



QUANTITATIVE FINANCE
RESEARCH CENTRE



UNIVERSITY OF
TECHNOLOGY SYDNEY



QUANTITATIVE FINANCE
RESEARCH CENTRE

UTS

THINK.CHANGE.DO

QUANTITATIVE FINANCE RESEARCH CENTRE

Research Paper 143

December 2004

Capital Asset Pricing for Markets with Intensity Based Jumps

Eckhard Platen

ISSN 1441-8010

www.qfrc.uts.edu.au

Capital Asset Pricing for Markets with Intensity Based Jumps

Eckhard Platen¹

December 2, 2004

Abstract. This paper proposes a unified framework for portfolio optimization, derivative pricing, modeling and risk measurement in financial markets with security price processes that exhibit intensity based jumps. It is based on the natural assumption that investors prefer more for less, in the sense that for two given portfolios with the same variance of its increments, the one with the higher expected increment is preferred. If one additionally assumes that the market together with its monetary authority acts to maximize the long term growth of the market portfolio, then this portfolio exhibits a very particular dynamics. In a market without jumps the resulting dynamics equals that of the growth optimal portfolio (GOP). Conditions are formulated under which the well-known capital asset pricing model is generalized for markets with intensity based jumps. Furthermore, the Markowitz efficient frontier and the Sharpe ratio are recovered in this continuous time setting. In this paper the numeraire for derivative pricing is chosen to be the GOP. Primary security account prices, when expressed in units of the GOP, turn out to be supermartingales. In the proposed framework an equivalent risk neutral martingale measure need not exist. Fair derivative prices are obtained as conditional expectations of future payoff structures under the real world probability measure. The concept of fair pricing is shown to generalize the classical risk neutral and the actuarial net present value pricing methodologies.

1991 *Mathematics Subject Classification*: primary 90A12; secondary 60G30, 62P20.

JEL Classification: G10, G13

Key words and phrases: benchmark model, jump diffusions, growth optimal portfolio, market portfolio, efficient frontier, Sharpe ratio, fair pricing, actuarial pricing.

¹University of Technology Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia

1 Introduction

This paper proposes an integrated approach that can be applied to portfolio optimization, credit risk and derivative and insurance pricing. It uses the *growth optimal portfolio* (GOP) as the benchmark or reference unit and establishes a class of benchmark models with intensity based jumps. In the case of diffusions without jumps, Long (1990) and Bajeux-Besnainou & Portait (1997) introduced the GOP, first considered in Kelly (1956) as the *numeraire portfolio*. It allows for the pricing of derivatives under the real world probability measure.

Under the standard risk neutral approach a major problem arises in modeling credit and insurance risk due to the difficulty in choosing an appropriate equivalent risk neutral pricing measure. Furthermore, actuarial approaches have focused over centuries on the modeling and pricing of insurance risk under the real world probability measure, see for instance, Gerber (1990) and Bühlmann (1992). On the other hand, in risk management and investment management the quantitative methods rely on the real world probability measure. It has been shown in Platen (2002) and Heath & Platen (2002a, 2002b, 2002c) that there exist reasonable market models that cannot be treated under the classical risk neutral approach. The currently very topical subject of credit risk provides an interesting case study on the conflict inherent in pricing under a risk neutral measure, while calculating risk statistics under the real world measure. Here the question of how to reconcile real world probabilities of default with credit spreads, which are often interpreted via risk neutral default probabilities, has become a technical minefield, see Duffie & Singleton (2003). Challenges are therefore emerging from the need to have a consistent approach to the modeling of continuous and event driven risk in the combined fields of finance and insurance.

In Markowitz (1959) the mean-variance portfolio theory with its well-known *efficient frontier* was introduced. This led to the *capital asset pricing model* (CAPM), see Sharpe (1964), Lintner (1965) and Merton (1973). The CAPM is based on the *market portfolio* as reference unit and represents an equilibrium model of exchange. In a continuous time setting Merton (1973) derived the *intertemporal* CAPM from the portfolio selection behavior of investors who maximize equilibrium expected utility. The current paper aims to avoid equilibrium and utility based arguments in deriving the CAPM for a jump diffusion market. It generalizes fundamental results on the Markowitz efficient frontier as well as the CAPM and the Sharpe ratio.

In this paper we construct a class of benchmark models, see Platen (2002, 2004a), for security prices that follow *diffusions with intensity based jumps*. One can refer to a wide range of literature on derivative pricing for jump diffusions, starting with Merton (1976) and leading to a considerable variety of papers and monographs, see, for instance, Cont & Tankov (2004). We will avoid the standard assumption on the existence of an equivalent risk neutral martingale measure. In this way important freedom is gained for financial modeling, as has become clear, for

instance, in Heath & Platen (2002a, 2002b, 2002c).

As in Long (1990), where prices denominated in units of the GOP are martingales, the fair pricing concept, which we advocate in this paper defines benchmarked fair derivative price processes as martingales under the real world probability measure. Therefore, derivative prices can be obtained as conditional expectations of future benchmarked prices without any measure transformation. We will show that fair prices coincide with the corresponding risk neutral prices if an equivalent risk neutral martingale measure exists. A natural generalization of the standard risk neutral framework is therefore obtained by fair pricing under the benchmark approach. Also, the classical actuarial pricing methodology turns out to be a particular case of fair pricing when the payoff is independent of the GOP.

Section 2 introduces a class of benchmark models with intensity based jumps. A portfolio choice theorem is presented in Section 3. Capital asset pricing with jumps is considered in Section 4. Fair contingent claim pricing is studied in Section 5. Finally, the evolution of the expectation of the market portfolio is analyzed in Section 6.

2 Benchmark Model with Jumps

2.1 Continuous and Event Driven Uncertainty

We consider a market containing *continuously evolving uncertainty* represented by m independent standard Wiener processes $W^k = \{W_t^k, t \in [0, T]\}$, $k \in \{1, 2, \dots, m\}$, $m \in \{1, 2, \dots, d\}$, $d \in \{1, 2, \dots\}$. These are defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0, \infty)$. We also consider events of certain types, for instance, corporate defaults, operational failures or specified insured events that are reflected in the movements of traded securities. Events of the k th type are counted by the $\underline{\mathcal{A}}$ -adapted k th *counting process* $p^k = \{p_t^k, t \in [0, T]\}$, whose *intensity* $h^k = \{h_t^k, t \in [0, T]\}$ is a given predictable, strictly positive process with

$$h_t^k > 0 \tag{2.1}$$

and

$$\int_0^T h_s^k ds < \infty \tag{2.2}$$

almost surely for $t \in [0, T]$ and $k \in \{m+1, \dots, d\}$. Furthermore, we introduce the k th *jump martingale* $W^k = \{W_t^k, t \in [0, T]\}$ with stochastic differential

$$dW_t^k = (dp_t^k - h_t^k dt) (h_t^k)^{-\frac{1}{2}} \tag{2.3}$$

for $k \in \{m+1, \dots, d\}$ and $t \in [0, T]$. It is assumed that the above jump martingales do not jump at the same time. They represent the compensated and normalized sources of *event driven uncertainty*.

Let us denote by A^\top the transpose of a vector or matrix A . The evolution of traded uncertainty is modeled by the vector process of independent $(\underline{\mathcal{A}}, P)$ -martingales $W = \{W_t = (W_t^1, \dots, W_t^d)^\top, t \in [0, T]\}$. Note that W^1, \dots, W^m are Wiener processes, while W^{m+1}, \dots, W^d are compensated normalized counting processes. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0, T]}$ is the augmentation under P of the natural filtration \mathcal{A}^W generated by the vector process W . It satisfies the usual conditions and \mathcal{A}_0 is the trivial σ -algebra, see Protter (2004). Note that the conditional variance of the k th source of uncertainty is

$$E\left((W_{t+\varepsilon}^k - W_t^k)^2 \mid \mathcal{A}_t\right) = \varepsilon \quad (2.4)$$

for all $t \in [0, T]$, $k \in \{1, 2, \dots, d\}$ and $\varepsilon \in [0, T-t]$.

2.2 Primary Security Accounts

A primary security account is a particular investment account, consisting only of one kind of security, with all proceeds reinvested. For the securitization of the d sources of uncertainty, let us introduce d *risky primary security accounts*, whose values at time t are denoted by $S_t^{(j)}$, for $j \in \{1, 2, \dots, d\}$. Each of these contains shares of some kind. These security accounts represent the evolution of wealth due to the ownership of assets, with all dividends or income reinvested. The 0th primary security account $S^{(0)} = \{S_t^{(0)}, t \in [0, T]\}$ is the riskless *savings account*, which continuously accrues the short term interest rate r_t . In this case the underlying asset is the domestic currency.

Without loss of generality we assume that the nonnegative j th primary security account value $S_t^{(j)}$ satisfies the stochastic differential equation (SDE)

$$dS_t^{(j)} = S_{t-}^{(j)} \left(a^j(t) dt + \sum_{k=1}^d b^{j,k}(t) dW_t^k \right) \quad (2.5)$$

for $t \in [0, T]$ with initial value $S_0^{(j)} > 0$ and $j \in \{0, 1, \dots, d\}$, see Protter (2004). Since $S_t^{(0)}$ is the savings account, we have

$$a^0(t) = r_t \quad (2.6)$$

and

$$b^{0,k}(t) = 0 \quad (2.7)$$

for $t \in [0, T]$ and $k \in \{1, 2, \dots, d\}$. One may interpret a roll-over treasure bill account as a suitable proxy for the savings account. We assume that the processes r , a^j , $b^{j,k}$ and h^k are finite and predictable, and that a unique strong solution for the system of SDEs (2.5) exists, see Protter (2004). To ensure nonnegativity for each primary security account we assume that

$$b^{j,k}(t) \geq -\sqrt{h_t^k} \quad (2.8)$$

for all $t \in [0, T]$, $j \in \{1, 2, \dots, d\}$ and $k \in \{m + 1, m + 2, \dots, d\}$.

To securitize the sources of uncertainty properly, we make the following assumption.

Assumption 2.1 *The generalized volatility matrix $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ is invertible for Lebesgue-almost-every $t \in [0, T]$.*

Assumption 2.1 allows us to introduce the *market price for risk* vector

$$\theta(t) = (\theta^1(t), \dots, \theta^d(t))^\top = b^{-1}(t) [a(t) - r_t \mathbf{1}] \quad (2.9)$$

for $t \in [0, T]$. Here $a(t) = (a^1(t), \dots, a^d(t))^\top$ is the *appreciation rate vector* and $\mathbf{1} = (1, \dots, 1)^\top$ the *unit vector*. Using (2.9), we can rewrite the SDE (2.5) in the form

$$dS_t^{(j)} = S_{t-}^{(j)} \left(r_t dt + \sum_{k=1}^d b^{j,k}(t) (\theta^k(t) dt + dW_t^k) \right) \quad (2.10)$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. For $k \in \{1, 2, \dots, m\}$, the quantity $\theta^k(t)$ expresses the *market price for risk* with respect to the k th Wiener process W^k . If $k \in \{m + 1, \dots, d\}$, then $\theta^k(t)$ can be interpreted as the *market price for k th event risk*. We will see later that the market prices for risk play a central role in our modeling framework, and that one needs a further condition on the market prices for event risk to avoid arbitrage.

The vector process $S = \{S_t = (S_t^{(0)}, \dots, S_t^{(d)})^\top, t \in [0, T]\}$ characterizes the evolution of all primary security accounts. We say that a predictable stochastic process $\delta = \{\delta(t) = (\delta^0(t), \dots, \delta^d(t))^\top, t \in [0, T]\}$ is a *strategy* if it is S -integrable, see Protter (2004). The j th component of δ denotes the number of units of the j th primary security account held at time $t \in [0, T]$ in a portfolio, $j \in \{0, 1, \dots, d\}$. For a strategy δ we denote by $S_t^{(\delta)}$ the value of the corresponding portfolio process at time t , when measured in units of the domestic currency. Thus, we set

$$S_t^{(\delta)} = \sum_{j=0}^d \delta^j(t) S_t^{(j)} \quad (2.11)$$

for $t \in [0, T]$.

Definition 2.2 *A strategy δ and the corresponding portfolio process $S^{(\delta)} = \{S_t^{(\delta)}, t \in [0, T]\}$ are called self-financing if*

$$dS_t^{(\delta)} = \sum_{j=0}^d \delta^j(t) dS_t^{(j)} \quad (2.12)$$

for all $t \in [0, T]$.

All changes in value of a self-financing portfolio process are due to changes in value of underlying primary security accounts. In what follows we will only consider self-financing portfolios. Therefore, from now on we omit the phrase “self-financing”.

2.3 Growth Optimal Portfolio

For a given strategy δ with strictly positive portfolio process $S_t^{(\delta)}$ let $\pi_\delta^j(t)$ denote the *fraction* of wealth that is invested in the j th primary security account at time t . It is defined by the relation

$$\pi_\delta^j(t) = \delta^j(t) \frac{S_t^{(j)}}{S_t^{(\delta)}} \quad (2.13)$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. Furthermore, by (2.11) these fractions always add to one. That is

$$\sum_{j=0}^d \pi_\delta^j(t) = 1 \quad (2.14)$$

for $t \in [0, T]$. In terms of the vector of fractions $\pi_\delta(t) = (\pi_\delta^1(t), \dots, \pi_\delta^d(t))^\top$ we obtain for $S_t^{(\delta)}$ from (2.12), (2.10), (2.7) and (2.13) the SDE

$$dS_t^{(\delta)} = S_{t-}^{(\delta)} \{r_t dt + \pi_\delta(t-)^\top b(t) (\theta(t) dt + dW_t)\} \quad (2.15)$$

for $t \in [0, T]$. Note that a portfolio process $S^{(\delta)}$ remains strictly positive if and only if

$$\sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) (h_t^k)^{-\frac{1}{2}} > -1 \quad (2.16)$$

a.s. for all $k \in \{m+1, m+2, \dots, d\}$ and $t \in [0, T]$. For a strictly positive portfolio process $S^{(\delta)}$ we obtain by an application of Itô's formula the following SDE for its logarithm

$$\begin{aligned} d \ln(S_t^{(\delta)}) &= g_\delta(t) dt + \sum_{k=1}^m \sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) dW_t^k \\ &+ \sum_{k=m+1}^d \ln \left(1 + \sum_{j=1}^d \pi_\delta^j(t-) \frac{b^{j,k}(t)}{\sqrt{h_t^k}} \right) \sqrt{h_t^k} dW_t^k \end{aligned} \quad (2.17)$$

for $t \in [0, T]$. The *growth rate* in this expression is given by

$$\begin{aligned}
g_\delta(t) = & r_t + \sum_{k=1}^m \left[\sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \theta^k(t) - \frac{1}{2} \left(\sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \right)^2 \right] \\
& + \sum_{k=m+1}^d \left[\sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \left(\theta^k(t) - \sqrt{h_t^k} \right) + \ln \left(1 + \sum_{j=1}^d \pi_\delta^j(t) \frac{b^{j,k}(t)}{\sqrt{h_t^k}} \right) h_t^k \right]
\end{aligned} \tag{2.18}$$

for $t \in [0, T]$. Note that for the first sum on the right hand side of (2.18) a unique maximum naturally exists because it has a quadratic form with respect to the fractions. Careful inspection of the terms in the second sum reveals that, in general, a unique maximum growth rate only exists if the market prices of event risks are less than the square roots of the corresponding jump intensities. This leads to the following assumption.

Assumption 2.3 *Assume that*

$$\sqrt{h_t^k} > \theta^k(t) \tag{2.19}$$

for $t \in [0, T]$ and $k \in \{m+1, \dots, d\}$.

Assumption 2.3 guarantees that the market is arbitrage free in the sense of Platen (2004a). Furthermore, it allows us to introduce the predictable vector process $c(t) = (c^1(t), \dots, c^d(t))^\top$ with components

$$c^k(t) = \begin{cases} \theta^k(t) & \text{for } k \in \{1, 2, \dots, m\} \\ \frac{\theta^k(t)}{1 - \theta^k(t) (h_t^k)^{-\frac{1}{2}}} & \text{for } k \in \{m+1, \dots, d\} \end{cases} \tag{2.20}$$

for $t \in [0, T]$. Note that a divergent jump intensity, with $h_t^k \rightarrow \infty$ a.s. for any $t \in [0, T]$ and $k \in \{m+1, \dots, d\}$, causes the corresponding component $c^k(t)$ to approach the market price for jump risk $\theta^k(t)$ asymptotically. In this case the component is similar to the market price for risk with respect to a Wiener process.

We now define the fractions of a portfolio $S^{(\delta_*)}$ by the relation

$$\pi_{\delta_*}(t) = (\pi_{\delta_*}^1(t), \dots, \pi_{\delta_*}^d(t))^\top = (c(t)^\top b^{-1}(t))^\top \tag{2.21}$$

for $t \in [0, T]$. By (2.15) and (2.20) it follows that $S_t^{(\delta_*)}$ satisfies the SDE

$$\begin{aligned}
dS_t^{(\delta_*)} &= S_{t-}^{(\delta_*)} \left(r_t dt + c(t)^\top (\theta(t) dt + dW_t) \right) \\
&= S_{t-}^{(\delta_*)} \left(r_t dt + \sum_{k=1}^m \theta^k(t) (\theta^k(t) dt + dW_t^k) \right. \\
&\quad \left. + \sum_{k=m+1}^d \frac{\theta^k(t)}{1 - \theta^k(t) (h_t^k)^{-\frac{1}{2}}} (\theta^k(t) dt + dW_t^k) \right)
\end{aligned} \tag{2.22}$$

for $t \in [0, T]$, with $S_0^{(\delta_*)} > 0$. Note from (2.22) that Assumption 2.3 keeps the portfolio process $S^{(\delta_*)}$ strictly positive. Let us now define a *growth optimal portfolio* (GOP).

Definition 2.4 *A portfolio process that maximizes the growth rate (2.18) among all positive portfolio processes is called a GOP.*

There is an increasing literature on the GOP and other related diversified portfolios. We refer the interested reader to Korn & Schäl (1999) and Platen (2002, 2004b, 2004c) for recent information on this topic. The following corollary is a consequence of results in Platen (2004a).

Corollary 2.5 *Under Assumptions 2.1 and 2.3 the portfolio process $S^{(\delta_*)} = \{S_t^{(\delta_*)}, t \in [0, T]\}$ satisfying (2.22), with wealth fractions given by (2.21), is a GOP. Furthermore, for any given fixed initial value $S_0^{(\delta_*)} > 0$, the GOP is uniquely determined.*

2.4 Benchmark Model with Intensity Based Jumps

We use the GOP $S^{(\delta_*)}$ as benchmark or numeraire, and call prices expressed in units of $S^{(\delta_*)}$ *benchmarked prices*. By the Itô formula, (2.15) and (2.22), a *benchmarked portfolio process* $\hat{S}^{(\delta)} = \{\hat{S}_t^{(\delta)}, t \in [0, T]\}$, with

$$\hat{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}} \quad (2.23)$$

for $t \in [0, T]$, satisfies the SDE

$$\begin{aligned} d\hat{S}_t^{(\delta)} = & \hat{S}_{t-}^{(\delta)} \left(\sum_{k=1}^m \left\{ \sum_{j=1}^d \pi_{\delta}^j(t) b^{j,k}(t) - \theta^k(t) \right\} dW_t^k \right. \\ & \left. + \sum_{k=m+1}^d \left\{ \left(\sum_{j=1}^d \pi_{\delta}^j(t-) b^{j,k}(t) \right) \left(1 - \frac{\theta^k(t)}{\sqrt{h_t^k}} \right) - \theta^k(t) \right\} dW_t^k \right) \end{aligned} \quad (2.24)$$

for $t \in [0, T]$. To obtain a simpler form of the above SDE we write $\sigma^{0,k}(t)$ instead of $\theta^k(t)$, for $t \in [0, T]$ and $k \in \{1, 2, \dots, d\}$. Now, define the matrix process $\sigma = \{\sigma(t) = [\sigma^{j,k}(t)]_{j,k=0,1}^d, t \in [0, T]\}$ by setting

$$\sigma^{0,k}(t) = \theta^k(t) \quad (2.25)$$

and

$$\sigma^{j,k}(t) = \begin{cases} \sigma^{0,k}(t) - b^{j,k}(t) & \text{for } k \in \{1, 2, \dots, m\} \\ \sigma^{0,k}(t) - b^{j,k}(t) \left(1 - \frac{\sigma^{0,k}(t)}{\sqrt{h_t^k}} \right) & \text{for } k \in \{m+1, \dots, d\} \end{cases} \quad (2.26)$$

for $t \in [0, T]$ and $j \in \{1, 2, \dots, d\}$. Using (2.9) and (2.25) one can then rewrite (2.24) as

$$d\hat{S}_t^{(\delta)} = -\hat{S}_{t-}^{(\delta)} \sum_{k=1}^d \sum_{j=0}^d \pi_{\delta}^j(t-) \sigma^{j,k}(t) dW_t^k \quad (2.27)$$

for $t \in [0, T]$. This SDE governs the dynamics of any benchmarked portfolio.

Note that the right hand side of (2.27) is driftless. Thus, a nonnegative benchmarked portfolio $\hat{S}^{(\delta)}$ forms an $(\underline{\mathcal{A}}, P)$ -local martingale. This also means that a nonnegative benchmarked portfolio process $\hat{S}^{(\delta)}$ is always an $(\underline{\mathcal{A}}, P)$ -supermartingale, see Rogers & Williams (2000), that is

$$\hat{S}_t^{(\delta)} \geq E\left(\hat{S}_{\tau}^{(\delta)} \mid \mathcal{A}_t\right) \quad (2.28)$$

for all $\tau \in [0, T]$ and $t \in [0, \tau]$. One can show that whenever a nonnegative supermartingale reaches the value zero it almost surely remains at zero. Based on this observation the above model is arbitrage free in the sense of Platen (2002, 2004a). We call a model of the form prescribed above, which is based on the Assumptions 2.1 and 2.3, a *benchmark model with intensity based jumps*. This notion acknowledges the fact that for this model the GOP exists and is used as the benchmark. A generalization of the benchmark model for event driven risk with respect to Poisson jump measures is given in Christensen & Platen (2004).

3 Maximizing the Portfolio Drift

3.1 Optimal Portfolios

Given a strictly positive portfolio $S^{(\delta)}$, its discounted value

$$\bar{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(0)}} \quad (3.1)$$

satisfies the SDE

$$d\bar{S}_t^{(\delta)} = \sum_{k=1}^d \psi_{\delta}^k(t) (\theta^k(t) dt + dW_t^k) \quad (3.2)$$

by (2.15) and an application of the Itô formula. Here

$$\psi_{\delta}^k(t) = \bar{S}_{t-}^{(\delta)} \sum_{j=1}^d \pi_{\delta}^j(t-) b^{j,k}(t) \quad (3.3)$$

for $k \in \{1, 2, \dots, d\}$ and $t \in [0, T]$, is called the k th *generalized portfolio diffusion coefficient*. Obviously, by (3.2) and (3.3) the discounted portfolio process $\bar{S}^{(\delta)}$ has

portfolio drift

$$\alpha_\delta(t) = \sum_{k=1}^d \psi_\delta^k(t) \theta^k(t) \quad (3.4)$$

for $t \in [0, T]$.

This drift measures the portfolio's time varying trend. The uncertainty of a discounted portfolio $\bar{S}^{(\delta)}$ can be measured by its *aggregate generalized diffusion coefficient*

$$\gamma_\delta(t) = \sqrt{\sum_{k=1}^d (\psi_\delta^k(t))^2} \quad (3.5)$$

at time $t \in [0, T]$. Note that by (2.4) we have normalized variances of increments of the driving martingales W^1, W^2, \dots, W^d .

For a given instantaneous level of the aggregate generalized diffusion coefficient $\gamma_\delta(t)$, any rational investor who prefers more for less can be assumed to aim to maximize the portfolio drift $\alpha_\delta(t)$. Building on the seminal works by Markowitz (1959) and Sharpe (1964) we now aim to capture this objective mathematically for the given benchmark model. More precisely, it is our aim to identify the typical structure of the SDEs for the total portfolios of investors who prefer more for less in the following sense:

Definition 3.1 *A strictly positive portfolio process that maximizes the portfolio drift (3.4) among all strictly positive portfolio processes with the same aggregate generalized diffusion coefficient (3.5) is called optimal.*

For the following analysis let us introduce the *total market price for risk*

$$|\theta(t)| = \sqrt{\sum_{k=1}^d (\theta^k(t))^2} \quad (3.6)$$

and the weighting factor

$$\Gamma^{(0)}(t) = \sum_{k=1}^d \sum_{j=1}^d \theta^k(t) b^{-1j,k}(t) \quad (3.7)$$

for $t \in [0, T]$. As we will see later, if the total market price for risk or the weighting factor are zero, then the savings account is the optimal portfolio that an investor would naturally prefer. The following natural assumption excludes this trivial case.

Assumption 3.2 *Assume that*

$$0 < |\theta(t)| < \infty \quad (3.8)$$

and

$$\Gamma^{(0)}(t) \neq 0 \quad (3.9)$$

almost surely for all $t \in [0, T]$.

We can now formulate a *portfolio choice theorem* in the sense of Markowitz (1959), which generalizes a result in Platen (2002) for continuous benchmark models. The following theorem identifies the structure of the drift and generalized diffusion coefficients in the SDE of an optimal portfolio.

Theorem 3.3 *Any discounted optimal portfolio $\bar{S}^{(\delta)}$ satisfies the SDE*

$$d\bar{S}_t^{(\delta)} = \bar{S}_t^{(\delta)} \frac{(1 - \pi_{\delta}^0(t))}{\Gamma^{(0)}(t)} \sum_{k=1}^d \theta^k(t) (\theta^k(t) dt + dW_t^k), \quad (3.10)$$

with optimal fractions

$$\pi_{\delta}^j(t) = \frac{(1 - \pi_{\delta}^0(t))}{\Gamma^{(0)}(t)} \sum_{k=1}^d \theta^k(t) b^{-1j,k}(t) \quad (3.11)$$

for all $t \in [0, T]$ and $j \in \{1, 2, \dots, d\}$.

This means that the family of discounted optimal portfolios is characterized by a single parameter, namely the fraction of wealth $\pi_{\delta}^0(t)$ held in the savings account. The proof of this theorem is given in the Appendix.

We obtain a particular optimal portfolio $S^{(\delta+)}$, which we call the *mutual fund*, by choosing

$$\pi_{\delta+}^0(t) = 1 - \Gamma^{(0)}(t) \quad (3.12)$$

for $t \in [0, T]$. By (3.10) the mutual fund satisfies the SDE

$$dS_t^{(\delta+)} = S_t^{(\delta+)} \left(r_t dt + \sum_{k=1}^d \theta^k(t) (\theta^k(t) dt + dW_t^k) \right) \quad (3.13)$$

for $t \in [0, T]$. This portfolio plays an important role in the remainder of the paper.

By Theorem 3.3 it follows that any efficient portfolio $S^{(\delta)}$ can be decomposed at any time into a fraction that is invested in the mutual fund $S^{(\delta+)}$ and a remaining fraction that is held in the savings account. Therefore, Theorem 3.3 can also be interpreted as a *mutual fund theorem* or *separation theorem*, see Merton (1973). We emphasize that the assumptions of Theorem 3.3 are rather weak and also realistic.

4 Capital Asset Pricing with Jumps

4.1 Markowitz Efficient Frontier and Sharpe Ratio

For a portfolio $S^{(\delta)}$ we introduce its *aggregate generalized volatility*

$$b_\delta(t) = \sqrt{\sum_{k=1}^d \left(\sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \right)^2} > 0 \quad (4.1)$$

and its *appreciation rate*

$$a_\delta(t) = r_t + \sum_{k=1}^d \sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \theta^k(t) \quad (4.2)$$

for all $t \in [0, T]$, by inspection of (2.15). If $S^{(\delta)}$ is in fact optimal, then it follows by the Itô formula, (3.10) and (3.6) that

$$b_\delta(t) = \frac{(1 - \pi_\delta^0(t))}{\Gamma^{(0)}(t)} |\theta(t)| \quad (4.3)$$

and

$$a_\delta(t) = r_t + b_\delta(t) |\theta(t)| \quad (4.4)$$

for $t \in [0, T]$. By analogy to the single-period mean-variance portfolio theory, developed in Markowitz (1959), we introduce the notion of an *efficient frontier*.

Definition 4.1 A portfolio $S^{(\delta)}$ is said to lie on the efficient frontier if its appreciation rate $a_\delta(t)$, as a function of squared aggregate generalized volatility $(b_\delta(t))^2$, is of the form

$$a_\delta(t) = a_\delta(t, (b_\delta(t))^2) = r_t + \sqrt{(b_\delta(t))^2} |\theta(t)| \quad (4.5)$$

for all times $t \in [0, T]$.

By relations (4.3), (4.4) and (4.5) the following result is directly obtained.

Corollary 4.2 An optimal portfolio is always located on the efficient frontier.

Corollary 4.2 can be interpreted as a “local in time” generalization of the seminal Markowitz efficient frontier to the jump diffusion setting. Note that due to Definition 3.1 and Theorem 3.3 it is not possible to form a positive portfolio that produces an appreciation rate located above the efficient frontier. The best that an investor can do, when searching for the maximum drift while maintaining a given generalized diffusion coefficient, is to form a portfolio on the efficient

frontier. The only remaining freedom is choosing the fraction of wealth that resides in the savings account. This fraction expresses the investor's *degree of risk aversion*. Note that this approach is more general than expected utility maximization, where the risk aversion at a certain time is indirectly specified via the chosen utility function and the given time horizon. In forthcoming work it will be shown how the above concept of optimal portfolios generalizes that of expected utility maximization.

4.2 Sharpe Ratio

For a portfolio $S^{(\delta)}$ the notation introduced in (4.1), (4.2), (3.4) and (3.5) leads to another important investment characteristic, the *Sharpe ratio* $s_\delta(t)$, which is defined by

$$s_\delta(t) = \frac{\alpha_\delta(t)}{\gamma_\delta(t)} = \frac{a_\delta(t) - r_t}{b_\delta(t)} \quad (4.6)$$

for $t \in [0, T]$, see Sharpe (1964). By (4.3), (4.4), (4.6) and Theorem 3.3 we obtain the following practically important result.

Corollary 4.3 *The maximum Sharpe ratio is obtained by optimal portfolios and equals the total market price for risk. For all strictly positive portfolios $S^{(\delta)}$, one has*

$$s_\delta(t) \leq |\theta(t)| \quad (4.7)$$

for all $t \in [0, T]$, with equality attained in (4.7) when $S^{(\delta)}$ is optimal.

The Markowitz efficient frontier and the Sharpe ratio are fundamental tools for investment management, whose natural meaning are preserved in the benchmark model with intensity based jumps. Note that we have not specified any particular dynamics for the stochastic quantities involved. In this sense the benchmark model presented so far provides a general jump diffusion framework for modeling event driven risk.

4.3 Capital Asset Pricing Model

Let us define the *market portfolio* $S^{(\delta_M)}$ as the portfolio consisting of all primary security accounts weighted according to market capitalization. The seminal *capital asset pricing model* (CAPM) was developed by Sharpe (1964), Lintner (1965) and Merton (1973) as a utility based equilibrium model of exchange with the market portfolio $S^{(\delta_M)}$ as reference unit. As we will demonstrate below we do not need to use any equilibrium or expected utility function arguments for generalizing the CAPM to the case of a continuous benchmark model with intensity based jumps. As in the classical CAPM, one can introduce the systematic risk

parameter $\beta_\delta(t)$, called the *portfolio beta*, for a portfolio $S^{(\delta)}$. It is here defined as the ratio of the time derivative of the covariation of the logarithms of the portfolio and the market portfolio over the time derivative of the quadratic variation of the market portfolio, that is,

$$\beta_\delta(t) = \frac{\frac{d}{dt} \langle \ln(S^{(\delta)}), \ln(S^{(\delta_M)}) \rangle_t}{\frac{d}{dt} \langle \ln(S^{(\delta_M)}) \rangle_t} \quad (4.8)$$

for all $t \in [0, T]$.

By (4.2) the instantaneous *risk premium* $p_\delta(t)$ of a portfolio $S^{(\delta)}$ is given by the expression

$$p_\delta(t) = a_\delta(t) - r_t = \sum_{k=1}^d \sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \theta^k(t) \quad (4.9)$$

for all $t \in [0, T]$. Note that by (2.15) and (3.13) the risk premium equals the covariance of the returns of the mutual fund with those of the portfolio.

As described in the literature, the CAPM states that the portfolio beta $\beta_\delta(t)$ equals the ratio of the portfolio risk premium over the market portfolio risk premium $\frac{p_\delta(t)}{p_{\delta_M}(t)}$, see Merton (1973). However, we note from (3.13) by using the mutual fund as reference unit that this fundamental CAPM relationship holds true when one uses the mutual fund as reference unit instead of the market portfolio, that is

$$\frac{\frac{d}{dt} \langle \ln(S^{(\delta)}), \ln(S^{(\delta_+)}) \rangle_t}{\frac{d}{dt} \langle \ln(S^{(\delta_+)}) \rangle_t} = \frac{\sum_{k=1}^d \sum_{j=1}^d \pi_\delta^j(t) b^{j,k}(t) \theta^k(t)}{\sum_{k=1}^d |\theta^k(t)|^2} = \frac{p_\delta(t)}{p_{\delta_+}(t)} \quad (4.10)$$

for all $t \in [0, T]$.

The form of the portfolio risk premium (4.9) and the portfolio beta (4.10) are exactly what the intertemporal CAPM suggests if the market portfolio equals the mutual fund. In what follows we will identify conditions which ensure that the market portfolio equals the mutual fund. This provides a basis for the derivation of the CAPM in the presence of intensity based jumps. The CAPM then arises purely out of the natural structure of a benchmark model with intensity based jumps. We make the following natural assumption.

Assumption 4.4 *Each market participant constructs an optimal portfolio with his or her total available wealth.*

This assumption essentially means that all investors are informed and prefer more for less. The total portfolio of the ℓ th market participant, which is optimal by Assumption 4.4, is denoted by $S^{(\delta_\ell)}$, $\ell \in \{1, 2, \dots, n\}$. The portfolio $S_t^{(\delta_M)}$ of all market participants is the market portfolio at time t with value given by the sum

$$S_t^{(\delta_M)} = \sum_{\ell=1}^n S_t^{(\delta_\ell)} \quad (4.11)$$

for all $t \in [0, T]$. It is reasonable to assume a strictly positive market portfolio. The dynamics of the discounted market portfolio $\bar{S}_t^{(\delta_M)} = \frac{S_t^{(\delta_M)}}{S_t^{(0)}}$ is characterized by the SDE

$$\begin{aligned}
d\bar{S}_t^{(\delta_M)} &= \sum_{\ell=1}^n d\bar{S}_t^{(\delta_\ell)} \\
&= \sum_{\ell=1}^n \frac{(\bar{S}_t^{(\delta_\ell)} - \delta_\ell^0)}{\Gamma^{(0)}(t)} \sum_{k=1}^d \theta^k(t) (\theta^k(t) dt + dW_t^k) \\
&= \bar{S}_t^{(\delta_M)} \frac{(1 - \pi_{\delta_M}^0(t))}{\Gamma^{(0)}(t)} \sum_{k=1}^d \theta^k(t) (\theta^k(t) dt + dW_t^k) \tag{4.12}
\end{aligned}$$

for $t \in [0, T]$, by Theorem 3.3 and (4.11). Thus, by Definition 3.1 and Theorem 3.3 one can show that the market portfolio $S_t^{(\delta_M)}$ is optimal. By (3.13) the fraction $\frac{1 - \pi_{\delta_M}^0(t)}{\Gamma^{(0)}(t)}$ of the market portfolio is invested in the mutual fund $S^{(\delta_+)}$ at time $t \in [0, T]$, and the fraction invested in the domestic savings account $S^{(0)}$ is given by the expression $\frac{\pi_{\delta_M}^0(t) - 1 + \Gamma^{(0)}(t)}{\Gamma^{(0)}(t)}$. By comparing the SDEs (3.13) and (4.12) we obtain the following result.

Corollary 4.5 *Given equal initial values $S_0^{(\delta_M)} = S_0^{(\delta_+)}$, the value of the market portfolio $S_t^{(\delta_M)}$ equals the value of the mutual fund $S_t^{(\delta_+)}$ at all times $t \in [0, T]$ if and only if their fractions held in the savings account are equal, that is*

$$\pi_{\delta_M}^0(t) = \pi_{\delta_+}^0(t) = 1 - \Gamma^{(0)}(t) \tag{4.13}$$

for all $t \in [0, T]$.

By (4.10), this leads to the following conclusion, which provides a generalization of the intertemporal CAPM of Merton (1973) to the case of jump diffusion markets.

Corollary 4.6 *As long as condition (4.13) is satisfied, a generalized intertemporal CAPM holds. That is for any given portfolio its systematic risk parameter, given by (4.10), captures the ratio of the risk premium of the portfolio over that of the market portfolio.*

This means that the well-known CAPM holds under a benchmark model with intensity based jumps if the fraction invested in the savings account of the market portfolio equals the fraction invested in the savings account of the mutual fund.

There are several lines of argument for justifying condition (4.13). Realistically, the monetary authorities can be assumed to control the fraction of the market

portfolio, which is held in the savings account, in such a way that the relationship (4.13) is obtained. This is typically achieved by influencing the short rate level or treasury bill supply. As we will see below, if jumps do not occur for the GOP, then condition (4.13) is equivalent to the case of having a monetary policy that maximizes the growth rate of the market portfolio and, thus, that of the economy. This seems to be a natural assumption. Because of the highly diversified nature of the market portfolio one could argue that jumps are either absent or not significant. Note however that this is an empirical issue which needs to be tested.

4.4 Mutual Fund and GOP

In the case where the GOP does not have jumps, it is clear from (2.22) and (3.13) that the GOP and the mutual fund coincide since this only happens when all market prices for event risks are zero. Suppose that the market prices for event risks are non-zero. From (2.20) and (2.21) we obtain the following expressions for the fractions of the GOP:

$$\pi_{\delta_*}^j(t) = \sum_{k=1}^m \theta^k(t) b^{-1j,k}(t) + \sum_{k=m+1}^d \frac{\theta^k(t)}{1 - \frac{\theta^k(t)}{\sqrt{h_t^k}}} b^{-1j,k}(t) \quad (4.14)$$

for all $t \in [0, T]$ and $j \in \{1, 2, \dots, d\}$. On the other hand, according to (3.11) the mutual fund is characterized by the fractions

$$\pi_{\delta_+}^j(t) = \sum_{k=1}^d \theta^k(t) b^{-1j,k}(t) \quad (4.15)$$

for all $t \in [0, T]$ and $j \in \{1, 2, \dots, d\}$. Consequently, one obtains

$$\pi_{\delta_+}^j(t) = \pi_{\delta_*}^j(t) + \sum_{k=m+1}^d \frac{(\theta^k(t))^2 b^{-1j,k}}{\sqrt{h_t^k} - \theta^k(t)} \quad (4.16)$$

for $t \in [0, T]$ and $j \in \{1, 2, \dots, d\}$. As already indicated, one notes from (4.16) that the mutual fund and the GOP coincide if the market prices for event risk are zero. The portfolios $S^{(\delta_+)}$ and $S^{(\delta_*)}$ are approximately similar if the intensities for the event risks are extremely high compared to their corresponding market prices for risk.

5 Fair Contingent Claim Pricing

5.1 Fair Pricing

It will now be shown that the direct observability of the GOP in the form of the market portfolio can be exploited for consistent derivative pricing. As demonstrated in Platen (2002, 2004a), for the class of benchmark models with intensity

based jumps under consideration, one does not, in general, have an equivalent risk neutral martingale measure. Therefore, the widely used risk neutral pricing methodology may break down for certain benchmark models. This is the case when the benchmarked savings account process $\hat{S}^{(0)}$ forms a *strict local martingale* and not a martingale. For realistic benchmark models where this happens see Platen (2002), Heath & Platen (2002a, 2002b, 2002c) and Miller & Platen (2004).

Since risk neutral pricing is not available, one needs a consistent and realistic alternative concept for pricing contingent claims that generalizes the standard risk neutral approach. To value derivatives uniquely, we apply the concept of *fair pricing*, as introduced in Platen (2002). It employs the GOP as benchmark or numeraire and forms conditional expectations under the real world probability measure. In some sense it generalizes the numeraire portfolio approach of Long (1990), as well as the well-known state price density, deflator, pricing kernel and discount factor approaches described in Constantinides (1992), Duffie (2001) and Cochrane (2001), for instance.

Definition 5.1 *We call a price process $U = \{U_t, t \in [0, T]\}$ fair if the corresponding benchmarked price process $\hat{U} = \{\hat{U}_t = \frac{U_t}{S_t^{(\delta_*)}}, t \in [0, T]\}$ forms an (\mathcal{A}, P) -martingale. That is, it satisfies the conditions*

$$E(|\hat{U}_T|) < \infty$$

and

$$\hat{U}_t = E\left(\hat{U}_s \mid \mathcal{A}_t\right) \quad (5.1)$$

for all $0 \leq t \leq s \leq T$.

Under the presented benchmark model with intensity based jumps we do *not* require the existence of an equivalent risk neutral martingale measure. Therefore, standard risk neutral pricing is, in general, not applicable. However, fair pricing generalizes standard risk neutral pricing, as shown in Platen (2002, 2004b, 2004c). Furthermore, in a benchmark model a *free lunch with vanishing risk*, in the sense of Delbaen & Schachermayer (1998), may arise for certain model specifications, see Heath & Platen (2002a, 2002b, 2002c). However, due to the supermartingale property of nonnegative benchmarked portfolios, see (2.28), one is unable to generate strictly positive terminal wealth from zero initial capital using a nonnegative portfolio. A benchmark model with intensity based jumps is arbitrage free in the sense that all nonnegative benchmarked portfolios are supermartingales, as described in Platen (2002).

Definition 5.2 *We define a contingent claim H_τ , which matures at a stopping time $\tau \in [0, T]$, as a nonnegative \mathcal{A}_τ -measurable random payoff with*

$$E\left(\frac{H_\tau}{S_\tau^{(\delta_*)}}\right) < \infty \quad (5.2)$$

almost surely.

With reference to Definition 5.1, we define the *fair price* of a contingent claim H_τ , as in Definition 5.2, by the process $U_{H_\tau} = \{U_{H_\tau}(t), t \in [0, T]\}$, determined by

$$U_{H_\tau}(t) = S_t^{(\delta_*)} E \left(\frac{H_\tau}{S_\tau^{(\delta_*)}} \middle| \mathcal{A}_t \right) \quad (5.3)$$

for $t \in [0, \tau]$. It will be shown below that if an equivalent risk neutral martingale measure exists, then the fair price coincides with the corresponding risk neutral price, see also Platen (2002). The benchmark approach enlarges the range of models that can be used if compared to what is possible under the risk neutral approach, see Heath & Platen (2002b).

5.2 Risk Neutral and Actuarial Pricing

Let us assume that condition (4.13) is satisfied and the last term in (4.16) is negligible. Then the market portfolio is a good proxy for the GOP. The direct observability of the market portfolio leads naturally to a practical fair pricing methodology, generalizing the well-known *arbitrage pricing theory* (APT) introduced by Ross (1976), Harrison & Kreps (1979) and Harrison & Pliska (1981). However, fair pricing does not require an equivalent risk neutral martingale measure to exist. Note that the Radon-Nikodym derivative process $\Lambda_Q = \{\Lambda_Q(t), t \in [0, T]\}$ for the presumed risk neutral measure Q can be expressed as inverse of the discounted GOP

$$\Lambda_Q(t) = \frac{dQ}{dP} \Big|_{\mathcal{A}_t} = \frac{\bar{S}_0^{(\delta_*)}}{\bar{S}_t^{(\delta_*)}} \quad (5.4)$$

for $t \in [0, T]$, see Karatzas & Shreve (1998). By the Itô formula and (2.22) we obtain the SDE

$$d\Lambda_Q(t) = -\Lambda_Q(t) \sum_{k=1}^d \theta^k(t) dW_t^k \quad (5.5)$$

for $t \in [0, T]$ with $\Lambda_Q(0) = 1$. This shows that Λ_Q is an $(\underline{\mathcal{A}}, P)$ -local martingale. Furthermore, by (2.27) it follows that $\bar{S}_t^{(\delta)} \Lambda_Q(t) = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}} = \hat{S}_t^{(\delta)}$ forms an $(\underline{\mathcal{A}}, P)$ -local martingale for any portfolio $S^{(\delta)}$. We emphasize that this does *not* mean that $\hat{S}^{(\delta)}$ is automatically an $(\underline{\mathcal{A}}, P)$ -martingale.

To demonstrate that the standard risk neutral approach is covered by the fair pricing concept of the benchmark approach, let us consider a fair portfolio $S^{(\delta)}$, where by Definition 5.1 and (5.4) we get

$$S_t^{(\delta)} = S_t^{(\delta_*)} E \left(\hat{S}_s^{(\delta)} \middle| \mathcal{A}_t \right) = E \left(\frac{\Lambda_Q(s) S_t^{(0)}}{\Lambda_Q(t) S_s^{(0)}} S_s^{(\delta)} \middle| \mathcal{A}_t \right) \quad (5.6)$$

for $0 \leq t \leq s \leq T$. Then, by application of the Girsanov theorem, see Protter (2004), one obtains the risk neutral pricing formula

$$S_t^{(\delta)} = E_Q \left(\frac{S_t^{(0)}}{S_s^{(0)}} S_s^{(\delta)} \mid \mathcal{A}_t \right) \quad (5.7)$$

for all $t \in [0, T]$ and $s \in [t, T]$, if Λ_Q is in fact an $(\underline{\mathcal{A}}, P)$ -martingale. Here E_Q denotes expectation under the risk neutral measure Q .

In the above sense, one recovers the risk neutral pricing methodology of the APT as a special case of fair pricing. Furthermore, this approach uses as numeraire an observable quantity in form of the market portfolio. As would be expected, this is rather important for realistic modeling and contingent claim pricing.

Let us briefly mention some empirical evidence which supports our view that we need to go beyond the APT. By (5.4) the putative Radon-Nikodym derivative Λ_Q for the candidate risk neutral measure equals the ratio of the savings account over the GOP. In the long run the market portfolio and thus the GOP is by rational investors expected to outperform the savings account. This has been also empirically confirmed by Dimson, Marsh & Staunton (2002) in a detailed empirical study of all major markets over the last century. This finding demonstrates that the trajectory of the process Λ_Q *decreases systematically* over long periods. The empirical fact of a systematic decline of this process for all major currency denominations surely cannot be ignored. As a consequence, it is not likely that Λ_Q can in reality be successfully modeled as an $(\underline{\mathcal{A}}, P)$ -martingale. This contradicts a core assumption of the APT. We emphasize that a decreasing graph for Λ_Q is still consistent with it being a nonnegative strict $(\underline{\mathcal{A}}, P)$ -local martingale and hence a supermartingale, see Protter (2004). The proposed benchmark approach can accommodate this fully. For derivative pricing under the benchmark model with intensity based jumps, where no equivalent risk neutral martingale measure is assumed to exist, we therefore advocate the fair pricing methodology.

Fair prices are uniquely determined even in incomplete markets. Under the existence of a minimal equivalent martingale measure, see Föllmer & Schweizer (1991), fair prices have been shown to correspond to local risk minimizing prices, see Platen (2004d). Fair pricing is practicable since one can model and calibrate the GOP when interpreted as the market portfolio. This enables us to calculate the real world expectations in (5.3) directly.

For the practically important case where a contingent claim H_T is *independent* of the GOP $S_T^{(\delta,*)}$, one obtains directly the following *actuarial pricing formula* from the fair pricing formula (5.3).

Corollary 5.3 *For a contingent claim H_T that is independent of the GOP*

value $S_T^{(\delta^*)}$, the fair price $U_{H_T}(t)$ satisfies the actuarial pricing formula

$$\begin{aligned} U_{H_T}(t) &= E \left(\frac{S_t^{(\delta^*)}}{S_T^{(\delta^*)}} \mid \mathcal{A}_t \right) E(H_T \mid \mathcal{A}_t) \\ &= P(t, T) E(H_T \mid \mathcal{A}_t), \end{aligned} \quad (5.8)$$

where $P(t, T)$ denotes the fair price at time $t \in [0, T]$ of a zero coupon bond with maturity date T .

One may regard (5.8) as a generalized net present value pricing formula. It is still valid when interest rates are stochastic. In various ways formulas of the type (5.8) have been used in insurance and other areas of risk management, see, for instance, Bühlmann (1995) and Gerber (1990). They appear here as a natural consequence of the benchmark approach.

6 Expected Discounted Mutual Fund

We conclude the paper by analyzing the dynamics of the mutual fund, which can be interpreted as the market portfolio under appropriate assumptions as previously discussed. The SDE (3.13) for the mutual fund reveals a close link between its drift and generalized diffusion coefficient. More precisely, the risk premium of the mutual fund equals the square of its total aggregate generalized volatility. To see this, one can rewrite the SDE (3.13) for the mutual fund in discounted form as

$$d\bar{S}_t^{(\delta_+)} = \bar{S}_t^{(\delta_+)} |\theta(t)| (|\theta(t)| dt + dW_t) \quad (6.1)$$

for $t \in [0, T]$. Here

$$dW_t = \frac{1}{|\theta(t)|} \sum_{k=1}^d \theta^k(t) dW_t^k \quad (6.2)$$

is the stochastic differential of an $(\underline{\mathcal{A}}, P)$ -martingale W with conditional variance

$$E((W_{t+\varepsilon} - W_t)^2 \mid \mathcal{A}_t) = \varepsilon \quad (6.3)$$

for all $t \in [0, T]$ and $\varepsilon \in (0, T - t]$, see Protter (2004). If W is continuous, then it is a standard Wiener process. Generally, W is a mixture of independent martingales that may exhibit some jumps. The SDE (6.1) reveals a useful structural relationship between the drift and the generalized diffusion coefficient of the mutual fund, which we will exploit below.

We reparameterize the mutual fund dynamics by using the average change per unit of time of the discounted mutual fund value, which is captured by the *discounted mutual fund drift*

$$\alpha(t) = \alpha_{\delta_+}(t) = \bar{S}_t^{(\delta_+)} |\theta(t)|^2 \quad (6.4)$$

for $t \in [0, T]$. Note that if there are no jumps in the GOP, then we have in (6.4) the drift of the discounted GOP. One can interpret $\alpha(t)$ as the change per unit time of the *accumulated underlying value* of the discounted market portfolio. Using the parametrization (6.4), we can express the total market price for risk in the form

$$|\theta(t)| = \sqrt{\frac{\alpha(t)}{\bar{S}_t^{(\delta_+)}}} \quad (6.5)$$

By substituting (6.4) and (6.5) into (6.1) we obtain the following SDE for the discounted mutual fund

$$d\bar{S}_t^{(\delta_+)} = \alpha(t) dt + \sqrt{\bar{S}_t^{(\delta_+)}} \alpha(t) dW_t \quad (6.6)$$

for $t \in [0, T]$. The solution of this SDE is a generalized time transformed squared Bessel process of dimension four. For the continuous version of this process, when W is a standard Wiener process, we refer the reader to Revuz & Yor (1999). In the current paper the process is driven by the normalized jump martingale W , given in (6.2).

The *transformed time* $\varphi(t)$ at time t for $\bar{S}^{(\delta_+)}$ in (6.6) is given by the expression

$$\varphi(t) = \varphi(0) + \int_0^t \alpha(s) ds \quad (6.7)$$

with $\varphi(0) \geq 0$ as a possibly unobserved random initial value. We emphasize the fact that $\varphi(t)$ is not just one arbitrarily selected time transformation. The increment $\varphi(t) - \varphi(0)$ expresses the *change of accumulated underlying value* in the discounted mutual fund. This is an important economic quantity, which appears here naturally in the benchmark setup. In the case where the discounted mutual fund does not exhibit jumps, which seems to be a realistic assumption, it can be shown, see Platen (2004c), that the increase in accumulated underlying value can be directly observed via the equalities

$$\varphi(t) - \varphi(0) = 4 \left\langle \sqrt{\bar{S}^{(\delta_+)}} \right\rangle_t = 4 \int_0^t \frac{\alpha(s)}{4} ds \quad (6.8)$$

for $t \in [0, T]$. This makes the transformed time or accumulated underlying value an observable quantity. It provides important information about the evolution of the average economic value of the market portfolio via some quadratic variation, which is readily observable, see Platen (2004c).

Let us decompose the discounted mutual fund value at time $t \in [0, T]$ as

$$\bar{S}_t^{(\delta_+)} = \bar{S}_0^{(\delta_+)} + \varphi(t) - \varphi(0) + M_t, \quad (6.9)$$

where $M = \{M_t, t \in [0, T]\}$ is the $(\underline{\mathcal{A}}, P)$ -local martingale

$$M_t = \int_0^t \bar{S}_s^{(\delta_+)} |\theta(s)| dW_s = \int_0^t \sqrt{\bar{S}_s^{(\delta_+)}} \alpha(s) dW_s \quad (6.10)$$

for $t \in [0, T]$. The discounted mutual fund value $\bar{S}_t^{(\delta+)}$ in (6.9) consists of a *noise part* M_t , which models the trading uncertainty of the discounted mutual fund and a systematic part $\varphi(t) - \varphi(0)$, which expresses the increase of its accumulated underlying value. As previously mentioned, the accumulated underlying value can be interpreted as a measure of the discounted wealth that has been generated by the companies listed in the stock market. The fluctuating share prices then express the perception of the market about the value of each company. Over long periods the evolution of this perceived value has to be in line with the corresponding accumulated underlying value. Hence, this provides some type of measure of the degree to which the mutual fund is over or undervalued.

Remarkably, when the accumulated underlying value of the market portfolio is used as time scale, then by (6.6) the dynamics of the discounted mutual fund turn out to be those of a very particular stochastic process, the generalized time transformed squared Bessel process of dimension four. This is a pleasing result, not only mathematically, but also economically.

The above relationships lead directly to the following statement, which exploits equations (6.9) and (6.7) and is obtained by a realistic martingale assumption on the local martingale M .

Corollary 6.1 *If the local martingale M in (6.10) is a true (\mathcal{A}, P) -martingale, then the expected change of the discounted mutual fund value over a given time period equals the expected change of its transformed time, that is,*

$$E \left(\bar{S}_s^{(\delta+)} - \bar{S}_t^{(\delta+)} \mid \mathcal{A}_t \right) = E \left(\varphi(s) - \varphi(t) \mid \mathcal{A}_t \right) \quad (6.11)$$

for all $t \in [0, T]$ and $s \in [t, T]$.

We emphasize that we have not made any major assumptions about the particular stochastic dynamics of the mutual fund. In principle, there is much modeling freedom that can be explored. However, as shown in Platen (2004a), in the case without jumps a natural dynamics emerges from the fact that the change of the transformed time or accumulated underlying value can be modeled realistically by a rather smooth increasing quantity. The dynamics of the discounted mutual fund is then in physical time that of a squared Bessel process of dimension four, see Platen (2002, 2004a). The resulting model with a slowly varying deterministic transformed time is called the *minimal market model* (MMM), see Platen (2001). It has major consequences for the nature of the dynamics of the supposed Radon-Nikodym derivative Λ_Q . This process is under the MMM a strict supermartingale and *not* a martingale and a core assumption of the APT is thus violated. The consequences of this fact were discussed in Section 5.2.

Conclusion

It has been shown that the growth optimal portfolio plays a central theoretical and practical role in finance. The paper assumes that investors always prefer more for less which leads them to hold optimal portfolios. A theorem is derived that characterizes any optimal portfolio as a mixture of some mutual fund and the savings account. Under the additional assumptions that the monetary authorities aim to maximize the long term growth of the market portfolio, assuming negligible jump risk, it has been shown that the market portfolio approximates the mutual fund and also the growth optimal portfolio. This observation provides a derivation of the capital asset pricing model for jump diffusion markets without requiring any equilibrium or expected utility maximization arguments. The Markowitz efficient frontier and Sharpe ratio follow naturally in a generalized form for the given benchmark model with intensity based jumps.

Without imposing any particular dynamics, the discounted mutual fund is identified as a generalized time transformed squared Bessel process of dimension four. The transformed time can be interpreted as the accumulated underlying value of the discounted market portfolio. Under appropriate assumptions the increase in expected discounted value of the market portfolio is shown to equal the expected increase of the transformed time.

For the pricing of contingent claims the GOP and under realistic assumptions its proxy, the market portfolio, can be used as numeraire, with expectations to be taken under the real world probability measure. The resulting fair pricing methodology applies also for market models where an equivalent risk neutral martingale measure does not exist and generalizes risk neutral and actuarial pricing.

Acknowledgement

The author likes to express his thanks to Morten Christensen, David Heath, Hardy Hulley and the participants at the Stochastic Finance 2004 Conference in Lisbon for their comments and stimulating interest in this work.

A Appendix

Proof of Theorem 3.3

According to Definition 3.1, to identify an optimal portfolio, one maximizes the drift (3.4) locally in time, while keeping the diffusion coefficient (3.5) as given. Suppressing time dependence, our task is to find $\psi_\delta^1, \psi_\delta^2, \dots, \psi_\delta^d$, which maximize

$\sum_{k=1}^d \psi_\delta^k \theta^k$ subject to the constraint $\sum_{k=1}^d (\psi_\delta^k)^2 = C$, for some given value $C > 0$. For this purpose we use the Lagrange multiplier λ and consider the function

$$G(\theta^1, \dots, \theta^k, C, \lambda, \psi_\delta^1, \dots, \psi_\delta^d) = \sum_{k=1}^d \psi_\delta^k \theta^k + \lambda \left(C - \sum_{k=1}^d (\psi_\delta^k)^2 \right). \quad (\text{A.1})$$

For $\psi_\delta^1, \psi_\delta^2, \dots, \psi_\delta^d$ to provide a maximum for $G(\theta^1, \dots, \theta^k, C, \lambda, \psi_\delta^1, \dots, \psi_\delta^d)$ it is necessary that the first-order conditions

$$\frac{\partial G(\theta^1, \dots, \theta^k, C, \lambda, \psi_\delta^1, \dots, \psi_\delta^d)}{\partial \psi_\delta^k} = \theta^k - 2\lambda \psi_\delta^k = 0 \quad (\text{A.2})$$

are satisfied for all $k \in \{1, 2, \dots, d\}$. Consequently, we must have

$$\psi_\delta^k = \frac{\theta^k}{2\lambda} \quad (\text{A.3})$$

for all $k \in \{1, 2, \dots, d\}$. We can now use the constraint together with (3.6) to obtain the relation

$$C = \sum_{k=1}^d (\psi_\delta^k)^2 = \left(\frac{|\theta|}{2\lambda} \right)^2 \quad (\text{A.4})$$

from (A.3). Then from (A.3) and (A.4) one obtains

$$\psi_\delta^k = \frac{\sqrt{C}}{|\theta|} \theta^k \quad (\text{A.5})$$

for $k \in \{1, 2, \dots, d\}$. Thus from (3.5), for $S^{(\delta)}$ to be an optimal portfolio, we must have

$$\psi_\delta^k(t) = \frac{|\gamma_\delta(t)|}{|\theta(t)|} \theta^k(t) \quad (\text{A.6})$$

for $t \in [0, T]$. Now, it follows from Assumption 2.1, (3.3) and (A.6) that

$$\begin{aligned} \pi_\delta^j(t) &= \frac{1}{\bar{S}_t^{(\delta)}} \sum_{k=1}^d \psi_\delta^k(t) b^{-1,j,k}(t) \\ &= \frac{|\gamma_\delta(t)|}{\bar{S}_t^{(\delta)} |\theta(t)|} \sum_{k=1}^d \theta^k(t) b^{-1,j,k}(t). \end{aligned} \quad (\text{A.7})$$

Therefore, by (2.14) we get

$$\begin{aligned} \pi_\delta^0(t) &= 1 - \sum_{j=1}^d \pi_\delta^j(t) \\ &= 1 - \frac{|\gamma_\delta(t)| \Gamma^{(0)}(t)}{\bar{S}_t^{(\delta)} |\theta(t)|} \end{aligned} \quad (\text{A.8})$$

with $\Gamma^{(0)}(t)$ as in (3.7), for $t \in [0, T]$. Thus, we have

$$|\gamma_\delta(t)| = \frac{(1 - \pi_\delta^0(t)) \bar{S}_t^{(\delta)} |\theta(t)|}{\Gamma^{(0)}(t)}. \quad (\text{A.9})$$

Substitution into (A.7) yields (3.11) and, with the aid of (3.7) and (3.2), proves the theorem. \square

References

- Bajoux-Besnainou, I. & R. Portait (1997). The numeraire portfolio: A new perspective on financial theory. *The European Journal of Finance* **3**, 291–309.
- Bühlmann, H. (1992). Stochastic discounting. *Insurance: Mathematics and Economics* **11**, 113–127.
- Bühlmann, H. (1995). Life insurance with stochastic interest rates. In G. Ottaviani (Ed.), *Financial Risk and Insurance*, pp. 1–24. Springer.
- Christensen, M. M. & E. Platen (2004). A general benchmark model for stochastic jump sizes. Technical report, University of Technology, Sydney. QFRC Research Paper 139.
- Cochrane, J. H. (2001). *Asset Pricing*. Princeton University Press.
- Constantinides, G. M. (1992). A theory of the nominal structure of interest rates. *Rev. Financial Studies* **5**, 531–552.
- Cont, R. & P. Tankov (2004). *Financial Modelling with Jump Processes*. Financial Mathematics Series. Chapman & Hall/CRC.
- Delbaen, F. & W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312**, 215–250.
- Dimson, E., P. Marsh, & M. Staunton (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*, Chapter 4. Princeton University Press.
- Duffie, D. (2001). *Dynamic Asset Pricing Theory* (3rd ed.). Princeton, University Press.
- Duffie, D. & K. Singleton (2003). *Credit Risk : Pricing, Measurement, and Management*. Princeton Series in Finance. Princeton University Press.
- Föllmer, H. & M. Schweizer (1991). Hedging of contingent claims under incomplete information. In M. Davis and R. Elliott (Eds.), *Applied Stochastic Analysis*, Volume 5 of *Stochastics Monogr.*, pp. 389–414. Gordon and Breach, London/New York.
- Gerber, H. U. (1990). *Life Insurance Mathematics*. Springer, Berlin.
- Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

- Harrison, J. M. & S. R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11**(3), 215–260.
- Heath, D. & E. Platen (2002a). Consistent pricing and hedging for a modified constant elasticity of variance model. *Quant. Finance.* **2**(6), 459–467.
- Heath, D. & E. Platen (2002b). Perfect hedging of index derivatives under a minimal market model. *Int. J. Theor. Appl. Finance* **5**(7), 757–774.
- Heath, D. & E. Platen (2002c). Pricing and hedging of index derivatives under an alternative asset price model with endogenous stochastic volatility. In J. Yong (Ed.), *Recent Developments in Mathematical Finance*, pp. 117–126. World Scientific.
- Karatzas, I. & S. E. Shreve (1998). *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.
- Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.
- Korn, R. & M. Schäl (1999). On value preserving and growth-optimal portfolios. *Math. Methods Oper. Res.* **50**(2), 189–218.
- Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Rev. Econom. Statist.* **47**, 13–37.
- Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.
- Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investment*. Wiley, New York.
- Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica* **41**, 867–888.
- Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *J. Financial Economics* **2**, 125–144.
- Miller, S. & E. Platen (2004). A two-factor model for low interest rate regimes. Technical report, University of Technology, Sydney. QFRC Research Paper 130, to appear in *Asia-Pacific Financial Markets* **11**(1).
- Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.
- Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540–558.
- Platen, E. (2004a). A class of complete benchmark models with intensity based jumps. *J. Appl. Probab.* **41**, 19–34.
- Platen, E. (2004b). Diversified portfolios with jumps in a benchmark framework. Technical report, University of Technology, Sydney. QFRC Research Paper 129, to appear in *Asia-Pacific Financial Markets* **11**(1).

- Platen, E. (2004c). Modeling the volatility and expected value of a diversified world index. *Int. J. Theor. Appl. Finance* **7**(4), 511–529.
- Platen, E. (2004d). Pricing and hedging for incomplete jump diffusion benchmark models. In *Mathematics of Finance*, Volume 351 of *Contemporary Mathematics*, pp. 287–301. American Mathematical Society.
- Protter, P. (2004). *Stochastic Integration and Differential Equations* (2nd ed.). Springer.
- Revuz, D. & M. Yor (1999). *Continuous Martingales and Brownian Motion* (3rd ed.). Springer.
- Rogers, L. C. G. & D. Williams (2000). *Diffusions, Markov Processes and Martingales: Itô Calculus* (2nd ed.), Volume 2 of *Cambridge Mathematical Library*. Cambridge University Press.
- Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.
- Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance* **19**, 425–442.