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A General Benchmark Model for Stochastic Jump Sizes

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Abstract

This paper extends the benchmark framework of Platen (2002) by introducing a sequence of incomplete markets, having uncertainty driven by a Wiener process and a marked point process. By introducing an idealized market, in which all relevant economical variables are observed, but may not all be traded, a generalized growth optimal portfolio (GOP) is obtained and calculated explicitly. The problem of determining the GOP is solved in a general setting which extends existing treatments and provides a clear link to the market prices of risk. The connection between traded securities, arbitrage and market incompleteness is analyzed. This provides a framework for analyzing the degree of incompleteness associated with jump processes, a problem well-known from insurance and credit risk modeling. By staying under the empirical measure, the resulting benchmark model has potential advantages for various applications in finance and insurance.

1 Introduction

In this paper a model driven by a marked point process and m independent Wiener processes is investigated. Including a marked point process rather than simple Poissonian counting processes allows some important extra degrees of freedom for modeling.

It is well-known that the introduction of a marked point process with a continuous mark space makes a finite market incomplete. For a discussion of this fact and the application of marked point processes, see for instance Aase (1988), Bardhan & Chao (1996), Björk et al. (1997) and Jarrow & Madan (1999). This feature, probably along with the fact that explicit solutions for jump diffusions are extremely few, has made the application of general point processes relatively limited in applied finance. However, as noted by Björk,

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Kabanov & Runggaldier (1997), interest term structure modelling provides a natural continuum of assets, namely zero coupon bonds of varying time to maturity, in which case a complete market may be obtainable. Another example would be option pricing with a continuum of strikes and maturities. Similarly, the insurance market also has a natural extension into some infinite traded asset case by the existence of reinsurance contracts, which limits the claim amounts to some prescribed size. In this paper, the case of infinitely many assets is restricted to the countable case, in the spirit of Ross (1976) and Kabanov & Kramkov (1998). In the paper Jensen (1999) incompleteness is studied using numerical methods, to determine the number of assets required to complete the market.

In order to create a realistic model, we consider a sequence of incomplete markets, since all markets will have a finite number of assets. By this assumption the introduction of a measure-valued portfolio strategy, as suggested by Björk et al. (1997) and Jarrow & Madan (1999), is avoided. It will still be possible to treat pricing, hedging and market incompleteness. Although market completeness, in general, will be unattainable in the sense of perfect replication, the notion of *approximate market completeness* can be obtained when investors are allowed to hold any given number of assets. This also extends results by Donno (2004), which are formulated in the context of mean square convergence and requires the machinery of cylindrical stochastic integration.

The main objective of this paper is to study the effects of incompleteness in such a sequence of markets as well as properties relating to no arbitrage. This will extend the benchmark framework suggested in Platen (2002), where the fundamental object is the growth optimal portfolio (GOP). Several other articles have considered the specific properties of a GOP, see for instance Long (1990), Bajeux-Besnaino & Portait (1997), Korn & Schäl (1999), Goll & Kallsen (2000, 2003), Becherer (2001), Bühlmann & Platen (2003), Korn, Oertel & Schäl (2003) and Platen (2004b). We define the GOP by its local characteristics, a definition which requires only very weak assumptions. It is characterized in each of the finite markets and calculated explicitly for an asymptotic market, which consists of all processes that can be approximated in a pathwise sense. Pricing and hedging are investigated in this setup.

The structure of the paper is as follows: Section 2 presents the mathematical foundation for the model and highlights the relationships between arbitrage, market price of risk and the GOP. Section 3 derives conditions for hedging and market completeness. Examples are provided in Section 4.

2 Benchmark Model

2.1 Modeling of Uncertainty

Let there be given a complete filtered probability space, $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$, with filtration $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, see Karatzas & Shreve (1988) or Protter (2003). It is assumed that $\mathcal{F} = \mathcal{F}_\infty = \sigma\{\mathcal{F}_t \mid t \in [0,\infty)\}$, which states that all uncertainty is revealed eventually. Event driven uncertainty is modeled by a marked point process, $p(\cdot,\cdot)$, see Bremaud (1981) and Protter (2003) and continuous uncertainty by an m-dimensional Wiener process, $W = \{W(t) = (W^1(t),\ldots,W^m(t))^T, t \in [0,\infty)\}$. The marked point process is represented by the jump measure $p(dv,dt)$. The corresponding compensated measure $q(dv,dt)$ is a martingale measure with respect to the filtration $\underline{\mathcal{F}}$ and the historical probability measure P. The mark space $\mathcal{E}$ is assumed to be a connected subset of $[0,\infty)$ equipped with the usual Borel sigma-algebra, $\mathbb{B}^{\mathcal{E}}$. It is assumed that the martingale measure, $q(\cdot,\cdot)$, admits a time-varying non-negative intensity measure, $\phi(\cdot,\cdot)$, such that, $q(dv,dt) = p(dv,dt) - \phi(dv,t)dt$.

It is assumed that W and p are independent and generate all uncertainty in the model. Hence, the filtration $\underline{\mathcal{F}} = (\mathcal{F}_t)_{t\geq 0}$ is assumed to be generated by the augmentation of the sigma-algebra

$$\mathbb{F}_t = \sigma\{W_s, p(A\times[0,s)) \mid \; A \in \mathbb{B}^{\mathcal{E}} \text{ and } s \in [0,t)\}. \tag{1}$$

This paper does not assume a fixed time horizon. This usually causes a problem, known from the theory of equivalent martingale measures, see Karatzas & Shreve (1998), due to the lack of uniform integrability of the Radon-Nikodym derivative. Under the benchmark approach, by using the GOP as numeraire, one can disregard this issue. Here fair prices when expressed in units of the GOP form martingales and can be used for consistent pricing. One only needs to ensure that the GOP does not explode at any finite time.

2.2 The Market

From these building blocks, we define a set of primary security accounts as non-negative stochastic processes on the given probability space. The first account $S^{(0)}$, is assumed to be locally risk free, which means that it is of finite variation and the solution to the differential equation

$$dS^{(0)}(t) = S^{(0)}(t)r(t)dt \tag{2}$$

for $t \in [0,\infty)$ with $S^{(0)}(0) = 1$. It is assumed that the interest rate process $r = \{r(t), t \in [0,\infty)\}$ is adapted and non-negative.

The remaining primary security accounts are risky and assumed to be given as solutions to the stochastic differential equations (SDEs)

$$dS^{(i)}(t) = S^{(i)}(t-)\left(a^i(t)dt + \sum_{j=1}^{m} \sigma^{i,j}(t)dW^j(t) + \int_{\mathcal{E}} b^i(v,t)q(dv,dt)\right) \quad (3)$$

for $t \in [0,\infty)$ with $S^{(i)}(0) > 0$ for $i \in \{1,2,\ldots\}$. In (3) it is assumed that the integrands $a^i, \sigma^{i,j}$ and b^i are all predictable stochastic processes such that a unique strong solution of these SDEs exists, see Protter (2003). In order to specify some minimum requirements the following assumptions are assumed to hold for any time $T > 0$:

$$\int_0^T \left(|r(s)| + |a^i(s)| + \sigma^{i,j}(s)^2 + \int_{\mathcal{E}} |b^i(v,s)|\phi(dv,s)\right) ds < \infty \quad (4)$$

almost surely for all $i,j \in \{1,2,\ldots\}$. To ensure non-negativity, it is assumed that $b^i(v,t) \geq -1$ almost surely. For simplicity, it can be assumed that $(\sigma^i(t), b^i(v,t)),\ i \in \{1,2,\ldots\}$, are linearly independent functions on $\mathbb{R}^m \times \mathcal{E}$ almost surely for all $t \in [0,\infty)$, which means that no asset can be formed as a portfolio of other assets. In a continuous diffusion model there can be at most m primary security accounts satisfying this property. In this framework, since the marked point process is, in principle, of infinite dimension, there can be infinitely many primary security accounts which cannot be formed as portfolios of each other. From no arbitrage considerations it will follow that either such a linear independence assumption is true or some primary security accounts are redundant, so the assumption can be made without loss of generality.

Define for each $N \in \mathbb{N}$ the market

$$\mathbb{S}^N \triangleq \{\mathbb{S}^N(t) = (S^{(0)}(t),\ S^{(1)}(t),\ S^{(2)}(t),\ \ldots,\ S^{(N)}(t))^T, t \in [0,\infty)\}$$

consisting of the first $N+1$ primary security accounts. The sequence of markets can be thought of as the gradual securitization of uncertainty, which naturally takes place in real markets as a consequence of the introduction of new securities and derivatives.

2.3 Strategies

A strategy, δ, in the market $\mathbb{S}^N$ is defined as a predictable vector process, $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(N)}(t))^T,\ t \in [0,\infty)\}$, in $\mathbb{R}^{N+1}$ such that the Itô integral $\int_0^T \delta^{(i)}(t)dS^{(i)}(t)$ is well-defined with finite mean for any $T > 0$ and $i \in \{0,1,\ldots,N\}$.

The value of the portfolio, $S^{(\delta)} = \{S^{(\delta)}(t),\ t \in [0,\infty)\}$, at any time $t \in [0,\infty)$ is then given as

$$S^{(\delta)}(t) = \sum_{i=0}^{N} \delta^{(i)}(t) S^{(i)}(t).$$

A strategy, δ, in the market $\mathbb{S}^N$ is called *self-financing* if

$$S^{(\delta)}(t) = S^{(\delta)}(0) + \sum_{i=0}^{N} \int_0^t \delta^{(i)}(s) dS^{(i)}(s) \tag{5}$$

for all $t \in [0,\infty)$. Let $\underline{\Theta}(\mathbb{S}^N)$ denote the non-negative, self-financing portfolios in $\mathbb{S}^N$. A portfolio is called *admissible* if it belongs to $\underline{\Theta}(\mathbb{S}^N)$ for some $N > 0$. Unless otherwise noted, all portfolios are assumed to be admissible. This does not constitute a real limitation on the model as negative valued portfolios can be constructed using short positions. Furthermore, lower bounded claims may be bought by any investor with sufficient initial funds to guarantee that the total wealth remains non-negative. The requirement that portfolio values should remain non-negative reflects the limited liability constraint for the total portfolio of each market participant.

For a strictly positive portfolio, $S^{(\delta)}$ in $\underline{\Theta}(\mathbb{S}^N)$, define the corresponding vector of fractions or portfolio weights, $\pi_\delta = \{\pi_\delta(t) = (\pi_\delta^{(0)}(t), \ldots, \pi_\delta^{(N)}(t))^T$, $t \in [0,\infty)\}$ with:

$$\pi_\delta^{(i)}(t) \quad \triangleq \quad \frac{\delta^{(i)}(t) S^{(i)}(t)}{S^{(\delta)}(t)} \tag{6}$$

for $i \in \{1, \ldots, N\}$. With these preliminaries it is convenient to introduce a space $\mathbb{S}$ of *generalized portfolios*. These are processes which can be approximated by traded portfolios in the markets $\mathbb{S}^N$. These processes will play a major role later on since, due to the incompleteness of the model, many interesting objects must be found here. For instance, the market can be assumed to contain diversified portfolios, which are not immediately tradeable by a single investor.

Definition 2.1 *A non-negative, right continuous stochastic process S with left hand limits satisfying the SDE (3) with general coefficients a, b, and σ, satisfying condition (4) is an element of the space $\mathbb{S}$ of generalized portfolios, if there exists a sequence $(S^{\delta_N})_{N \in \{1,\ldots\}}$ of portfolio processes, $S^{(\delta_N)} \in \Theta(\mathbb{S}^N)$, such that $S^{(\delta_N)}(t) \to S(t)$ in probability for all $t \in [0,\infty)$ when $N \to \infty$.*

2.4 Arbitrage

As usual, it should be impossible to create "something from nothing" using a non-negative portfolio. The following definition of arbitrage is similar to that in Platen (2004b). It reflects the limited liability of agents, in particular, an agent applying an arbitrage should always remain solvent. This demands more than merely having the wealth process bounded from below and hence assuming no arbitrage in this sense is a mild restriction on investor behavior. As will become evident later on, this is sufficient for the model.

Definition 2.2 *Let $S^{(\delta)}$ belong to $\underline{\Theta}(\mathbb{S}^N)$. The market $\mathbb{S}^N$ is said to permit arbitrage and δ is called an arbitrage strategy if $S^{(\delta)}(0) = 0$ and there exists a time $T \in [0, \infty)$, such that $P(S^{(\delta)}(T) > 0) > 0$. The space $\mathbb{S}$ is said to permit arbitrage, if there exists an $S \in \mathbb{S}$ and a $T \in [0, \infty)$ such that $S(0) = 0$, $S(t) \geq 0$ for all $t \in [0, T)$ and $P(S(T) > 0) > 0$.*

Obviously, no arbitrage in $\mathbb{S}$ implies no arbitrage in $\mathbb{S}^N$. Note that an approximate arbitrage in the sense of Duffie (2001) becomes an arbitrage in $\mathbb{S}$. Hence excluding, for instance, this type of arbitrage is necessary if one wishes to avoid arbitrage in $\mathbb{S}$. However, it is not necessary to exclude arbitrage in $\mathbb{S}$ to obtain a reasonable theory. Consequently this assumption is not made. The above arbitrage definition resembles the definition of asymptotic arbitrage of the first kind given by Kabanov & Kramkov (1998), see also the *No Free Lunch With Vanishing Risk* concept and the fundamental theorem of asset pricing in Delbaen & Schachermayer (1994) for another type of asymptotic arbitrage definition.

For the remainder of the paper it is assumed that no arbitrage in any of the markets $\mathbb{S}^N$ exists.

2.5 Growth Optimal Portfolio

No arbitrage is often associated with the existence of a state price density or equivalent martingale measure, see Duffie (2001). In the benchmark framework, the object used as a reference unit for pricing purposes is a portfolio having a maximal growth rate, the GOP. Under the standard assumptions of risk neutral pricing the inverse of the discounted GOP is the Radon-Nikodym derivative of the risk-neutral measure with respect to the empirical measure. However, under the weaker assumptions made in the benchmark framework, in particular, the definition of arbitrage, a risk-neutral equivalent martingale measure may not exist, whereas it is still possible to define the GOP and apply it for pricing purposes as done in Platen (2002, 2004b, 2004c).

In this paper the growth rate has a slightly more general meaning than is usual in the literature. By the Itô formula, the logarithm of a strictly positive portfolio decomposes naturally into a drift term, a continuous local martingale and a finite variation *sigma martingale*. The concept of a sigma martingale plays no major role in this setting. It can be thought of as the solution to a driftless SDE or simply the processes that appear when using a local martingale as integrator, see Delbaen & Schachermayer (1998), Kallsen (2004) and Protter (2003) for more information.

Definition 2.3 *The growth rate $g^{\delta}(t)$ of a strictly positive portfolio $S^{(\delta)}$ is defined as the infinitesimal drift of the SDE of* $\log(S^{(\delta)}(t))$.

More precisely, if $S^{(\delta)}$ is in Definition 2.3 any strictly positive portfolio in $\mathbb{S}^N$, then by the Itô formula, the definition of q and (3) it follows that

$$\begin{aligned} d(\log(S^{(\delta)}(t))) &= g^{\delta}(t)dt + \sum_{j=1}^{m}\sum_{i=1}^{m} \pi_{\delta}^{(i)}(t)\sigma^{i,j}(t)dW^{j}(t) \\ &\quad + \int_{\mathcal{E}} \log(1 + \sum_{i=m+1}^{N} \pi_{\delta}^{(i)}(t)b^{i}(v,t))q(dv,dt) \end{aligned} \tag{7}$$

with $g^{\delta}(t)$ given as

$$\begin{aligned} g^{\delta}(t) &= \sum_{i=1}^{m} \pi_{\delta}^{(i)}(t)a^{i}(t) - \frac{1}{2}\sum_{j=1}^{m}\Big(\sum_{i=1}^{m} \pi_{\delta}^{(i)}(t)\sigma^{i,j}(t)\Big)^{2} \\ &\quad + \sum_{i=m+1}^{N}\int_{\mathcal{E}} \pi_{\delta}^{(i)}(t)b^{i}(v,t)\phi(dv,t) \end{aligned}$$

for $t > 0$. Any kind of conditional expectation is purposely left out of Definition 2.3 as the growth rate otherwise becomes unclear whenever the local martingale and sigma martingale parts of $S^{(\delta)}$ are not true martingales. Furthermore, this will allow a very broad class of models to be covered, since $\log\left(S^{(\delta)}(t)\right)$ does not even have to be locally P-integrable. Note that since a semimartingale can be represented by its local characteristics, see Jacod & Shiryaev (1987), the above definition of a growth rate can be applied to a very general class of semimartingales. The following definition of a GOP appears to be natural:

Definition 2.4 *A GOP in the market $\mathbb{S}^N$ is a self-financing, strictly positive portfolio maximizing the growth rate of all portfolios in $\mathbb{S}^N$. A* generalized GOP $S^{(\underline{\delta})} = \{S^{(\underline{\delta})}(t), t \in [0,\infty)\} \in \mathbb{S}$ *is defined by the condition that for any $N \in \{1,2,\ldots\}$ and any strictly positive portfolio $S^{(\delta)}$ in $\mathbb{S}^N$, the growth rate of $S^{(\underline{\delta})}$ is greater than or equal to that of $S^{(\delta)}$, that is $g^{\underline{\delta}}(t) \geq g^{\delta}(t)$ almost surely for Lebesgue almost every $t \in [0,\infty)$.*

If a GOP exists in $\mathbb{S}^N$ and $\mathbb{S}$ it will be denoted $S^{(\underline{\delta}^N)}$ and $S^{(\underline{\delta})}$, respectively. Similarly, the corresponding fractions of portfolio weights will be denoted by $\pi_{\underline{\delta}^N}$ and $\pi_{\underline{\delta}}$, respectively. For simplicity, assume that $S^{(\underline{\delta}^N)}(0) = S^{(\underline{\delta})}(0) = 1$ a.s.

One might have expected the GOP in $\mathbb{S}$ to be the process that has a higher growth rate than any other asset in $\mathbb{S}$. There are, however, important reasons for the chosen definition. As will become clear, the existence of a GOP implies no arbitrage in the markets $\mathbb{S}^N$ of traded securities. However, since no trading is possible in $\mathbb{S}$ there is no practically relevant reason for assuming no arbitrage in $\mathbb{S}$ and requiring the growth rate of the generalized GOP to dominate any possible growth rate in $\mathbb{S}$. This would impose unnecessary restrictions on $\mathbb{S}$, which would translate into implicit and complex restrictions on the finite markets. Similarly, there is no reason why the GOP should be a traded portfolio.

The following lemma shows how the supermartingale property is a defining characteristic of the GOP. It is a generalization of results in Breiman (1960), Long (1990), Becherer (2001), Platen (2002, 2004a), Bühlmann & Platen (2003), Goll & Kallsen (2003).

Lemma 2.5 *Assume a strictly positive process $X = \{X(t), t \in [0,\infty)\} \in \mathbb{S}$ exists, such that, for any $N \in \{1, 2, \ldots\}$ and any portfolio $S^{(\delta)} \in \underline{\Theta}(\mathbb{S}^N)$, the benchmarked portfolio process $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t), t \in [0,\infty)\}$ defined by $\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{X(t)}$ is an $(\underline{\mathcal{F}}, P)-$supermartingale. Then X is the generalized GOP, that is $X(t) = S^{(\underline{\delta})}(t)$ almost surely for all $t \in [0,\infty)$.*

Proof: The lemma follows from the observation that if $\frac{S^{(\delta)}(t)}{X(t)}$ forms a supermartingale, then the growth rate of X is never lower than that of $S^{(\delta)}$. More precisely, consider for an admissible portfolio process $S^{(\delta)}$ in $\mathbb{S}^N$, the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t-)\left(a_\delta(t)dt + \sum_{i=1}^{m} \sigma_\delta^i(t)dW^i(t) + \int_{\mathcal{E}} b_\delta(v,t)q(dv,dt)\right) \quad (8)$$

and assume that X is given as the solution of the SDE

$$dX(t) = X(t-)\left(a_X(t)dt + \sum_{i=1}^{m} \sigma_X^i(t)dW^i(t) + \int_{\mathcal{E}} b_X(v,t)q(dv,dt)\right) \quad (9)$$

for all $t \in [0,\infty)$. Assume that $\hat{S}^{(\delta)}$, as defined above, is an $(\underline{\mathcal{F}}, P)$-supermartingale. By the Itô formula, the appreciation rate $\hat{\mu}_\delta(t)$ from the SDE of $\hat{S}^{(\delta)}(t)$ equals:

$$\begin{aligned}\hat{\mu}_\delta(t) \quad &= \quad a_\delta(t) - a_X(t) + \sum_{i=1}^{m} \sigma_X^i(t)^2 - \frac{1}{2}\sum_{i=1}^{m} \sigma_\delta^i(t)\sigma_X^i(t) \\ &\quad + \int_{\mathcal{E}} \left[b_X(v,t) - b_\delta(v,t) + \frac{b_\delta(v,t) - b_X(v,t)}{1 + b_X(v,t)}\right]\phi(dv,t)\end{aligned}$$

for all $t \in [0,\infty)$. As $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{F}}, P)$-supermartingale, the appreciation rate must be non-positive $P \otimes dt$ a.e., see Protter (2003). Using Definition

2.3 it follows that the difference between the corresponding growth rates $g^{\delta}(t)$ and $g^{X}(t)$ equals

$$
\begin{aligned}
g^{\delta}(t) - g^{X}(t) &= a_{\delta}(t) - a_X(t) + \frac{1}{2}\sum_{i=1}^{m}\left(\sigma_X^i(t)^2 - \sigma_{\delta}^i(t)^2\right) \\
&\quad + \int_{\mathcal{E}}\Big[b_X(v,t) - b_{\delta}(v,t) + \log(1 + b_{\delta}(v,t)) \\
&\quad - \log(1 + b_X(v,t))\Big]\phi(dv,t)
\end{aligned}
$$

for $t \in [0,\infty)$. By noting that

$$
\begin{aligned}
\frac{1}{2}(\sigma_X^2 - \sigma_{\delta}^2) &= \frac{1}{2}(\sigma_X - \sigma_{\delta})(\sigma_X + \sigma_{\delta}) \\
&\leq \sigma_X^2 - \sigma_{\delta}\sigma_X
\end{aligned}
$$

and rewriting

$$
\begin{aligned}
b_X - b_{\delta} + \log(1 + b_{\delta}) - \log(1 + b_X) &= b_X - b_{\delta} + \log(\frac{1 + b_{\delta}}{1 + b_X}) \\
&\leq b_X - b_{\delta} + \frac{1 + b_{\delta}}{1 + b_X} - 1 \\
&= b_X - b_{\delta} + \frac{b_{\delta} - b_X}{1 + b_X}
\end{aligned}
$$

it follows that

$$
g^{\delta}(t) - g^{X}(t) \leq \hat{\mu}_{\delta}(t) \leq 0.
$$

This proves the lemma.

□

The same argument applies if X is a portfolio process in $\mathbb{S}^N$:

Corollary 2.6 *Assume that a strictly positive portfolio, $X = \{X(t), t \in [0,\infty)\} \in \mathbb{S}^N$, exists, such that for any portfolio $S^{(\delta)} \in \underline{\Theta}(\mathbb{S}^N)$, the benchmarked portfolio process $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\ t \in [0,\infty)\}$ defined by $\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{X(t)}$ is an $(\underline{\mathcal{F}}, P)$-supermartingale. Then $X(t) = S^{(\underline{\delta}^N)}(t)$ almost surely for all $t \in [0,\infty)$.*

Lemma 2.5 and Corollary 2.6 provide a useful criterion for verifying that a given candidate process is in fact growth optimal. It also makes clear, that having infinite *expected* returns in the market, although unrealistic, is not really problematic from a technical point of view since it may still allow a GOP. As usual, the supermartingale property described above ensures that a GOP in $\mathbb{S}^N$ and the generalized GOP will be unique, see Becherer (2001) or Platen (2002). Note the fact that the process $\hat{S}^{(\delta)}(t)$ may be a strict supermartingale and not just a local martingale. The reader is referred to Bühlmann & Platen (2003) and Christensen & Larsen (2004) for a discussion of this phenomenon.

The numeraire property expressed in Lemma 2.5 makes the GOP well-suited for asset pricing purposes. As asset pricing is centered around the market price of risk, the next section will explore the relationship between this object and the GOP. Furthermore, it will be argued that the weak no arbitrage requirement imposed in this paper will be sufficient to obtain a market price of risk.

2.6 The GOP and the Market Price of Risk

We now obtain an explicit characterization of the GOP in $\mathbb{S}^N$ and the generalized GOP in $\mathbb{S}$. The following concept of a market price of risk could be made more general, see for instance Schweizer (1992), Delbaen & Schachermayer (1995) and Christensen & Larsen (2004), but we emphasize the interpretation as a density on the space $L^1(\mathcal{E})$ to allow full generality within the setting of a marked point process. Define the set

$$\mathcal{B}_N = \big\{(\sigma(t), b(v,t)) \in \mathbb{R}^m \times L^1(\mathcal{E}) | (\sigma(t), b(v,t)) = \sum_{i=1}^{N} \pi^i(t)(\sigma^i(t), b^i(v,t))\big\},$$

where $\pi_i(t)$ are predictable processes defining an admissible strategy in $\mathbb{S}^N$, $N \in \{1, 2 \ldots\}$. The union of the sets $\mathcal{B}_N$ is denoted by $\mathcal{B}$.

For each $N \in \mathbb{N}$ and $i \in \{0, 1, \ldots, N\}$ consider a continuous, linear functional $\Gamma : \mathcal{B}_N \to \mathbb{R}$ such that:

$$\Gamma(\{\sigma^{i,j}(t)\}_{j=1}^m, b^i(v,t)) = a^i(t) - r(t). \tag{10}$$

Such a functional will be called a *risk premium functional* and being a continuous functional on a subset of $(\mathbb{R}^m, L^1(\mathcal{E}))$, Γ can be represented by the processes $\theta = \{\theta(t) = (\theta^1(t),\ \ldots,\ \theta^m(t))^T,\ t \in [0,\infty)\}$ and $\psi_\theta = \{\psi_\theta(t,v),\ t \in [0,\infty)\}$, such that

$$\Gamma(\{\sigma^{i,j}(t)\}_{j=1}^m, b^i(v,t)) = \sum_{j=1}^{m} \theta^j(t)\sigma^{i,j}(t) + \int_{\mathcal{E}} b^i(v,t)\psi_\theta(v,t)\phi(dv,t) \tag{11}$$

for $i \in \{1, 2, \ldots, N\}$, where $\theta^j,\ j \in \{1, 2, \ldots, m\}$, are predictable processes that are assumed to be square integrable in t. The process ψ_θ is bounded on $\mathcal{E}$ and predictable. It is assumed, for simplicity, that ψ_θ is also bounded in t. In most papers it is assumed that $\psi_{\theta(v,t)} > -1$. This is not unreasonable, but it does not follow from no arbitrage, when the markets $\mathbb{S}^N$ are considered. However, it follows that

$$\int_{\mathcal{E}} b(v,t)\psi_\theta(v,t)\phi(dv,t) < \int_{\mathcal{E}} b(v,t)\phi(dv,t) \tag{12}$$

for Lebesgue almost every $t \in [0, \infty)$ and every non-negative b appearing as second coordinate in $\mathcal{B}_N$, if there should be no arbitrage. Note that Bardhan & Chao (1996) obtain a similar condition to the one above in their search for equivalent martingale measures, although they impose stronger assumptions on the jump volatilities. Note that if there exists a risk premium functional, Γ, satisfying (10), it is unique but the representation of this functional is not unique because Γ is only defined on a subset of $L^1(\mathcal{E})$. Any extension of this functional to the entire space of integrable functions on $\mathcal{E}$, taking values in $(-1, \infty)$, will have identical properties on $\mathbb{S}^N$ and can be represented by a distinct function ψ_θ.

From the market price of risk representation, the GOP can be characterized. Aase (1984) has derived a Hamilton-Jacobi-Bellman equation characterizing the GOP in a similar setting. Goll & Kallsen (2003) provide a general Hamilton-Jacobi-Bellman equation for obtaining a GOP in a semi-martingale setting, where consumption is allowed according to a stochastic clock. Korn, Oertel & Schäl (2003) have studied necessary and sufficient conditions in the setting of a finite-dimensional jump process, which are similar to the first part of the following theorem proved in Appendix A.

Theorem 2.7 *A sufficient condition for a portfolio, $S^{(\underline{\delta}^N)}$, to be growth optimal in the market $\mathbb{S}^N$ is given by the equation:*

$$\sum_{j=1}^{m} \theta^j(t)\sigma^{i,j}(t) - \sum_{j=1}^{m} \sigma^{i,j}(t)\Big(\sum_{k=1}^{m} \pi^k_{\underline{\delta}^N}(t)\sigma^{k,j}(t)\Big) + \int_{\mathcal{E}} b^i(v,t)\left(\psi_\theta(v,t) - 1 + \frac{1}{1+\sum_{k=1}^{N}\pi^k_{\underline{\delta}^N}(t)b^k(v,t)}\right)\phi(dv,t) = 0 \tag{13}$$

for $i \in \{1, \ldots, N\}$ and $t \in [0, \infty)$. If such a portfolio, $S^{(\underline{\delta}^N)}$, exists for all $N \in \{1, 2, \ldots\}$ and if

$$(\theta(t), \frac{\psi_\theta(t)}{1-\psi_\theta(t)}) \in cl(\mathcal{B}) \tag{14}$$

for almost every t, where the closure, $cl(\mathcal{B})$ is taken with respect to the Euclidian norm in $\mathbb{R}^m$ and the L^1-norm, respectively, then a unique generalized GOP, $S^{(\underline{\delta})}(t)$, exists, which is given by the SDE

$$\begin{aligned} dS^{(\underline{\delta})}(t) &= S^{(\underline{\delta})}(t-)\Bigg(\Bigg(r(t) + \sum_{i=1}^{m}(\theta^i(t))^2 + \int_{\mathcal{E}} \frac{\psi_\theta(v,t)^2}{1-\psi_\theta(v,t)}\phi(dv,t)\Bigg)dt \\ &\quad + \sum_{i=1}^{m}\theta^i(t)dW^i(t) + \int_{\mathcal{E}} \frac{\psi_\theta(v,t)}{1-\psi_\theta(v,t)}q(dv,dt)\Bigg) \end{aligned} \tag{15}$$

for all $t \in [0, \infty)$ and with $S^{(\underline{\delta})}(0) = 1$.

Note that a solution to equation (13) does not provide a necessary condition, since it corresponds to the optimal growth rate being obtained in an inner point of the set of admissible strategies. In cases where the market price of risk is large and the chance of jumps to low values is reasonably small, the solution to equation (13) does not exist in the class of admissible portfolio choices. In this case the optimal strategy is a boundary solution. The consequence of a boundary solution is that benchmarked price processes become strict supermartingales which are not local martingales, see Christensen & Larsen (2004) for precise statements. From the structure of (13) it follows that the GOP strategy is obtained as a complex projection in the space $(\mathbb{R}^m, L^1)$.

Solving the equation (13) is generally difficult and the generalized GOP does not necessarily appear as a solution to this equation. However, in an important special case, the solutions coincide as seen by the following result, which follows immediately from Theorem 2.7.

Corollary 2.8 *The GOP of the market $\mathbb{S}^N$ is the generalized GOP if and only if* $(\theta(t), \frac{\psi_\theta(v,t)}{1-\psi_\theta(v,t)}) \in \mathcal{B}_N$.

Due to the supermartingale property of non-negative benchmarked portfolios, the following is a well-known consequence, see Platen (2004a, Platen (2004b).

Corollary 2.9 *If the market $\mathbb{S}^N$ permits a growth optimal portfolio, then there is no arbitrage in $\mathbb{S}^N$.*

Thus, the local martingale property obtained when condition (13) is satisfied is not necessary. The existence of a risk premium functional, Γ, and some regularity conditions is sufficient to guarantee the existence of a GOP in $\mathbb{S}^N$, even in the situation with no equivalent (local/sigma) martingale measures. The reader is referred to Christensen & Larsen (2004) for a general proof. Under additional assumptions, the proof follows directly from Kramkov & Schachermayer (1999). The next theorem will complete the circle, by showing that if there is no arbitrage in the market $\mathbb{S}^N$, then a risk premium functional, Γ, can be constructed on the set of volatilities from traded assets. Furthermore, it is under certain conditions possible to extend this functional to the closure of the space $\mathcal{B}$. Similar theorems on local market prices of risk exist in the literature, see for instance Back (1991), Schweizer (1992, 1995) or Delbaen & Schachermayer (1995) but here the existence of ψ_θ is obtained under weaker assumptions and the underlying structure is more evident.

Theorem 2.10 *For each $N \in \{1, 2 \ldots\}$ assume that there is no arbitrage in $\mathbb{S}^N$. Then there exists a continuous, linear, functional,*

$$\Gamma_N : \ \mathcal{B}_N \to \mathbb{R},$$

such that

$$\Gamma_N(\{\sigma^{i,j}(t)\}_{j=1}^m, b^i(v,t)) = a^i(t) - r(t) \tag{16}$$

for $i \in \{1,2,\ldots,N\}$. *If* $(\{\theta_N^i\}_{i=1}^m, \psi_{\theta_N})$ *is a representation of this functional, then*

$$\int_{\mathcal{E}} b(v,t)\psi_{\theta_N}(v,t)\phi(dv,t) < \int_{\mathcal{E}} b(v,t)\phi(dv,t) \tag{17}$$

for any non-negative function $b(v,t) \in \mathcal{B}^N$. *If,*

$$\inf_N \psi_{\theta_N}(v,t) \geq K(t) \tag{18}$$

$P \otimes dt$*-almost everywhere, and if the linear subspace generated by* $\mathcal{B}$ *is dense in* $\mathbb{R}^m \times L^1(\mathcal{E})$*, then there exists a unique extension* Γ *of the functionals* Γ_N *to* $cl(\mathcal{B})$. Γ *satisfies (16) and has a representation* $(\{\theta^i\}_{i=1}^m, \psi_\theta)$ *where* $\psi_\theta(v,t) < 1$ $\phi(dv,t) \otimes dt \otimes P$ *almost everywhere.*

Proof: As the existence of the market price of risk for the Wiener process is well-known, see e.g. Karatzas & Shreve (1998), and the extension to the general case is straightforward, we assume for simplicity that $m = 0$.

Assume that there is a functional satisfying (16) and remember that if $b(v,t)$ is a non-negative element of $\mathcal{B}_N$, then (17) must hold, otherwise there will be an arbitrage.

The proof of existence will use induction. For $N = 1$, define the functional $\Gamma_1(kb^1(v,t)) = k(a^1(t) - r(t))$ for any $k \in \mathbb{R}$ and $\Gamma_1(x) = 0$ if $x \notin \mathcal{B}_1$. Now assume the statement holds for the markets $\mathbb{S}^1, \mathbb{S}^2, \ldots, \mathbb{S}^{N-1}$. If $b^N \in \mathcal{B}_{N-1}$, then there is an arbitrage if

$$\Gamma_{N-1}(b^N(v,t)) \neq a^N(t) - r(t) \tag{19}$$

for some predictable set $A \subseteq [0,\infty) \times \Omega$ having a positive measure. To see this, consider the admissible portfolio that invests in S^N at times when the drift of S^N is higher than the drift of the replicating portfolio. If the set $\{(t,\omega) \mid \Gamma_{N-1}(b^N(v,t)) \neq a^N(t) - r(t)\}$ is not a null set, then this constitutes an arbitrage. Hence, it can be assumed that $b^N \notin \mathcal{B}_N$.

Denote the smallest subspace containing $\mathcal{B}_N$ by $\mathcal{H}_{N-1}$. Because $\mathcal{H}_{N-1}$ is closed in $L^1(\mathcal{E})$, there exists a linear functional, $\hat{\Gamma}_N$, such that $\hat{\Gamma}_N(b) = 0$ on $\mathcal{H}_{N-1}$ and $\hat{\Gamma}_N(b^N) > 0$. This is a consequence of the Hahn-Banach extension theorem, see Rudin (1987). Without loss of generality, $\hat{\Gamma}_N$ can be chosen such that $\hat{\Gamma}_N(b^N(v,t)) = a^N(t) - r(t)$. Now define $\Gamma_N = \Gamma_{N-1} + \hat{\Gamma}_N - a^N(t) - r(t)$ to get the desired functional.

Assuming no arbitrage in any of the markets, one can follow the construction above to get a sequence of functionals, Γ_N. Define

$$\hat{\Gamma}(b) \triangleq \lim_{N\to\infty} \Gamma_N(b) \tag{20}$$

almost surely, for any $b \in \mathcal{B}$. The function $\hat{\Gamma}$ is well-defined, because due to the construction above, the sequence $\left(\Gamma_N(b)\right)_{N\in\{1,\ldots,\}}$ will be constant from a certain N upward. By construction it is well-defined on the smallest subspace containing $\mathcal{B}$. Evidently the function is linear, and using denseness, (17) and the assumption that $\inf_N \phi_{\theta_N}$ is lower bounded, there is a unique, continuous expansion of the functional $\hat{\Gamma}$ to a functional Γ defined on all of $L^1(\mathcal{E})$. As the dual space of $L^1(\mathcal{E})$ is $L^\infty(\mathcal{E})$, there exists a function, $\psi_\theta \in L^\infty(\mathcal{E})$, such that

$$\Gamma(b) = \int_{\mathcal{E}} b(v,t)\psi_\theta(v,t)\phi(dv,t) \tag{21}$$

for $t \in [0,\infty)$. Because of the continuity of Γ, the extension must be unique whenever the subspace containing $\mathcal{B}$ is dense in $L^1(\mathcal{E})$, which is the case when $\mathcal{B}$ is dense in the subset of $L^1(\mathcal{E})$, containing functions with values larger than -1.

□

2.7 Singularities in the Compensating Measure

For fixed v, it has been assumed that $\phi(dv,t)dt$ is a measure, which is absolutely continuous with respect to the Lebesgue measure on the time axis $[0,\infty)$. However, the statements of the first section will carry over if singularities are allowed. Considers the canonical Lebesgue decomposition and assume that singularities only occur at fixed points in time, where

$$\phi(dv,dt) = \phi^{abs}(dv,t)dt + \phi^{sing}(dv,t)1_{(t\in\{t_1,t_2,\ldots,t_h\})}. \tag{22}$$

That is, there are some jumps which only occur at certain times given by the set $\{t_1,t_2,\ldots,t_n\}$. For simplicity, the same mark space, $\mathcal{E}$, is used, but the jump coefficients should, of course, be separated into two terms, one represents the unexpected jumps and a second term models the anticipated jumps. In order to have a unique generalized GOP it then becomes necessary to span $\mathcal{E}$ at these instants using the discrete jump coefficients. However, if the jumps are of this type only, then the jump coefficients need not span $L^1(\mathcal{E})$ for Lebesgue almost all t, but only at times $t \in \{t_1,t_2,\ldots,t_h\}$. As the portfolio weights can be chosen to have a singularity at these instants, these discrete jumps will *not* impact on the continuous dynamics in any way, since the set of discrete jumps is a subset of Lebesgue measure zero. Hence, the portfolio weights in the cases of discrete jumps will have to satisfy

$$\int_{\mathcal{E}} b^i(v,t_i)\left(\psi_\theta(v,t_i) - 1 + \frac{1}{1+\sum_{i=m+1}^N \pi^i_{\underline{\delta}}(t_i)b^i(v,t_i)}\right)\phi(dv,t_i) = 0 \tag{23}$$

for $i \in \{1,2,\ldots,h\}$. From this, it is possible to capture well-known discrete time models, including for instance binomial trees, which were introduced by Cox, Ross & Rubinstein (1979).

3 Hedging and Complete Markets

In the context of marked point processes with marks taking values in an infinite dimensional space, uniqueness of a martingale measure is not equivalent to being able to hedge any appropriately integrable claim, see Björk, Kabanov & Runggaldier (1997) and Björk et al. (1997). This is in contrast to the standard continuous case, see Karatzas & Shreve (1998) and Duffie (2001). For marked point processes, approximate market completeness is the natural extension and a key result is the link between this notion and uniqueness of a martingale measure.

Similarly, in the benchmark case the existence and uniqueness of a generalized GOP follows if the set of volatilities and jump coefficients satisfies a certain spanning condition. Here, one can never hope for market completeness in the usual sense, since an investor is only allowed to hold a finite number of assets. However, approximate market completeness is indeed attainable as will be shown below. Throughout this section, the following assumptions will be applied:

Assumption 3.1 *For any $T \in [0, \infty)$, the following holds: The market price of jump risk density is assumed to be uniformly bounded in (v, t), that is ψ_θ satisfies*

$$-K(\omega) \leq \psi_\theta(v, t) < 1 \tag{24}$$

for $\phi(dv, t) \otimes dt$-almost all $(v, t) \in [0, T] \times \mathcal{E}$, and where $K(\omega)$ is a positive random variable. The closure $cl(\mathcal{B})$, of the set $\mathcal{B}$ satisfies

$$\begin{aligned} cl(\mathcal{B}) \quad = \quad & \mathbb{R}^m \times \big\{ b(v,t) \mid b(v,t) \in L^1(\mathcal{E}, \mathbb{B}^{\mathcal{E}}, \phi(dv,t)), \\ & b(v,t) > -1 \ \phi(dv,t) - \text{almost everywhere} \big\} \end{aligned} \tag{25}$$

for almost every (t, ω). Second, the marked point process is assumed to satisfy

$$\int_0^T \int_{\mathcal{E}} p(dv, dt) < \infty \tag{26}$$

almost surely.

Definition 3.2 *For $T \in (0, \infty)$, let H_T be any non-negative $\mathcal{F}_T$-measurable, random variable, such that*

$$E\left[\frac{H_T}{S^{(\underline{\delta})}(T)}\right] < \infty. \tag{27}$$

If there exists a sequence of portfolios, $\big(S^{(\delta_N)}\big)_{N \in \{1,2,\ldots\}}$, with $\delta_N \in \underline{\Theta}(\mathbb{S}^N)$, such that the value of the portfolio $S^{(\delta_N)}(T)$ converges in probability to the value of H_T as $N \to \infty$, then the market is said to be approximately complete.

The element $S^{(\delta_N)}$ of the sequence $\left(S^{(\delta_N)}\right)_{N\in\{1,2,...\}}$ from Definition 3.2 will be termed an approximating portfolio. The benchmarked value of the contingent claim mentioned in (27) uses the generalized GOP in $\mathbb{S}$ as the benchmark.

The fundamental mathematical tool required for hedging purposes is the martingale representation property, well-known in the Brownian motion case, see e.g. Karatzas & Shreve (1988). In the case of marked point processes, the martingale representation theorem is somewhat similar: if the usual conditions are satisfied for the stochastic basis $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$ and if $\mathcal{F}_t = \sigma(\mathcal{F}_0, p(A \times [0,t]), A \in \mathbb{B}^{\mathcal{E}})$ and p is a multivariate point process, then the martingale representation holds for p as well, see Jacod & Shiryaev (1987). Hence, the martingale representation property applies for p and W separately and, as noted in Björk et al. (1997), to combine them one needs to show that the measure P is indeed the unique measure under which W is a Wiener process and p is a Poisson measure with compensator ϕ. This is the case in the given setup, as the sigma-algebra is generated by p and W and it leads to the following theorem:

Theorem 3.3 *Assume $M = \{M(t),\ t \in [0,\infty)\}$ is a local martingale, on the probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$. Then there exist predictable processes $b(v)$ and $\{\beta^j\}_{j=1}^m$, such that*

$$E\left(\int_0^t \int_{\mathcal{E}} b(v,s)\phi(dv,t)ds\right) < \infty,$$

$$\int_0^t \sum_{j=1}^m \beta^j(s)^2 ds < \infty$$

almost surely, and

$$M(t) = M(0) + \sum_{j=1}^m \int_0^t \beta^j(s)dW^j(s) + \int_0^t \int_{\mathcal{E}} b(v,s)q(dv,ds) \tag{28}$$

almost surely for $t \in [0,\infty)$.

Often, it is just assumed that p and W have this representation property, in which case the assumption in equation (26) is superfluous. It means that the number of jumps does not explode and it is a multivariate point process in the terminology of Jacod & Shiryaev (1987).

Now assume that the contingent claim H_T is $\mathcal{F}_T$ measurable and satisfies (27). Furthermore, define

$$\hat{U}_{H_T}(t) \triangleq E[\frac{H_T}{S^{(\underline{\delta})}(T)} \mid \mathcal{F}_t] \tag{29}$$

and note that $\hat{U}_{H_T} = \{\hat{U}_{H_T}(t), t \in [0,T)\}$ is a martingale. The following theorem states that if an investor is allowed to hold all finite portfolios, consisting of assets from any market, $\mathbb{S}^N$, then this investor will be able to hedge any payoff as closely as desired.

Theorem 3.4 *The combined market, $\bar{\mathbb{S}}$, given by*

$$\bar{\mathbb{S}} \triangleq \bigcup_{N\in\mathbb{N}} \big\{S^{(\delta^N)} \in \underline{\Theta}(\mathbb{S}^N),\ t \in [0,\infty)\big\}.$$

is approximately complete.

Proof: Since $\hat{U}_{H_T}$ is a martingale, it follows from Theorem 3.3 that

$$\hat{U}_{H_T}(t) = \hat{U}_{H_T}(0) + \sum_{j=1}^{m} \int_0^t \tilde{\beta}^j(s)dW^j(s) + \int_0^t \int_{\mathcal{E}} \tilde{b}(v,s)q(dv,ds) \qquad (30)$$

almost surely for all $t \in [0,T)$ and some predictable processes $\tilde{b}(t)$ and $\tilde{\beta}$. Since by approximation $\hat{U}_{H_T}$ can be assumed to be bounded away from zero, this process can be written as the solution to the SDE

$$d\hat{U}_{H_T}(t) = \hat{U}_{H_T}(t-) \left(\sum_{j=1}^{m} \beta^j(t)dW^j(t) + \int_{\mathcal{E}} b(v,t)q(dv,dt) \right),$$

where $\beta = \frac{\tilde{\beta}}{\hat{U}_{H_T}}$ and $b = \frac{\tilde{b}}{\hat{U}_{H_T}}$. By the product rule $U_{H_T}(t) = \hat{U}_{H_T}(t)S^{(\underline{\delta})}(t)$ satisfies the SDE

$$dU_{H_T}(t) = U_{H_T}(t-) \left(a(t)dt + \sum_{j=1}^{m} \sigma^j(t)dW^j(t) + \int_{\mathcal{E}} e(v,t)q(dv,dt) \right),$$

where

$$\begin{aligned} a(t) &= r(t) + \Gamma\big(\{\sigma^j(t)\}_{j=1}^m, e(v,t)\big) \\ \sigma^j(t) &= \beta^j(t) + \theta^j(t) \\ e(v,t) &= b(v,t) + \frac{\psi_\theta(v,t)}{1-\psi_\theta(v,t)}(1+b(v,t)). \end{aligned}$$

From (24) and since the GOP by approximation can be assumed to be locally bounded, $\frac{\psi_\theta(t)}{1-\psi_\theta(t)}$ is bounded for almost every t. By Lemma A.1 in the Appendix it follows that a sequence of admissible strategies, $\big(\delta_N\big)_{N\in\{1,2,\ldots\}}$, exists such that $S^{(\delta_N)}(t) \to U_{H_T}(t)$ in probability for all $t \in [0,T]$, which proves the theorem.

□

It is useful to compare this theorem to Proposition 6.1 of Björk et al. (1997), which states that the market is approximately complete if and only if the closure of the image of this operator contains $\mathbb{R}^m \times L^2(\mathcal{E})$. Note that the so-called *hedging* operator of the mentioned article is mapping a portfolio into its resulting volatility functions.

Theorem 3.4 is simpler in some aspects, as it avoids the heavy technical machinery from operator theory and integration with respect to measure valued strategies. It removes the requirements of square integrability for the jump coefficients although this comes at a cost as it assumes that the market price of jump risk density is uniformly bounded. This assumption could be softened, but some restrictions must be made on the jump coefficients to ensure the existence of a market price of jump risk functional. We only deal with the original measure, P, and the absence of references to martingale measures allow for a broader class of models. Furthermore, we eliminate the requirement of a "sufficiently rich" stochastic basis. This is because the benchmark framework does not depend on a fixed time horizon and the existence of one single equivalent martingale measure.

4 Examples

The following simple examples indicate how the above framework can be applied and how it incorporates existing models.

Poissonian Jumps: To see how the parameter ψ_θ relates to the familiar market price of jump risk and how the above theorem is linked to the case of Poissonian jumps, consider the following example:

Put $m = 0$ and $d = N$. Let $\mathcal{E} = (0, 1]$, $\phi(dv, t) = \varphi(dv)$ and define the functions $b^i(v, t) \triangleq 1_{A_i}$, where A_i represents disjoint sets covering $\mathcal{E}$. This means that one basically discretizes the mark space and collects the jumps in "buckets". In this case, (13) becomes

$$\int_{A_i} \left(\psi_\theta(v, t) - 1 + \frac{1}{1 + \pi^i_{\underline{\delta}^n}(t)} \right) \varphi(dv) = 0 \tag{31}$$

for $i \in \{1, 2, \ldots, N\}$, the solution of which is given by

$$\pi^i_{\underline{\delta}^N}(t) = \frac{\int_{A_i} \psi(v, t)\varphi(dv)}{\varphi(A_i) - \int_{A_i} \psi_\theta(v, t)\varphi(dv)}. \tag{32}$$

Note that the denominator in equation (32) is strictly positive due to equation (12). Furthermore, comparing this equation to Platen (2004b) it becomes clear how $\psi_\theta(v, t)$ can be interpreted as the marginal market price of risk attributed to the risk of having a jump with mark v at time t. Note that when comparing to Platen (2004b), the jump sizes in the example above have been normalized to one. This explains why the process $\psi_\theta(v, t)$ is termed a market price of jump risk *density* and it shows how, in the case of discrete jumps, it can be reconciled with the usual definition.

A Simple Jump Volatility: Suppose that $N = 1$ so that there is only one risky asset. Moreover, assume that $m = 0$ and

$$b(v,t) = v, \quad \phi(dv,t) = 1_{]0,1[}(v) \text{ and } \psi_\theta(v,t) = k$$

where k is a constant. Then solving (13) yields

$$\begin{aligned} 0 &= \int_{\mathcal{E}} b(v,t)(\psi_\theta(v,t) - 1 + \frac{1}{1+\pi b(v,t)})\phi(dv,t) \\ &= \int_0^1 v(k - 1 + \frac{1}{1+\pi v})dv \\ &= \frac{1}{2}(k-1) + \frac{1}{\pi} - \frac{\ln(1+\pi)}{\pi^2}. \end{aligned}$$

From this simple example we make the following observations:

- If $k = 0$, then there is no market price on event risk and $\pi = 0$, that is the investor invests everything in the risk free asset.
- If $k \to 1$, then $\pi \to \infty$, since there will be an arbitrage for $k = 1$.
- If $k \to -\infty$, then $\pi \to -1$.

Note that $\pi < -1$ is not feasible, since wealth may become negative in this case. If one replaces the jump volatility $b(v,t) = v$ with a function having a lower chance of jumping to a value near one, then the equation may not have a solution and the GOP is attained by setting $\pi = -1$. For an example in this direction see Christensen & Larsen (2004).

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A Appendix

A.1 Denseness and the Proof of Theorem 2.7

The market price of risk $\theta(t)$ associated with the Wiener process is assumed to be square integrable in t, that is

$$\int_0^T \sum_{j=1}^m (\theta^j(t))^2 dt < \infty$$

almost surely for $i \in \{1, \ldots, m\}$. Furthermore, assume the market price of risk density associated with jumps, $\psi_\theta(v,t,\omega)$, is less than one and uniformly bounded as a function of (v,t), almost surely. This means, there exists a finite, random variable, $H(\omega)$, such that

$$|\psi_\theta(v,t,\omega)| \leq H(\omega).$$

Lemma A.1 *Suppose there exists a sequence of strategies,* $\left(\delta^N\right)_{N\in\{1,2,\ldots\}}$, *such that the corresponding volatilities and jump coefficients* $\left((\sigma^N(t,\omega), b^N(v,t,\omega))\right)_{N\in\{1,2,\ldots\}}$ *in* $\mathcal{B}$ *are such that for* $P \otimes dt$*-almost every* (t,ω) *one has*

$$|\sigma^N(t,\omega) - \sigma(t,\omega)| \to 0 \tag{33}$$

in the Euclidian norm on $\mathbb{R}^m$ *and*

$$\int_{\mathcal{E}} |b^N(v,t,\omega) - b(v,t,\omega)| \phi(dv,t,\omega) \to 0. \tag{34}$$

In this case, there exists a sequence of strategies $\left(\hat{\delta}^N\right)_{N\in\{1,2,\ldots\}}$, *such that* $S^{(\hat{\delta}^N)}(t)$ *converges to* $S^{(\delta)}(t)$ *in probability for all* $t \in [0,\infty)$, *where* $S^{(\delta)}(t) \in \mathbb{S}$ *is given by the SDE*

$$dS^{(\delta)}(t) = S^{(\delta)}(t-)\left(a(t)dt + \sum_{j=1}^m \sigma^j(t)dW^j(t) + \int_{\mathcal{E}} b(v,t)q(dv,dt)\right) \tag{35}$$

and $a(t) = \Gamma\big(\sigma(t,\omega), b(v,t,\omega)\big)$. *Moreover,* $\hat{\delta}^N(t)$ *converges to* $\delta^N(t)$ *for almost every* (t,ω) *as* $N \to \infty$.

Proof: Consider the set

$$K^N \triangleq \{(t,\omega) \mid \ |\sigma^N(t,\omega) - \sigma(t,\omega)| + \int_{\mathcal{E}} |b^N(v,t,\omega) - b(v,t,\omega)|\phi(dv,t) \leq M\}$$

for some fixed $M > 0$ and $N \in \{1, 2, \ldots, \}$. Then each K^N is a predictable set, such that

$$\int_{K^N} dt dP \to 0.$$

Choose a strategy $\hat{\delta}^N(t,\omega) = \delta^N(t,\omega)1_{K^N}(t,\omega)$ and make the portfolio self-financing by adjusting the investment in the savings account. It follows that $\hat{\delta}^N(t,\omega)$ is admissible and if one denotes the volatility and jump coefficients resulting from this set of strategies by $\hat{\sigma}^N(t,\omega)$ and $\hat{b}^N(v,t,\omega)$, respectively, it follows that

$$\begin{aligned}
& \int_0^T |\hat{\sigma}^N(t,\omega) - \sigma(t,\omega)|dt + \int_0^t \int_{\mathcal{E}} |\hat{b}^N(v,t,\omega) - b(v,t,\omega)|\phi(dv,t,\omega)dt \\
= & \int_{K^N(\omega)} |\hat{\sigma}^N(t,\omega) - \sigma(t,\omega)|dt \\
& + \int_{K_N(\omega)} \int_{\mathcal{E}} |\hat{b}^N(v,t,\omega) - b(v,t,\omega)|\phi(dv,t,\omega)dt \\
& + \int_{(K^N(\omega))^c} |\sigma(t,\omega)|dt + \int_{(K^N(\omega))^c} \int_{\mathcal{E}} |b(v,t,\omega)|\phi(dv,t,\omega)dt
\end{aligned}$$

converges to zero by dominated convergence, almost surely. From this it follows that

$$\int_0^T \sum_{j=1}^m \hat{\sigma}^{j,N}(t,\omega)dW^j(t) \to \int_0^T \sum_{j=1}^m \sigma^j(t,\omega)dW^j(t)$$

in probability, see Karatzas & Shreve (1988). Moreover,

$$\begin{aligned}
& E\left| \int_0^T \int_{\mathcal{E}} \hat{b}^N(v,t,\omega)p(dv,dt) - \int_0^T \int_{\mathcal{E}} b(v,t,\omega)p(dv,dt) \right| \\
\leq & E\left[\int_0^T \int_{\mathcal{E}} |\hat{b}^N(v,t,\omega) - b(v,t,\omega)|p(dv,dt) \right] \\
= & E\left[\int_0^T \int_{\mathcal{E}} |\hat{b}^N(v,t,\omega) - b(v,t,\omega)|\phi(dv,t)dt \right]
\end{aligned}$$

also converges to zero as $N \to \infty$, since the integrals

$$\int_0^T \int_{\mathcal{E}} \hat{b}^N(v,t,\omega)p(dv,dt)$$

are bounded in N and

$$\int_0^T \int_{\mathcal{E}} b(v,t,\omega)p(dv,dt)$$

are P-integrable. Finally, by the assumptions on $\{\theta^j\}_{j=1}^m$ and ψ_θ, the expression

$$\begin{aligned}
\int_0^T |\hat{a}^N(t) - a(t)|dt \quad \leq & \quad \int_0^T |\hat{\sigma}^N(t,\omega) - \sigma(t,\omega)|\theta(t,\omega)dt \\
& + \int_0^T |\hat{b}^N(v,t) - b(v,t)|\psi_\theta(v,t)dt
\end{aligned}$$

converges to zero by application of the Hölder inequality. Solving the SDEs for the portfolio process $S^{(\delta^N)}$ using the Doleans-Dade exponential formula, see Jacod & Shiryaev (1987), yields

$$\begin{aligned} S^{(\hat{\delta}^N)}(t) &= \exp\Bigg(\int_0^t \hat{a}^N(s)ds - \int_0^t \int_{\mathcal{E}} \hat{b}^N(v,s)\phi(dv,s)ds \\ &\quad -\frac{1}{2}\int_0^t \sum_{j=1}^m \big(\hat{\sigma}^{N,j}(t)\big)^2 dt\Bigg) \prod_{t_k \le t} \big(1+\hat{b}^N(v_k,t_k)\big), \end{aligned}$$

which by the arguments above converges in probability to $S^\delta(t)$ for almost every t. By choosing the right-continuous modification of the limit process, the convergence is true for all $t \in [0,\infty)$.

□

Proof of Theorem 2.7: Assume the existence of a market price of risk functional with a representation given by equation (10). Let $S^{(\delta)}$ be any strictly positive portfolio in $\underline{\Theta}(\mathbb{S}^N)$. By Definition 2.3 and equations (7), (10) and (11), the growth rate can be expressed as:

$$\begin{aligned} g^\delta(t) &= r(t) + \sum_{j=1}^m \sum_{i=1}^m \pi^i_\delta(t)\sigma^{i,j}(t)\theta^j(t) - \frac{1}{2}\sum_{j=1}^m \big(\sum_{i=1}^m \pi^i_\delta(t)\sigma^{i,j}(t)\big)^2 \\ &\quad + \sum_{i=1}^N \pi^i_\delta(t) \int_{\mathcal{E}} b^i(v,t)(\psi_\theta(v,t)-1)\phi(dv,t) \\ &\quad + \int_{\mathcal{E}} \log(1+\sum_{i=1}^N \pi^i_\delta(t)b^i(v,t))\phi(dv,t). \end{aligned} \tag{36}$$

To find the maximum value of g on $\mathbb{R}^N$ one starts out by finding a stationary point and by differentiating with respect to π^i_δ, necessary conditions are obtained and given by (13). It will be shown in Lemma A.2 that equation (13) is also sufficient.

For the next part of the theorem, consider the case when no incompleteness associated with the Wiener processes exists. If one could choose a portfolio with an infinite number of assets, $(\pi^1_{\underline{\delta}}, \pi^2_{\underline{\delta}}, \ldots)$, it would be a reasonable idea to choose it such that:

$$\sum_{i=1}^\infty \pi^i_{\underline{\delta}^N}(t)\sigma^{i,j}(t) = \theta^j(t) \tag{37}$$

for $j \in \{1,\ldots,m\}$. Moreover, one needs to solve the set of integral equations:

$$\int_{\mathcal{E}} b^i(v,t)\left(\psi_\theta(v,t) - 1 + \frac{1}{1+\sum_{i=1}^N \pi^i_{\underline{\delta}^N}(t)b^i(v,t)}\right)\phi(dv,t) = 0 \tag{38}$$

for $t \in [0, \infty)$ and $i \in \{m+1, \dots, N\}$. Note that by the assumption that the set of processes $b^i(v,t)$ is dense in $L^1(\mathcal{E})$ in a neighborhood around zero, the set of integral equations (38) implies that one should search for a solution to

$$\psi_\theta(v,t) - 1 + \frac{1}{1 + \sum_{i=m+1}^{\infty} \pi^i_{\underline{\delta}}(t) b^i(v,t)} = 0 \tag{39}$$

$dt \otimes \phi(dv,t) - a.e.$ if $S^{(\underline{\delta})}$ is to be growth optimal. Hence, a reasonable candidate for growth optimality in $\mathbb{S}$ is the portfolio having jump volatility

$$b^{\underline{\delta}}(v,t) = \frac{\psi_\theta(v,t)}{1 - \psi_\theta(v,t)} \tag{40}$$

for $t \in [0, \infty)$. To apply this insight, consider the element $(\theta(t), \frac{\psi_\theta(v,t)}{1-\psi_\theta(v,t)}) \in \mathbb{R}^m \times L^1(\mathcal{E})$. The denseness assumption on $\mathcal{B}$ implies the existence of a sequence $\left(\sigma^N(t), b^N(v,t)\right)_{N \in \{1,2,\dots\}}$ such that

$$|\sigma^N(t) - \theta(t)| \to 0, \tag{41}$$

where $|\cdot|$ is the Euclidian norm on $\mathbb{R}^m$ and

$$\int_{\mathcal{E}} |b^N(v,t) - \frac{\psi_\theta(v,t)}{1 - \psi_\theta(v,t)}| \phi(dv,t) \to 0 \tag{42}$$

for $P \otimes dt$ almost every $(t, \omega) \in [0, \infty) \times \Omega$. The remaining part of Theorem 2.7 now follows by applying Lemma A.1

□

Lemma A.2 *Assume that $\underline{\delta}$ is a strategy satisfying (13). If δ is any other admissible strategy in $\mathbb{S}^N$, the process $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t) \triangleq \frac{S^{(\delta)}(t)}{S^{(\underline{\delta})}(t)}, t \in [0, \infty)\}$ is a supermartingale.*

The proof is given by applying the Itô formula and it is also straightforward to show that the generalized GOP given by the SDE (15) has the property that benchmarked prices become supermartingales.

Note that in the general set-up, there is no restriction on the jump sizes, except that the involved integrals be well-defined. For this reason, even though the compensated random measure is a local martingale, integration with respect to q will not, in general, result in a new local martingale. However, it is by definition a sigma martingale, see Protter (2003). For benchmarked portfolio processes, due to the requirement of non-negativity, sigma martingales will indeed become local martingales and, in particular, supermartingales, so the concept of sigma martingales is not really relevant here.

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