot rate of interest is 0.02 in the U.S market, doubles to 0.043 in the Australian market, and increases to 0.065 in the U.K market. However, this volatility factor dies out quickly as the contract maturity lengthens. The two contracts only need to be 9.6 days apart in the U.S market for this volatility factor to halve. In the other two markets, the distance between two contracts for this second volatility factor to halve is smaller, at 6.3 days for the U.K and 6.8 days for the Australian market. The third volatility factor has a hump shape. We find that the hump appears at a relatively small time to maturity. The volatility factor obtains its maximum at contracts that have 44 days time-to-maturity in the U.S market, and at contracts that have around 19 days time-to-maturity in the other two markets.

The model's in-sample interest rate prediction error is reasonably small. Table 2 reports the prediction errors for different markets across different contract maturities. The mean absolute error is less than 5.5 basis points in the U.S. and U.K market, and slightly higher at 6.5 basis points in the Australian markets. Overall, there are no clear differences in the prediction errors across maturities.

4.2.2. *Models with jump component.*

We added a jump component to the 3-factor smooth volatility model. In all of the markets, the estimates of the parameters specifying the jump volatility component are insignificant. To ascertain that it is the jump component that adds no further explanatory power to the model, we estimated various jump models with less number of factors

TABLE 2. Interest rate prediction error

This table reports the errors in interest rate prediction for the different contract maturities (measured in years) used in the estimation. The numbers of contracts used in the estimation for each market are different due to liquidity constraints. All of the values are reported as basis points.

Contract Maturity	1.25 yrs	2 yrs	2.75 yrs	3.5 yrs	4.25 yrs	5 yrs
	U.S. market					
Mean error	-1.6	-6.7	-5.7	-5.1	-4.4	-4.3
Standard deviation of error	7.5	7.6	7.1	6.9	6.9	6.8
Mean absolute error	5.5	5.5	5.2	5.1	5.2	5.1
Stdev (absolute error)	5.3	5.3	4.9	4.6	4.6	4.5
	U.K. market					
Mean error	1.4	1.0	1.2			
Standard deviation of error	8.7	7.4	6.0			
Mean absolute error	5.6	5.2	4.5			
Stdev (absolute error)	6.8	5.4	4.2			
	Australian market					
Mean error	1.5	1.2				
Standard deviation of error	8.8	9.0				
Mean absolute error	6.3	6.5				
Stdev (absolute error)	6.4	6.4				

in the smooth volatility component, as reported in Table 3. However, under all of those specifications, the jump component is not significant. It therefore can be concluded that if we can successfully estimate a multi-factor smooth volatility model, a jump component is not needed. From a practical point of view, this is a good news, as the analysis and computation for models without jump components is much less demanding. It is easier to apply different estimation techniques, eg. the Kalman filter, to use the estimated parameters to forecast, or to price different instruments, such as options on futures.

5. CONCLUSION

We have demonstrated that a computational technique is very important in empirical work. For the liquid but short term interest rate futures markets, the estimation of a multi-factor stochastic differential equation model has always been difficult, and it was argued previously that different factors were not distinguishable. The estimation difficulties arise from the fact that the standard hill-climbing optimization procedures

TABLE 3. Specific models considered

The models are formed by combining different volatility functions. The 3 smooth volatility components are $\sigma_1(t,T) = \gamma_1$, $\sigma_2(t,T) = \gamma_2 \exp(-\kappa_2(T - t))$, $\sigma_3(t,T) = \gamma_3(T-t)\exp(-\kappa_3(T-t))$, and the jump volatility component is $\delta(t,T) = \gamma \exp(-\kappa(T - t))$.

Model code	$\sigma_1(t,T)$	$\sigma_2(t,T)$	$\sigma_3(t,T)$	$\delta(t,T)$
J1F1	✓			✓
J1F1B		✓		✓
J1F1C			✓	✓
J1F2	✓	✓		✓
J1F2C		✓	✓	✓
J1F3	✓	✓	✓	✓

cannot cope with the non-smooth likelihood surfaces that occur in multifactor models. To overcome these difficulties we use a genetic algorithm and have been able to satisfactorily estimate a 3-factor model for the futures markets. The inclusion of a multifactor volatility obviates the need to include a jump volatility component, thus making the models more attractive to practical uses. Even though the estimation is of some degree computationally intensive, it is worthwhile given the benefit of not having to include the jump component for later uses of this interest rate modelling framework, such as pricing options or making forecasts.

APPENDIX A. DENSITY FOR THE STATE VARIABLE

Assume that for each underlying pure-discount interest rate instrument, there are K futures contracts maturing at times T_{Fk} ($k = 1, 2, \ldots, K$). The (observable) quoted futures price in the market is $G(t, T_{Fk}, T_{Bk})$, which is linked with $F(t, T_{Fk}, T_{Bk})$ via a function η, i.e. $F(t, T_{Fk}, T_{Bk}) \equiv \eta\big(G(t, T_{Fk}, T_{Bk})\big)$, which depends on the quoting convention of each exchange. Let $X(t, T_{Fk}, T_{Bk})$ be a state variable defined by $X(t, T_{Fk}, T_{Bk}) = \ln(F(t, T_{Fk}, T_{Bk}))$.

The evolution of our state variable X, incorporating a random measurement error ε_k, is

$$\begin{aligned} X(t,T_{Fk},T_{Bk}) &= X(0,T_{Fk},T_{Bk}) + \int_0^t \left(\mu_k(u,\cdot) - \frac{1}{2}\sigma_\varepsilon^2\right) du \\ &+ \int_0^t \beta_k(u,\cdot) dW(u) + \sum_{m=1}^{M} \int_0^t \nu_{m,k}(u,\cdot) dN_m(u) + \sigma_\varepsilon \varepsilon_k. \end{aligned} \tag{A.1}$$

The transitional likelihood function for X is

$$\begin{aligned} & p_{\boldsymbol{X}}(\boldsymbol{x}_j, t_j|\boldsymbol{x}_{j-1}, t_{j-1}; \boldsymbol{\theta}) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_M=0}^{\infty}\left(\mathrm{e}^{-\lambda_1\Delta t}\,\frac{(\lambda_1\Delta t)^{n_1}}{n_1!}\right)\cdots\left(\mathrm{e}^{-\lambda_M\Delta t}\,\frac{(\lambda_M\Delta t)^{n_M}}{n_M!}\right)\Phi\left(\boldsymbol{x}_j;\boldsymbol{a}_j,\boldsymbol{\Omega}_j\right), \end{aligned}$$

where $\Delta t = t_j - t_{j-1}$ and Φ is a multivariate Gaussian density defined by

$$\begin{aligned} \Phi\left(\boldsymbol{x}_j;\boldsymbol{a}_j,\boldsymbol{\Omega}_j\right) &= (2\pi)^{-\frac{K}{2}}\left|\boldsymbol{\Omega}_j\right|^{-\frac{1}{2}} \\ &\exp\left(-\frac{1}{2}\left(\boldsymbol{x_j}-\boldsymbol{x}_{j-1}-\boldsymbol{a}_j\right)'\boldsymbol{\Omega}_j^{-1}\left(\boldsymbol{x_j}-\boldsymbol{x}_{j-1}-\boldsymbol{a}_j\right)\right), \end{aligned}$$

whose mean and variance are

$$\boldsymbol{a}_j = (a_{j1} \quad a_{j2} \quad \ldots \quad a_{jK})',$$

$$\boldsymbol{\Omega}_j = \begin{pmatrix} b_{j(11)} & b_{j(12)} & \ldots & b_{j(1K)} \\ b_{j(21)} & b_{j(22)} & \ldots & b_{j(2K)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j(K1)} & b_{j(K2)} & \ldots & b_{j(KK)} \end{pmatrix},$$

and the matrix elements are defined by

$$a_{jk} = \int_{t_{j-1}}^{t_j}\left(\mu_k(u) - \frac{1}{2}\sigma_\varepsilon^2 + \sum_{m=1}^{M} n_m\nu_{mk}(u)\right) du,$$

$$b_{j(kk)} = \int_{t_{j-1}}^{t_j}\left(\beta_k^2(u) + \sigma_\varepsilon^2\right) du, \tag{A.2}$$

$$b_{j(k_1k_2)} = \int_{t_{j-1}}^{t_j}\beta_{k_1}(u)\beta_{k_2}(u) du \quad \text{for } k_1 \neq k_2. \tag{A.3}$$

The transitional likelihood function for X can be approximated by a Bernoulli mixture of Gaussian densities, where all of the values of order higher than $(dt)^2$ is ignored,

$$
\begin{aligned}
& p_{\boldsymbol{X}}(\boldsymbol{x}_j, t_j | \boldsymbol{x}_{j-1}, t_{j-1}; \boldsymbol{\theta}) \\
={}& \sum_{m=1}^{M} \lambda_m \Delta t \, \Phi(\boldsymbol{x}_j; \boldsymbol{a}_{j(m)}, \boldsymbol{\Omega}_j) + \left(1 - \sum_{m=1}^{M} \lambda_m \Delta t\right) \Phi(\boldsymbol{x}_j; \boldsymbol{a}_{j(0)}, \boldsymbol{\Omega}_j),
\end{aligned}
\tag{A.4}
$$

where

$$
\boldsymbol{a}_{j(m)} = (a_{j1(m)} \quad a_{j2(m)} \quad \ldots \quad a_{jK(m)})', \quad \text{for } m = 0, 1, \ldots, M,
$$

and the matrix elements are defined by

$$
a_{jk(0)} = \int_{t_{j-1}}^{t_j} \left(\mu_k(u) - \frac{1}{2}\sigma_\varepsilon^2\right) du, \tag{A.5}
$$

$$
a_{jk(m)} = \int_{t_{j-1}}^{t_j} \left(\mu_k(u) - \frac{1}{2}\sigma_\varepsilon^2 + \nu_{mk}(u)\right) du \quad \text{for } m \neq 0. \tag{A.6}
$$

Using the likelihood function of X and the maximum likelihood transformation method, the likelihood function for the quoted futures prices can be easily found.

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