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A NOTE ON THE BIAS OF USING FUTURES RATES AS A PROXY FOR THE INSTANTANEOUS FORWARD RATE

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A NOTE ON THE BIAS OF USING FUTURES RATES AS A PROXY FOR THE INSTANTANEOUS FORWARD RATE

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ABSTRACT. The note shows that there is a non-negligible bias in using the futures rates as a proxy for the instantaneous forward rates in the estimation of forward rate models. It is therefore desirable to derive the evolution of observable rates, then use the distributional properties of this evolution to do the estimation. In a general case where these properties are hard to obtained, a filtering technique is required.

Key words: Heath-Jarrow-Morton; Forward rate; Futures; Estimation bias;

JEL classifications: C51, E43, G12, G13

1. INTRODUCTION

Interest rate modelling has long been of interest to researchers. The yield curve is usually characterized in terms of the stochastic processes followed by the instantaneous spot rate or the instantaneous forward rate, of which spot rate is a special case. However, these instantaneous forward rates are not directly observable in the market, and attempt to use proxies for these rates might result in estimation bias, and consequently unattractive properties for pricing or risk management purposes.

In the case where the interest rate model is specified in terms of the instantaneous spot rate, the most used proxy is the short term treasury bond rates. The bias arisen due to this proxy is significant, as has been documented in Chapman et al. (1999). In the case where the interest rate model is specified in terms of the instantaneous forward

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rate, a natural candidate is the short term futures yields. This paper investigates the bias involved in this approximation and its implications.

2. THE INTEREST RATE MODEL

The framework of Heath, Jarrow and Morton ((1992), hereafter HJM), and later of Brace, Gatarek and Musiela (1997), is a popular framework within the arbitrage-free class of interest rate models. The HJM models start with the evolution of the instantaneous forward rate

$$f(t,T) = f(0,T) + \int_0^t \mu(u,T,\cdot)du + \int_0^t \sigma(u,T)dW(u), \tag{2.1}$$

where the $W(t)$ is a standard Wiener process under the historical probability measure $\mathcal{Q}$, and $\mu(t,T,\cdot)$ and the $\sigma(t,T)$ are respectively the drift and the diffusion coefficient for the instantaneous forward rate to maturity T. Here we shall assume that the $\sigma(t,T)$ is time deterministic functions.

From the evolution of the instantaneous forward rate, the evolutions of other economic variables such as the instantaneous spot rate, the bond price, can be derived accordingly. HJM show that in order to eliminate arbitrage opportunities among traded securities in the market, the drift term in (2.1) is uniquely determined by the volatility term via the market price of risk

$$\mu(t,T,\cdot) = -\sum_i \sigma_i(t,T)\left[\phi_i(t) - \int_t^T \sigma_i(t,s)ds\right]. \tag{2.2}$$

The evolution for the instantaneous forward rate becomes

$$df(t,T) = \sum_i \left[\sigma_i(t,T)\left(\int_t^T \sigma_i(t,s)ds - \phi_i(t)\right)dt + \sigma_i(t,T)dW(t)\right]. \tag{2.3}$$

Estimating the parameters of this instantaneous forward rate becomes crucial in pricing traded instrument and risk management. Since the instantaneous rates do not exist in the market, practitioners may be tempted to use the market futures yields as a proxy for the instantaneous forward rate. In the next section, we will examine the theoretical relationship between these two variables.

3. THE LINK BETWEEN INSTANTANEOUS FORWARD RATE AND FUTURES YIELDS

Let $F(t, T_F, T_B)$ be the price at time t of a futures contract maturing at time $T_F(> t)$. The contract is written on a pure discount instrument which has a face value of \$1 and matures at time $T_B(> T_F)$. The evolution of this futures price is derived in Musiela et al. (1992)[1]

$$\begin{aligned}\frac{dF(t,T_F,T_B)}{F(t,T_F,T_B)} &= -\sum_i \left(\int_{T_F}^{T_B} \sigma_i(t,s)ds\right) d\widetilde{W}_i(t) \\ &\equiv \sum_i \sigma_{Fi}(t,T_F,T_B) d\widetilde{W}_i(t),\end{aligned} \tag{3.1}$$

where $\widetilde{W}$ is a standard Wiener process in the equivalent probability measure $\widetilde{\mathcal{Q}}$.

Let $y(t, T_F, T_B)$ be the market quoted "futures yield" corresponding to the futures price $F(t, T_F, T_B)$, ie. the quantity defined according to [2]

$$F(t,T_F,T_B) = 1 - y(t,T_F,T_B)(T_B - T_F). \tag{3.2}$$

Application of Itô's lemma gives the stochastic differential equation for $y(t, T_F, T_B)$ under $\widetilde{\mathcal{Q}}$, viz

$$dy(t,T_F,T_B) = \left(\frac{1}{T_B - T_F} - y\right) \sum_i \sigma_{Fi}(t,T_F,T_B) d\widetilde{W}_i(t), \tag{3.3}$$

or in the historical measure

$$dy(t,T_F,T_B) = \left(\frac{1}{T_B - T_F} - y\right) \sum_i \left[-\phi_i \sigma_{Fi}(t,T_F,T_B)dt + \sigma_{Fi}(t,T_F,T_B) d\widetilde{W}_i(t)\right]. \tag{3.4}$$

It follows from (2.3) that the forward rate $f(t, T_F)$ is distributed normally, whereas it is clear from (3.4) that the futures yield $y(t, T_F, T_B)$ is distributed log-normally. Using one as a proxy for another will result in estimation bias. The question is that whether this bias is negligible.

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[1] Musiela et al. (1992) derive the evolution under a single noise case. A straight forward extension to multiple noise case can be found in Chiarella and Tô (2004).

[2] This is the "futures yield" quoted as a discount rate, which is appropriate in the U.S. market. In some other markets such as the Australian market, it may be more appropriate to use the "futures yield" quoted as a yield-to-maturity, i.e. according to the formula $F(t, T_F, T_B) = \frac{1}{1+y(t,T_F,T_B)(T_B-T_F)}$. The lines of argument follow similarly.

In order to examine this question, we look at the first two moments of the two distributions. The mean of each distribution is clearly dependent on the variance of that distribution. Therefore, we focus on the variances first.

From (2.1), the variance of the instantaneous forward rate is

$$\operatorname{var}\big(f(t,T_B)\big) = \sum_i \int_{t_0}^{t} \sigma_i^2(u,T_B)du, \tag{3.5}$$

whereas the variance of the fixed-maturity futures yield is (see Appendix A)

$$\begin{aligned}\operatorname{var}\big(y(t,T_F,T_B)\big) = &\left(\frac{1}{T_B-T_F} - y(0,T_F,T_B)\right)^2 \times \\ &\exp\left(\sum_i \int_{t_0}^{t} 2\phi_i \sigma_{Fi}(u)du\right)\left(e^{\bar{\sigma}_f^2 (T_B-T_F)^2} - 1\right),\end{aligned} \tag{3.6}$$

where

$$\bar{\sigma}_f^2 = \frac{1}{(T_B-T_F)^2} \sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u,T_F,T_B)du.$$

The difference between the two variance measures is the overall bias, which can be decomposed into two components, maturity bias and convexity bias, as illustrated in Figure 1, where we have denoted by $f(t,T_F,T_B)$ the discrete-period forward rate, which is the holding period return between time T_F and $T_B (> T_F)$ of a bond maturing at time T_B, ie. $f(t,T_F,T_B)$ satisfies

$$P(t,T_F) = P(t,T_B)\exp\big(f(t,T_F,T_B)(T_B-T_F)\big).$$

FIGURE 1. Bias Decomposition

$$\begin{tikzcd}
\boxed{f(t,T_F)} \arrow[rr, "\text{Overall Bias}"] \arrow[dr, "\text{Maturity Bias}"'] & & \boxed{y(t,T_F,T_B)} \\
& \boxed{f(t,T_F,T_B)} \arrow[ur, "\text{Convexity Bias}"'] &
\end{tikzcd}$$

In Appendix B, we show that the variance of this discrete-period forward rate is

$$\operatorname{var}\big(f(t,T_F,T_B)\big) = \bar{\sigma}_f^2. \tag{3.7}$$

Therefore, from (3.5) and (3.7), the maturity bias component, which arises from approximating the instantaneous forward rate by the discrete-period forward rate, is given by

$$\text{Maturity Bias} = \bar{\sigma}_f^2 - \sum_i \int_{t_0}^{t} \sigma_i^2(u, T_F)\,du. \tag{3.8}$$

This bias component is negligible when the discrete period is short (ie. $\tau = T_B - T_F \rightarrow 0$). This is in agreement with Chapman et al. (1999) who study the bias induced by using fixed tenor short rates as a proxy for the instantaneous spot rate. They also conclude that the bias is not economically significant in the class of linear short rate models, to which the HJM with deterministic volatility belongs.

The convexity bias component, which arises from approximating the fixed-maturity forward rate by a fixed-maturity futures rate, is given by (see (3.6) and (3.7))

$$\begin{aligned}\text{Convexity Bias} = {} & \left(\frac{1}{T_B - T_F} - y(0, T_F, T_B)\right)^2 \times \\ & \exp\left(\sum_i \int_{t_0}^{t} 2\phi_i \sigma_{Fi}(u)\,du\right)\left(e^{\overline{\sigma}_f^2 (T_B - T_F)^2} - 1\right) - \overline{\sigma}_f^2.\end{aligned} \tag{3.9}$$

which is non-negligible due to the presence of the initial futures yield value, the market price of interest rate risk and the convexity of the exponential function. The difference between forward rates and futures rates results from the difference between forward contract prices and futures contract prices. The marking-to-market feature of futures contracts causes their prices to differ from forward contract prices under a stochastic interest rate environment.

4. MONTE CARLO SIMULATIONS

We have run a Monte Carlo simulation in order to gauge the level of bias when the futures yield is used as a proxy for the instantaneous forward rate in estimation. The simulation was run for a single factor HJM model with a humped forward volatility curve

$$\sigma(t, T) = [\sigma_0 + \sigma_1(T - t)]\exp(-\kappa(T - t)),$$

and a constant market price of risk ϕ. The model was simulated (50,000 times) for a time period of one year (252 observations) from an assumed true parameter set[3]. First, we simulated the futures price according to its dynamics (3.1) (transformed into the historical measure). Then we used this futures price series as the proxy for the instantaneous forward rate, and estimated the model via the likelihood function based on the instantaneous forward rate evolution (2.3). The results of the simulation are displayed in Table 1. It can be clearly seen that the proxy method results in quite high mean bias and root mean squared error. Thus it is advisable that in empirical work the futures yields should not be used as a proxy for the instantaneous forward rates.

TABLE 1. Estimation Bias from the Proxy Method

This table reports the bias resulting from using "futures yields" calculated from the futures price as a proxy for the instantaneous forward rate in estimation. The simulation is run for 50,000 experiments. "Mean MC" is the mean for all simulated estimates. "MCSD" is the standard deviation of the simulated estimates. "Mean Bias" is the difference between the "Mean MC" and the true parameter value. "RMSE" is the root of the mean squared errors.

Parameters	True value	Mean MC	MCSD	Mean Bias	RMSE
σ_0	0.01	0.0074	0.0036	-0.0026	0.0045
σ_1	0.004	0.0107	0.0121	0.0067	0.0138
κ	0.25	0.5271	0.3873	0.2771	0.4762
ϕ	0.7	1.3128	2.5650	0.6128	2.6372

5. CONCLUSION

This note shows that using a futures rates as a proxy for the instantaneous forward rates to estimate forward rate models will result in significant estimation bias. It is therefore desirable to derive the evolution for observed rates and use this evolution to estimate the underlying forward rate model. In general cases, where the distribution properties of this evolution cannot be derived, a filtering technique is required.

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[3] This assumed true set was chosen to coincide with the estimated values found in the empirical analysis of Bhar et al. (2004)

APPENDIX A. VARIANCE OF FUTURES YIELD

The futures yield follows a stochastic differential equation

$$dy(t,T_F,T_B)=\sum_i\left(\frac{1}{T_B-T_F}-y\right)\Big(-\phi_i\sigma_{Fi}(t)dt+\sigma_{Fi}(t)dW_i(t)\Big).$$

Let $z(t,T_F,T_B)=\frac{1}{T_B-T_F}-y(t,T_F,T_B)$, then $\operatorname{var}(y(t,T_F,T_B))=\operatorname{var}(z(t,T_F,T_B))$, and

$$\frac{dz(t,T_F,T_B)}{z}=\sum_i\left(\phi_i\sigma_{Fi}(t)dt-\sigma_{Fi}(t)dW_i(t)\right).$$

With a view to calculating $\mathbb{E}_0[z(t,T_F,T_B)]$ and $\operatorname{var}_0[z(t,T_F,T_B)]$ we set

$$m(t)=\ln z(t,T_F,T_B) \tag{A.1}$$

and

$$n(t)=\ln\big(z(t,T_F,T_B)\big)^2=2m(t). \tag{A.2}$$

Application of Itô’s lemma to (A.1) followed by an integration yields

$$m(t)=m(t_0)+\sum_i\int_{t_0}^{t}\left(\phi_i\sigma_{Fi}(u)-\frac{1}{2}\sigma_{Fi}^2(u)\right)du-\sum_i\int_{t_0}^{t}\sigma_{Fi}(u)dW_i(u), \tag{A.3}$$

and it follows from (A.2) that

$$n(t)=2m(t_0)+\sum_i\int_{t_0}^{t}\left(2\phi_i\sigma_{Fi}(u)-\sigma_{Fi}^2(u)\right)du-\sum_i\int_{t_0}^{t}2\sigma_{Fi}(u)dW_i(u). \tag{A.4}$$

Since $\sigma_{Fi}(t)$ are deterministic functions of time, (A.3) and (A.4) imply that both $m(t)$ and $n(t)$ are normally distributed and we readily calculate that

$$m(t)\sim\mathcal{N}\left(m(t_0)+\sum_i\int_{t_0}^{t}\left(\phi_i\sigma_{Fi}(u)-\frac{1}{2}\sigma_{Fi}^2(u)\right)du\ ,\ \sum_i\int_{t_0}^{t}\sigma_{Fi}^2(u)du\right), \tag{A.5}$$

and

$$n(t)\sim\mathcal{N}\left(2m(t_0)+\sum_i\int_{t_0}^{t}\left(2\phi_i\sigma_{Fi}(u)-\sigma_{Fi}^2(u)\right)du\ ,\ 4\sum_i\int_{t_0}^{t}\sigma_{Fi}^2(u)du\right). \tag{A.6}$$

We recall that if a random variable $v(t)$ is distributed $\mathcal{N}\left(\mu(t), \sigma^2(t)\right)$ then

$$\mathbb{E}\left[\mathrm{e}^{v(t)}\right] = \mathrm{e}^{\mu(t)+\frac{1}{2}\sigma^2(t)}\,.$$

Using this result we calculate from (A.5) and (A.6) that

$$\mathbb{E}_0\left[z(t, T_F, T_B)\right] = \mathbb{E}_0\left[\mathrm{e}^{m(t)}\right] = \exp\left(m(0) + \sum_i \int_{t_0}^{t} \phi_i \sigma_{Fi}(u)du\right), \qquad \text{(A.7)}$$

and

$$\begin{aligned}\mathbb{E}_0\left[z^2(t, T_F, T_B)\right] &= \mathbb{E}_0\left[\mathrm{e}^{n(t)}\right] \\ &= \exp\left(2m(0) + \sum_i \int_{t_0}^{t} \left(2\phi_i \sigma_{Fi}(u) + \sigma_{Fi}^2(u)\right)du\right). \qquad \text{(A.8)}\end{aligned}$$

Using (A.7) and (A.8) and the relationship

$$\begin{aligned}\operatorname{var}\left[y(t, T_F, T_B)\right] &= \operatorname{var}\left[z(t, T_F, T_B)\right] \\ &= \mathbb{E}_0\left[z^2(t, T_F, T_B)\right] - \Big(\mathbb{E}_0\left[z(t, T_F, T_B)\right]\Big)^2,\end{aligned}$$

and some minor manipulations, we obtain

$$\begin{aligned}\operatorname{var}\big(y(t, T_F, T_B)\big) = &\left(\frac{1}{T_B - T_F} - y(0, T_F, T_B)\right)^2 \times \exp\left(\sum_i \int_{t_0}^{t} 2\phi_i \sigma_{Fi}(u)du\right) \\ &\times \left[\exp\left(\sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du\right) - 1\right]. \qquad \text{(A.9)}\end{aligned}$$

If we define

$$\bar{\sigma}_f^2 \equiv \frac{1}{(T_B - T_F)^2} \sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du,$$

then (3.6) is obtained.

APPENDIX B. FIXED-MATURITY FORWARD RATE EVOLUTION

Consider an investor who holds a bond maturing at T_B and seek the return he or she would earn between T_F and $T_B (> T_F)$ by contracting now at time t. The required rate of return is the discrete period forward rate $f(t, T_F, T_B)$ defined by

$$P(t, T_F) = P(t, T_B) \exp\big(f(t, T_F, T_B)(T_B - T_F)\big),$$

ie.

$$\begin{aligned} f(t,T_F,T_B) &= \frac{1}{T_B-T_F}\ln\left[\frac{P(t,T_F)}{P(t,T_B)}\right] \\ &= \frac{1}{T_B-T_F}\int_{T_F}^{T_B} f(t,s)ds. \end{aligned}$$

Recall that the evolution of the instantaneous forward rate is

$$f(t,T_B) = f(0,T_B) + \sum_i \left[\int_0^t \sigma_i(u,T_B)\int_u^{T_B}\sigma_i(u,v)dv\,du + \int_0^t \sigma_i(u,T_B)d\widetilde{W}_i(u)\right].$$

Therefore, the discrete period forward rate $f(t,T_F,T_B)$ evolves according to

$$\begin{aligned} f(t,T_F,T_B) &= \frac{1}{T_B-T_F}\int_{T_F}^{T_B} f(0,s)ds \\ &\quad + \frac{1}{T_B-T_F}\sum_i \int_{T_F}^{T_B}\int_0^t \sigma_i(u,s)\int_u^s \sigma_i(u,v)dv\,du\,ds \\ &\quad + \frac{1}{T_B-T_F}\int_{T_F}^{T_B}\int_0^t \sigma_i(u,s)d\widetilde{W}_i(u)\,ds \\ &= \frac{1}{T_B-T_F}\int_{T_F}^{T_B} f(0,s)ds \\ &\quad + \frac{1}{T_B-T_F}\sum_i \int_0^t\int_{T_F}^{T_B} \sigma_i(u,s)\int_u^s \sigma_i(u,v)dv\,ds\,du \\ &\quad + \frac{1}{T_B-T_F}\int_0^t\int_{T_F}^{T_B} \sigma_i(u,s)ds\,d\widetilde{W}_i(u). \end{aligned}$$

The variance of the discrete period forward rate is thus readily calculated as

$$\begin{aligned} \operatorname{var}\big(f(t,T_F,T_B)\big) &= \frac{1}{(T_B-T_F)^2}\sum_i \int_{t_0}^t \left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)^2 du \\ &= \overline{\sigma}_f^2. \end{aligned}$$

This variance is calculated under the equivalent measure. However, the variance of $f(t,T_F,T_B)$ is preserved under the change of measure, therefore it is also equal to $\overline{\sigma}_f^2$ under the historical measure.

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