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A Benchmark Approach to Finance

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A Benchmark Approach to Finance

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Abstract. This paper derives a unified framework for portfolio optimization, derivative pricing, financial modeling and risk measurement. It is based on the natural assumption that investors prefer more for less, in the sense that given two portfolios with the same diffusion coefficient value, the one with the higher drift is preferred. Each such investor is shown to hold an efficient portfolio in the sense of Markowitz with units in the market portfolio and the savings account of his or her home currency. If the market portfolio is diversified or monetary authorities aim to maximize the growth rates of the portfolios of their market participants through corresponding interest policies, then the market portfolio is the growth optimal portfolio (GOP). In this setup the capital asset pricing model follows without the use of expected utility functions or equilibrium assumptions. The expected increase of the discounted value of the GOP is shown to coincide with the expected increase of its discounted underlying value. The discounted GOP has the dynamics of a time transformed squared Bessel process of dimension four. The time transformation is given by the discounted underlying value of the GOP. The squared volatility of the GOP equals the discounted GOP drift, when expressed in units of the discounted GOP. Risk neutral derivative pricing and actuarial pricing are generalized by the fair pricing concept, which uses the GOP as numeraire and the real world probability measure as pricing measure. An equivalent risk neutral martingale measure does not exist under the derived minimal market model.

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1 Introduction

The aim of this paper is to derive under the benchmark approach several relationships that are fundamental for the understanding of financial markets. This approach uses the *growth optimal portfolio* (GOP) as central building block.

In Markowitz (1959) a mean-variance theory with its well-known *efficient frontier* was introduced, thus opening the avenue to modern portfolio theory. This led to the *capital asset pricing model* (CAPM), see Sharpe (1964), Lintner (1965) and Merton (1973), which is based on the *market portfolio* as reference unit and represents an equilibrium model of exchange. For the continuous time setting Merton (1973) developed the *intertemporal* CAPM from the portfolio selection behavior of investors who maximize equilibrium expected utility. It is apparent that the choice of utility functions with particular time horizons introduces a subjective element into the analysis, which will be avoided in this paper. A practical problem for applications of the CAPM arises from the fact that the dynamics of the market portfolio with its stochastic volatility are difficult to specify and consequently not easily modeled. The identification of the market portfolio and the dynamics of the GOP is of critical importance and the focus of this paper.

Of particular significance to derivative pricing has been the *arbitrage pricing theory* (APT) proposed in Ross (1976) and further developed in an extensive literature that includes Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991), Delbaen & Schachermayer (1994, 1998), Karatzas & Shreve (1998) and references therein. Under the APT, several authors refer to the *state price density*, *pricing kernel*, *deflator* or *discount factor* for the modeling of asset price dynamics, see, for instance, Constatinides (1992), Duffie (2001), Rogers (1997) or Cochrane (2001). The state price density is known to be the inverse of the discounted *numeraire portfolio*, introduced in Long (1990). The numeraire portfolio equals in a standard risk neutral setting the *growth optimal portfolio* (GOP), see Bajeux-Besnainou & Portait (1997) and Karatzas & Shreve (1998). By using the numeraire portfolio as reference unit or *benchmark*, it makes sense to define benchmarked contingent claim prices as expectations of benchmarked contingent claims under the real world probability measure. The current paper emphasizes that this *fair pricing concept*, see Platen (2002), does not require the existence of an equivalent risk neutral martingale measure. It avoids any measure transformations, but is consistent with the APT when changes of numeraire with corresponding equivalent martingale measure changes can be performed, see Geman, El Karoui & Rochet (1995). To apply under the benchmark approach fair pricing effectively, the GOP needs to be observed and modeled. This leads outside the standard pricing methodologies and is a challenge that will be addressed in the current paper.

The GOP was discovered in Kelly (1956) and is defined as the portfolio that maximizes expected logarithmic utility from terminal wealth. It has a myopic strategy and appears in a stream of literature including, for instance, Long (1990), Artzner (1997), Bajeux-Besnainou & Portait (1997), Karatzas & Shreve (1998), Kramkov & Schachermayer (1999), Becherer (2001), Platen (2002) and Goll & Kallsen (2003). Collectively, this literature demonstrates that the GOP plays a natural unifying role in derivative pricing, risk management and portfolio optimization. The aim of this paper is to establish the relationship between the GOP and the market portfolio, under the natural assumption that every investor prefers *more for less*. Some of the resulting consequences are well known, but are derived here under weaker assumptions than usual, while others, as the interpretation of the discounted GOP drift as rate of increase of *underlying value*, are likely to stimulate further research.

The paper is structured as follows. Section 2 introduces a continuous benchmark model. Section 3 discusses the market portfolio in relation to the GOP and studies some applications. In Section 4 the expected value of the GOP is characterized. Finally, in Section 5 the dynamics of the GOP are modeled.

2 Continuous Benchmark Model

2.1 Primary Security Accounts

For the modeling of a financial market we rely on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0, \infty)$. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,T]}$ is assumed to satisfy the usual conditions, see Karatzas & Shreve (1991). We consider that $\underline{\mathcal{A}}$ describes the structure of information entering the market, in the sense that the sigma-algebra $\mathcal{A}_t$ expresses the information available in the market at time t. For simplicity, we restrict ourselves to markets with continuous security prices. Trading uncertainty is expressed by the independent standard Wiener processes $W^k = \{W^k_t,\, t \in [0,T]\}$ for $k \in \{1, 2, \ldots, d\}$ and $d \in \{1, 2, \ldots\}$. The increments $W^k_{t+\varepsilon} - W^k_t$ are assumed to be independent of $\mathcal{A}_t$ for all $t \in [0,T]$, $\varepsilon > 0$ and $k \in \{1, 2, \ldots, d\}$.

We consider a continuous financial market model that comprises $d+1$ primary security accounts. These include a savings account $S^{(0)} = \{S^{(0)}(t),\, t \in [0,T]\}$, which is a riskless primary security account whose value at time t is given by

$$S^{(0)}(t) = \exp\left\{\int_0^t r(s)\, ds\right\} \tag{2.1}$$

for $t \in [0,T]$, where $r = \{r(t),\, t \in [0,T]\}$ denotes the adapted short rate process. They also include d nonnegative, risky primary security account processes $S^{(j)} = \{S^{(j)}(t),\, t \in [0,T]\}$, $j \in \{1, 2, \ldots, d\}$, each of which contains units of one type of security with all proceeds reinvested. Typically, these securities are stocks. However, derivatives, including options, as well as foreign savings accounts and bonds, may also, form primary security accounts.

To specify the dynamics of primary securities in the given financial market we assume, without loss of generality, that the jth primary security account value $S^{(j)}(t)$, $j \in \{1, 2, \ldots, d\}$, satisfies the stochastic differential equation (SDE)

$$dS^{(j)}(t) = S^{(j)}(t) \left(a^j(t)\, dt + \sum_{k=1}^{d} b^{j,k}(t)\, dW_t^k \right) \tag{2.2}$$

for $t \in [0, T]$ with $S^{(j)}(0) > 0$. Here the process $b^{j,k} = \{b^{j,k}(t),\, t \in [0, T]\}$ can be interpreted as the *volatility* of the (j, k)th security account with respect to the kth Wiener process W^k. Suppose that the (j, k)th volatility $b^{j,k}$ is a given predictable process that satisfies the integrability condition

$$\int_0^T \sum_{j=1}^{d} \sum_{k=1}^{d} \left(b^{j,k}(t)\right)^2 dt < \infty \tag{2.3}$$

almost surely, for all $j, k \in \{1, 2, \ldots, d\}$. Furthermore, we assume that the jth *appreciation rate* $a^j = \{a^j(t),\, t \in [0, T]\}$, $j \in \{1, 2, \ldots, d\}$, is a predictable process such that

$$\int_0^T \sum_{j=0}^{d} |a^j(s)|\, ds < \infty \tag{2.4}$$

almost surely.

It is reasonable to use the same number d of Wiener processes as there are risky primary security accounts. If the number of securities is bigger than the number of Wiener processes, then we have redundant securities that can be removed from the set of primary security accounts. Alternatively, if there are fewer risky securities than Wiener processes, then the market is incomplete. The core analysis of this paper is then still valid. However, some additional considerations arise which are not the focus of the current paper. The following assumption avoids redundant primary security accounts.

Assumption 2.1 *Assume that the volatility matrix $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ is invertible for Lebesgue-almost every $t \in [0, T]$ with inverse matrix $b^{-1}(t) = [b^{-1\,j,k}(t)]_{j,k=1}^d$.*

This allows us to introduce the kth *market price for risk* $\theta^k(t)$ with respect to the kth Wiener process W^k, according to the relation

$$\theta^k(t) = \sum_{j=1}^{d} b^{-1\,j,k}(t) \left(a^j(t) - r(t)\right) \tag{2.5}$$

for $t \in [0, T]$ and $k \in \{1, 2, \ldots, d\}$. Now, we can rewrite the SDE (2.2) for the jth primary security account in the form

$$dS^{(j)}(t) = S^{(j)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} b^{j,k}(t) \left(\theta^k(t)\, dt + dW_t^k\right) \right) \tag{2.6}$$

for $t \in [0, T]$ and $j \in \{1, 2, \ldots, d\}$.

2.2 Portfolios

We call a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0,T]\}$ a *strategy* if for each $j \in \{0, 1, \ldots, d\}$ the Itô stochastic integral

$$\int_0^t \delta^{(j)}(s)\, dS^{(j)}(s) \tag{2.7}$$

exists, see Karatzas & Shreve (1991). Here $\delta^{(j)}(t)$, $j \in \{0, 1, \ldots, d\}$, is the number of units of the jth primary security account that are held at time $t \in [0,T]$ in the corresponding portfolio. We denote by

$$S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, S^{(j)}(t) \tag{2.8}$$

the time t value of the portfolio process $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,T]\}$. A strategy δ and the corresponding portfolio $S^{(\delta)}$ are said to be *self-financing* if

$$dS^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, dS^{(j)}(t) \tag{2.9}$$

for $t \in [0,T]$. This means that all changes in the portfolio value are due to gains or losses from trade in primary security accounts. In what follows we consider only self-financing strategies and portfolios and will therefore omit the phrase "self-financing".

Let S be a portfolio strategy whose value $S^{(\delta)}(t)$ is strictly positive for all times $t \in [0,T]$. It is convenient to introduce the jth *fraction* $\pi_\delta^{(j)}(t)$ of $S^{(\delta)}(t)$ that is invested in the jth primary security account $S^{(j)}(t)$, $j \in \{0, 1, \ldots, d\}$, at time $t \in [0,T]$. This fraction is given by the expression

$$\pi_\delta^{(j)}(t) = \delta^{(j)}(t)\, \frac{S^{(j)}(t)}{S^{(\delta)}(t)} \tag{2.10}$$

for $t \in [0,T]$ and $j \in \{0, 1, \ldots, d\}$. Note that the fractions can be negative and always sum to one, that is

$$\sum_{j=0}^{d} \pi_\delta^{(j)}(t) = 1 \tag{2.11}$$

for $t \in [0,T]$. By (2.9), (2.6) and (2.10) we get the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} b_\delta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \right) \tag{2.12}$$

for a strictly positive portfolio. The kth *portfolio volatility* in (2.12) is given by

$$b_\delta^k(t) = \sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t) \tag{2.13}$$

and its appreciation rate is of the form

$$a_\delta(t) = r(t) + \sum_{k=1}^{d} b_\delta^k(t)\,\theta^k(t) \tag{2.14}$$

for $t \in [0,T]$ and $k \in \{1,2,\ldots,d\}$. Given a strictly positive portfolio $S^{(\delta)}$, its discounted value

$$\bar{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(0)}(t)} \tag{2.15}$$

satisfies the SDE

$$d\bar{S}^{(\delta)}(t) = \sum_{k=1}^{d} \psi_\delta^k(t)\left(\theta^k(t)\,dt + dW_t^k\right) \tag{2.16}$$

by (2.1), (2.12) and an application of the Itô formula with *kth portfolio diffusion coefficient*

$$\psi_\delta^k(t) = \bar{S}^{(\delta)}(t)\,b_\delta^k(t) \tag{2.17}$$

for $k \in \{1,2,\ldots,d\}$ and $t \in [0,T]$. Obviously, by (2.16) and (2.17) the discounted portfolio process $\bar{S}^{(\delta)}$ has *instantaneous portfolio drift*

$$\alpha_\delta(t) = \sum_{k=1}^{d} \psi_\delta^k(t)\,\theta^k(t) \tag{2.18}$$

at time $t \in [0,T]$, which measures its trend at that time. The uncertainty of a discounted portfolio $\bar{S}^{(\delta)}$ can be measured by its *instantaneous portfolio diffusion coefficient*

$$\gamma_\delta(t) = \sqrt{\sum_{k=1}^{d}\left(\psi_\delta^k(t)\right)^2} \tag{2.19}$$

at time $t \in [0,T]$.

2.3 Growth Optimal Portfolio

It is well known, see Kelly (1956), Long (1990) and Karatzas & Shreve (1998), that the *growth optimal portfolio* (GOP), which maximizes expected logarithmic utility from terminal wealth, plays a central role in finance theory. To identify this important portfolio we apply the Itô formula to obtain the SDE for $\ln(S^{(\delta)}(t))$ in the form

$$d\ln(S^{(\delta)}(t)) = g_\delta(t)\,dt + \sum_{k=1}^{d} b_\delta^k(t)\,dW_t^k \tag{2.20}$$

with *portfolio growth rate*

$$g_\delta(t) = r(t) + \sum_{k=1}^{d}\left(\sum_{j=1}^{d}\pi_\delta^{(j)}(t)\,b^{j,k}(t)\,\theta^k(t) - \frac{1}{2}\left(\sum_{j=1}^{d}\pi_\delta^{(j)}(t)\,b^{j,k}(t)\right)^2\right) \tag{2.21}$$

for $t \in [0,T]$.

Definition 2.2 *A strictly positive portfolio process* $S^{(\delta_*)} = \{S^{(\delta_*)}(t),\, t \in [0,T]\}$ *is called a* GOP *if, for all* $t \in [0,T]$ *and all strictly positive portfolios* $S^{(\delta)}$*, the inequality*

$$g_{\delta_*}(t) \geq g_\delta(t) \tag{2.22}$$

holds almost surely.

By using the first order conditions one can determine the *optimal fractions*

$$\pi_{\delta_*}^{(j)}(t) = \sum_{k=1}^{d} \theta^k(t)\, b^{-1\,j,k}(t) \tag{2.23}$$

for all $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$, which maximize the portfolio growth rate (2.21). It is straightforward to show in the given continuous financial market, see Long (1990), Karatzas & Shreve (1998) or Platen (2002), that the GOP value $S^{(\delta_*)}(t)$ satisfies the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} \theta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \right) \tag{2.24}$$

for $t \in [0,T]$ with some appropriate initial value $S^{(\delta_*)}(0) > 0$. Obviously, up to a constant factor the GOP is uniquely determined by (2.23) or (2.24). From now on we use the GOP as *benchmark* and refer to the above financial market model as *benchmark model.*

We call any security expressed in units of the GOP a *benchmarked security.* For a portfolio $S^{(\delta)}$ the corresponding benchmarked portfolio value

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} \tag{2.25}$$

satisfies the SDE

$$d\hat{S}^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, \hat{S}^{(j)}(t) \sum_{k=1}^{d} \left(b^{j,k}(t) - \theta^k(t) \right) dW_t^k \tag{2.26}$$

for $t \in [0,T]$. It follows from the driftless SDE (2.26) that any benchmarked portfolio is an $(\underline{\mathcal{A}}, P)$-local martingale. On the other hand, any nonnegative local martingale is an $(\underline{\mathcal{A}}, P)$-supermartingale, see Karatzas & Shreve (1991). The nonnegativity of portfolios of market participants is naturally given, since it reflects the *limited liability* of investors, in the sense that they are excluded from trading as soon as their total portfolio becomes zero or slightly negative.

For any nonnegative benchmarked portfolio $\hat{S}^{(\delta)}$ with $\hat{S}^{(\delta)}(\tau) = 0$ at any stopping time $\tau \in [0, T]$ we have by the supermartingale property of $\hat{S}^{(\delta)}$ the relations

$$0 = \hat{S}^{(\delta)}(\tau) \geq E\left(\hat{S}^{(\delta)}(T) \,\middle|\, \mathcal{A}_\tau\right) \geq 0$$

and therefore

$$P(S^{(\delta)}(T) > 0) = P(\hat{S}^{(\delta)}(T) > 0) = 0. \tag{2.27}$$

This guarantees that any nonnegative portfolio process $S^{(\delta)}$, which reaches at any stopping time $\tau \in [0, T]$ the value zero, remains at any later time $s \in [\tau, T]$ at the level zero. Therefore, under the above described benchmark model it is impossible to construct any nonnegative portfolio process $S^{(\delta)}$ with strictly positive wealth $S^{(\delta)}(T)$ at the terminal time T if the portfolio has before time T reached the bankruptcy level zero. This means, the above benchmark framework leads via the supermartingale property of benchmarked securities to a natural no-arbitrage concept, which is discussed in more detail in Platen (2004a). Since the existence of an equivalent risk neutral martingale measure is not required, the benchmark approach provides a richer modeling framework than is given under the fundamental theorem of asset pricing derived in Delbaen & Schachermayer (1995, 1998). We will need to use this modeling freedom towards the end of the paper when we derive a parsimonious, realistic financial market model.

3 More for Less

3.1 Optimal Portfolios

It is now our aim to identify the typical structure of the SDE of a, so called, *optimal portfolio*. To describe an optimal portfolio we introduce the following definition similar as in Platen (2002).

Definition 3.1 *We call a positive portfolio* $S^{(\tilde{\delta})}$ optimal*, if for all* $t \in [0, T]$ *and all positive portfolios* $S^{(\delta)}$ *when*

$$\gamma_{\tilde{\delta}}(t) = \gamma_\delta(t) \tag{3.1}$$

we have

$$\alpha_{\tilde{\delta}}(t) \geq \alpha_\delta(t). \tag{3.2}$$

Definition 3.1 specifies that a positive portfolio is optimal if at all times the drift of its discounted value is greater than or equal to the drift of every other discounted positive portfolio, with the same value of the diffusion coefficient. Essentially, we are simplifying our analysis by discounting since this will not have an impact on our optimization procedure. Furthermore, our analysis is strongly simplified by comparing only locally in time drift and diffusion coefficients.

This definition encapsulates a precise and simple characterization of what means *more for less*. It is clear that all rational investors would prefer more for less and therefore an optimal portfolio.

Let us introduce the *total market price for risk*

$$|\theta(t)| = \sqrt{\sum_{k=1}^{d} \left(\theta^k(t)\right)^2} \tag{3.3}$$

for $t \in [0, T]$. By methods described in the Appendix, it can be shown that if the total market price for risk is zero, then the savings account is the only optimal portfolio. We exclude this unrealistic case with the following assumption.

Assumption 3.2 *The total market price for risk is strictly greater than zero and finite, that is*

$$0 < |\theta(t)| < \infty \tag{3.4}$$

almost surely for all $t \in [0, T]$.

We can now formulate for our continuous benchmark model a *portfolio choice theorem* in the sense of Markowitz (1959) that identifies the structure of the drift and diffusion coefficients in the SDE of an optimal portfolio.

Theorem 3.3 *The value* $\bar{S}^{(\tilde{\delta})}(t)$ *at time* t *of a discounted optimal portfolio satisfies the SDE*

$$d\bar{S}^{(\tilde{\delta})}(t) = \bar{S}^{(\tilde{\delta})}(t) \frac{\left(1 - \pi^{(0)}_{\tilde{\delta}}(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \sum_{k=1}^{d} \theta^k(t) \left(\theta^k(t)\, dt + dW^k_t\right), \tag{3.5}$$

with optimal fractions

$$\pi^{(j)}_{\tilde{\delta}}(t) = \frac{\left(1 - \pi^{(0)}_{\tilde{\delta}}(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \pi^{(j)}_{\delta_*}(t) \tag{3.6}$$

for all $t \in [0, T]$ *and* $j \in \{1, 2, \ldots, d\}$.

This means that the family of discounted, optimal portfolios $\bar{S}^{(\tilde{\delta})}$ is parameterized by the fractions $\pi^{(0)}_{\tilde{\delta}}(t)$ to be held at time t in the savings account $S^{(0)}(t)$. The proof of this theorem is given in the Appendix.

By Theorem 3.3 and Itô's formula applied to (2.24), any optimal portfolio value $S^{(\tilde{\delta})}(t)$ can be decomposed into a fraction of wealth that is invested in the GOP and a remaining fraction that is held in the savings account. Therefore, Theorem 3.3 can also be interpreted as a *mutual fund theorem* or *separation theorem*, see Merton (1973). We emphasize again that the assumptions for Theorem 3.3 are very natural and also general when considering continuous markets.

3.2 Markowitz Efficient Frontier and Sharpe Ratio

It follows from the SDE (3.5) that at time t the *volatility* $b_{\tilde{\delta}}(t)$ of an optimal portfolio $S^{(\tilde{\delta})}$ equals

$$b_{\tilde{\delta}}(t) = \frac{\left(1 - \pi^{(0)}_{\tilde{\delta}}(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \, |\theta(t)| \tag{3.7}$$

and its *risk premium* $p_{\tilde{\delta}}(t)$ is

$$p_{\tilde{\delta}}(t) = b_{\tilde{\delta}}(t) \, |\theta(t)| \tag{3.8}$$

for $t \in [0, T]$. Note that the risk premium of $S^{(\tilde{\delta})}$ is the appreciation rate of $\bar{S}^{(\tilde{\delta})}$. By analogy to the mean-variance theory in Markowitz (1959), one can introduce an *efficient portfolio.*

Definition 3.4 *An* efficient portfolio $S^{(\delta)}$ *is one whose appreciation rate* $a_\delta(t)$*, as a function of squared volatility* $(b_\delta(t))^2$*, lies on the* efficient frontier*, in the sense that*

$$a_\delta(t) = a_\delta(t, (b_\delta(t))^2) = r(t) + \sqrt{(b_\delta(t))^2} \, |\theta(t)| \tag{3.9}$$

for all times $t \in [0, T]$.

By relations (3.7), (3.8) and (3.9) the following result is obtained.

Corollary 3.5 *An optimal portfolio is efficient.*

Corollary 3.5 can be interpreted as a "local in time" version of the Markowitz efficient frontier in a continuous time setting. Due to Definition 3.1 and Theorem 3.3 it is not possible to form a positive portfolio that produces an appreciation rate above the efficient frontier.

Another important investment characteristic is the *Sharpe ratio* $s_\delta(t)$, see Sharpe (1964). It is defined for a portfolio $S^{(\delta)}$ with total volatility $b_\delta(t) = \sqrt{\sum_{k=1}^{d} (b^k_\delta(t))^2} > 0$ at time t as the ratio of the risk premium

$$p_\delta(t) = b_\delta(t) \, |\theta(t)| \tag{3.10}$$

over the total volatility, that is,

$$s_\delta(t) = \frac{p_\delta(t)}{b_\delta(t)} = \frac{\alpha_\delta(t)}{\gamma_\delta(t)} \tag{3.11}$$

for $t \in [0, T]$, see (2.18) and (2.19). By (3.7), (3.8), (3.11) and Theorem 3.3 we obtain the following important result.

Corollary 3.6 *The maximum achievable Sharpe ratio is that of an optimal portfolio and equals the total market price for risk. That is,*

$$s_{\tilde{\delta}}(t) = \frac{p_{\tilde{\delta}}(t)}{b_{\tilde{\delta}}(t)} = |\theta(t)| \geq \frac{p_\delta(t)}{b_\delta(t)} \tag{3.12}$$

for all $t \in [0, T]$*. Here* $S^{(\delta)}$ *ranges over all portfolios with a given strictly positive diffusion coefficient and* $S^{(\tilde{\delta})}$ *is the optimal portfolio with that diffusion coefficient.*

The Markowitz efficient frontier and the Sharpe ratio are important tools for long term investment management. For short term investments they can probably only be efficiently exploited if the stochastic nature of the volatility of an efficient portfolio is understood. This is a problem that we address towards the end of the paper.

3.3 Capital Asset Pricing Model

The seminal *capital asset pricing model* (CAPM) was developed by Sharpe (1964), Lintner (1965) and Merton (1973) as an equilibrium model of exchange. Under the benchmark approach we need not to rely on any equilibrium or expected utility function argument. By (3.10), (2.20) and (2.24) the risk premium $p_\delta(t)$ of a portfolio $S^{(\delta)}$ is given by

$$p_\delta(t) = \sum_{k=1}^{d} b_\delta^k(t)\, \theta^k(t) = \frac{d\langle \ln(S^{(\delta)}), \ln(S^{(\delta_*)})\rangle_t}{dt} \tag{3.13}$$

at time t, where the kth portfolio volatility $b_\delta^k(t)$ is given in (2.13). Here $\langle \ln(S^{(\delta)}), \ln(S^{(\delta_*)})\rangle_t$ denotes the covariation at time t of the stochastic processes $\ln(S^{(\delta)})$ and $\ln(S^{(\delta_*)})$, see Karatzas & Shreve (1991). The time derivative of the covariation is the local in time analogue for continuous time processes of the covariance of log-returns, as used in the CAPM. The systematic risk parameter $\beta_\delta(t)$, the *portfolio beta*, can then be expressed as the ratio of the portfolio risk premium over the GOP risk premium. That is,

$$\beta_\delta(t) = \frac{p_\delta(t)}{p_{\delta_*}(t)} = \frac{\sum_{k=1}^{d} b_\delta^k(t)\, \theta^k(t)}{\sum_{k=1}^{d} |\theta^k(t)|^2} \tag{3.14}$$

for $t \in [0, T]$. The above forms of the portfolio risk premium and the portfolio beta are exactly what the intertemporal CAPM suggests, should the market portfolio equal the GOP, see Merton (1973). The *market portfolio* is here defined as the portfolio consisting of all primary security accounts weighted according to market capitalization. In what follows we will identify conditions which ensure that the market portfolio equals or approximates the GOP. This then provides a firm basis for the CAPM without equilibrium or utility based arguments.

Assume that there are $n \in \{1, 2, \dots, d\}$ foreign currencies in the market. Let $S^{(i)}$, $i \in \{0, 1, \dots, n\}$, be the savings account of the ith currency, denominated in units of the domestic currency. In other words, the first n primary security accounts $S^{(1)}, \dots, S^{(n)}$ after the domestic savings account $S^{(0)}$ are foreign savings accounts expressed in terms of the domestic currency. For each currency we assume the existence of $n_i \in \{1, 2, \dots\}$ market participants. The following assumption is natural.

Assumption 3.7 *Each market participant holds his or her invested wealth in a portfolio that is optimal with respect to his or her home currency and the assumptions made with respect to the domestic currency are extended to all currency denominations.*

The portfolio of the ℓth market participant which is optimal with respect to the ith currency denomination is denoted by $S^{(\tilde{\delta}_{i,\ell})}$, $\ell \in \{1, 2, \dots, n_i\}$. The ith total portfolio $S^{(\delta_i)}(t)$ of all such market participants is given by

$$S^{(\delta_i)}(t) = \sum_{\ell=1}^{n_i} S^{(\tilde{\delta}_{i,\ell})}(t) \tag{3.15}$$

at time $t \in [0, T]$. It is natural to assume that $S^{(\delta_i)}(t) > 0$ for all $t \in [0, T]$ and $i \in \{0, 1, \dots, n\}$. The discounted total portfolio of all market participants in the domestic market is by Theorem 3.3 and (3.15) determined by the SDE

$$\begin{aligned} d\bar{S}^{(\delta_0)}(t) &= \sum_{\ell=1}^{n_0} d\bar{S}^{(\tilde{\delta}_{0,\ell})}(t) \\ &= \sum_{\ell=1}^{n_0} \frac{\left(\bar{S}^{(\tilde{\delta}_{0,\ell})}(t) - \tilde{\delta}^{(0)}_{0,\ell}\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\left(\theta^k(t)\,dt + dW^k_t\right) \\ &= \bar{S}^{(\delta_0)}(t)\, \frac{\left(1 - \pi^{(0)}_{\delta_0}(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\left(\theta^k(t)\,dt + dW^k_t\right) \end{aligned} \tag{3.16}$$

for $t \in [0, T]$. This shows that $S^{(\delta_0)}(t)$ has the fraction $\frac{1-\pi^{(0)}_{\delta_0}(t)}{1-\pi^{(0)}_{\delta_*}(t)}$ invested in the GOP and the fraction $\frac{\pi^{(0)}_{\delta_0}(t)-\pi^{(0)}_{\delta_*}(t)}{1-\pi^{(0)}_{\delta_*}(t)}$ invested in the domestic savings account. By analogy, such a result holds with respect to each currency. The benchmarked value $\hat{S}^{(\delta_i)}(t) = \frac{S^{(\delta_i)}(t)}{S^{(\delta_*)}(t)}$ of the ith total portfolio then satisfies the SDE

$$d\hat{S}^{(\delta_i)}(t) = \hat{S}^{(\delta_i)}(t) \left(\frac{\pi^{(i)}_{\delta_*}(t) - \pi^{(i)}_{\delta_i}(t)}{1 - \pi^{(i)}_{\delta_*}(t)} \right) \sum_{k=1}^{d} \theta^k_i(t)\, dW^k_t \tag{3.17}$$

for all $t \in [0,T]$ and $i \in \{0,1,\dots,n\}$. Here $\theta_i^k(t)$ denotes the market price for risk for the ith savings account denomination with respect to the kth Wiener process. According to Assumption 3.7 and Assumption 3.2 we have $0 < |\theta_i(t)| < \infty$ a.s., for all $i \in \{0,1,\dots,d\}$ and $t \in [0,T]$.

The *market portfolio* $S^{(\underline{\delta})}(t)$ then equals

$$S^{(\underline{\delta})}(t) = \sum_{i=0}^{n} S^{(\delta_i)}(t) \tag{3.18}$$

at time $t \in [0,T]$, when expressed in terms of the domestic currency. We assume that $S^{(\delta_*)}(0) = S^{(\underline{\delta})}(0)$. This leads to the following SDE for the benchmarked market portfolio

$$d\hat{S}^{(\underline{\delta})}(t) = \sum_{i=0}^{n} \hat{S}^{(\delta_i)}(t) \left(\frac{\pi_{\delta_*}^{(i)}(t) - \pi_{\delta_i}^{(i)}(t)}{1 - \pi_{\delta_*}^{(i)}(t)} \right) \sum_{k=1}^{d} \theta_i^k(t)\, dW_t^k \tag{3.19}$$

for $t \in [0,T]$ by (3.17). Since the benchmarked GOP is constant, $\hat{S}^{(\delta_i)}(t) > 0$ and $|\theta_i(t)| > 0$, we get the following crucial result.

Theorem 3.8 *The market portfolio $S^{(\underline{\delta})}(t)$ equals the GOP $S^{(\delta_*)}(t)$ at all times $t \in [0,T]$ if and only if*

$$\pi_{\delta_i}^{(i)}(t) = \pi_{\delta_*}^{(i)}(t) \tag{3.20}$$

for all $t \in [0,T]$ and $i \in \{0,1,\dots,n\}$.

This leads by (3.14) under the benchmark approach to the following conclusion in relation to the intertemporal CAPM, see Merton (1973).

Corollary 3.9 *For a given portfolio $S^{(\delta)}$ the intertemporal CAPM describes with the systematic risk parameter*

$$\beta_\delta(t) = \frac{\frac{d\langle \ln(S^{(\delta)}), \ln(S^{(\underline{\delta})})\rangle_t}{dt}}{\frac{d\langle \ln(S^{(\underline{\delta})})\rangle_t}{dt}} \tag{3.21}$$

the relationship between the given portfolio and the market portfolio, as long as condition (3.20) *is satisfied for all $t \in [0,T]$.*

This means that the CAPM holds if the market portfolio equals the GOP. There are several lines of argument for justifying condition (3.20). For instance, the monetary authority of the ith currency can be assumed to be able to control the fraction of the total portfolio $S^{(\delta_i)}$, held in the ith savings account $S^{(i)}$ by participants in that market. This is typically achieved by influencing the ith short rate or short term money supply. It would be reasonable to assume that the ith monetary authority aims to maximize the long term net growth of the ith total portfolio $S^{(\delta_i)}$. This leads by Theorem 3.8 and a straightforward calculation of the optimal growth rate for $\frac{S^{(\delta_i)}(t)}{S^{(i)}(t)}$ to the following result.

Corollary 3.10 *If the growth rates of the discounted portfolios $\frac{S^{(\delta_i)}(t)}{S^{(i)}(t)}$, $i \in \{0, 1, \ldots, n\}$, are maximized for all $t \in [0, T]$, then the market portfolio equals the GOP.*

Alternatively, to secure a basis for the CAPM one can naturally argue that the GOP $S^{(\delta_*)}$ and the total portfolios $S^{(\delta_i)}$, $i \in \{0, 1, \ldots, n\}$, are such that the fraction invested in any savings account are small. Then condition (3.20) is approximately satisfied and the market portfolio approximates the GOP. This leads to the conclusion that simply by preferring *more for less* and small total holdings in the savings accounts for the market portfolio and the GOP both portfolios approximate each other.

Now, let us consider the case when the CAPM does **not** hold. We can show that in the case where condition (3.20) is not satisfied, the GOP can still be observed. To be precise, from (3.19) it follows that the GOP is obtained from the market portfolio by subtracting appropriate units of the savings accounts.

Corollary 3.11 *The GOP has the representation*

$$S^{(\delta_*)}(t) = S^{(\underline{\delta})}(t) - \sum_{i=0}^{n} \xi_i(t)\, S^{(i)}(t), \tag{3.22}$$

where

$$\xi_i(t) = \frac{S^{(\delta_i)}(t)}{S^{(i)}(t)} \, \frac{(\pi^{(i)}_{\delta_i}(t) - \pi^{(i)}_{\delta_*}(t))}{(1 - \pi^{(i)}_{\delta_*}(t))} \tag{3.23}$$

for $t \in [0, T]$ *and* $i \in \{0, 1, \ldots, n\}$.

With this explicit formula for the GOP it is possible to compute, for instance, the beta of a portfolio according to (3.14), even if the intertemporal CAPM does not apply because the market portfolio does in this case not equal the GOP. Assumption 3.7 may at present not perfectly hold in reality. However, one can argue that with more and more sophisticated risk management technology the market will come rather closer and closer to a situation that makes this is a very realistic assumption. Corollary 3.11 facilitates a range of risk management applications where the GOP is used as benchmark. Some applications in the area of contingent claim pricing are demonstrated in the next section.

3.4 Risk Neutral and Actuarial Pricing

The direct observability of the GOP by formula (3.22) allows us to generalize in a practical way the well-known *arbitrage pricing theory* (APT) introduced by Ross (1976) and further developed by Harrison & Kreps (1979), Harrison & Pliska

(1981) and many others. Under the benchmark approach one can use the GOP $S^{(\delta_*)}$ as numeraire along the lines of Long (1990). Note that the Radon-Nikodym derivative process $\Lambda_Q = \{\Lambda_Q(t),\, t \in [0,T]\}$ for the candidate risk neutral measure Q can be obtained as inverse of the discounted GOP

$$\Lambda_Q(t) = \frac{dP}{dQ}\Big|_{\mathcal{A}_t} = \frac{\bar{S}^{(\delta_*)}(0)}{\bar{S}^{(\delta_*)}(t)} \tag{3.24}$$

for $t \in [0,T]$, see Karatzas & Shreve (1998). For $\Lambda_Q(t)$ we obtain the SDE

$$d\Lambda_Q(t) = -\Lambda_Q(t) \sum_{k=1}^{d} \theta^k(t)\, dW_t^k \tag{3.25}$$

for $t \in [0,T]$ with $\Lambda_Q(0) = 1$ by the Itô formula and (2.24). This demonstrates that Λ_Q is an $(\underline{\mathcal{A}}, P)$-local martingale. Furthermore, by (2.26) it follows that $\bar{S}^{(\delta)}(t)\Lambda_Q(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} = \hat{S}^{(\delta)}(t)$ is an $(\underline{\mathcal{A}}, P)$-local martingale for any portfolio $S^{(\delta)}$. We emphasize that under the benchmark approach this does *not* mean that $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-martingale. By application of the Girsanov theorem, see Karatzas & Shreve (1998), one obtains the following result.

Corollary 3.12 *If an equivalent risk neutral martingale measure Q exists and $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-martingale, then the* risk neutral pricing formula

$$\begin{aligned} S^{(\delta)}(t) &= S^{(\delta_*)}(t)\, E\left(\hat{S}^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{\Lambda_Q(t)}{\Lambda_Q(s)}\, \frac{S^{(0)}(t)}{S^{(0)}(s)}\, S^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) \\ &= E_Q\left(\frac{S^{(0)}(t)}{S^{(0)}(s)}\, S^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) \end{aligned} \tag{3.26}$$

holds for all $t \in [0,T]$ and $s \in [t,T]$. Here E_Q denotes expectation under the risk neutral measure Q.

According to Corollary 3.12, $\hat{S}^{(\delta)}$ needs to be an $(\underline{\mathcal{A}}, P)$-martingale for $\bar{S}^{(\delta)}$ to be an $(\underline{\mathcal{A}}, Q)$-martingale, such that the risk neutral pricing formula holds. Note that even if an equivalent martingale measure exists, then *not all* benchmarked portfolios are automatically $(\underline{\mathcal{A}}, P)$-martingales and not all discounted portfolios are $(\underline{\mathcal{A}}, P)$-martingales. This point is sometimes overlooked in the literature. In the above sense one recovers the risk neutral pricing methodology of the APT when assuming the existence of a risk neutral equivalent martingale measure Q. Relations similar to (3.26) also appear in the literature in connection with pricing kernels, state price densities, deflators, discount factors and numeraire portfolios, see, for instance, Long (1990), Constatinides (1992), Duffie (2001) and Cochrane (2001).

Since a benchmark model does not require the existence of an equivalent risk neutral martingale measure it provides a more general modeling framework than the standard risk neutral setup. In particular, not all benchmarked securities need to be true martingales under the benchmark approach. As we will see, this is important for realistic modeling.

An indication for the need to go beyond the APT is given by the fact that by (3.24) the Radon-Nikodym derivative Λ_Q for a market equals the ratio of the savings account over the GOP. In the long run the GOP is expected by investors to outperform the savings account. This means that the trajectory of the Radon-Nikodym derivative should decrease systematically. Empirical evidence supports such systematic decline for all major currency denominations, when using the MSCI world stock accumulation index as proxy for the GOP. Therefore, it is not likely that Λ_Q is in reality well modeled as a true $(\underline{\mathcal{A}}, P)$-martingale, thereby contradicting the standard APT assumptions. Note that a decreasing graph for Λ_Q is still consistent with it being a nonnegative, strict $(\underline{\mathcal{A}}, P)$-local martingale and hence a strict supermartingale, see Karatzas & Shreve (1991). We will come back to this point towards the end of the paper.

For derivative pricing in a benchmark model, where no equivalent risk neutral martingale measure exists, the *fair pricing concept* has been proposed in Platen (2002).

Definition 3.13 *A benchmarked portfolio process* (2.25) *is called* fair *if it forms an* $(\underline{\mathcal{A}}, P)$*-martingale.*

In practice, it appears that fair pricing is appropriate for determining the competitive price of a contingent claim.

Definition 3.14 *The fair price* $U_{H_\tau}(t)$ *at time* $t \in [0, \tau]$ *of an* $\mathcal{A}_\tau$*-measurable contingent claim* H_τ*, payable at a stopping time* τ*, is defined by the* fair pricing formula

$$U_{H_\tau}(t) = E\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(\tau)} \, H_\tau \,\Big|\, \mathcal{A}_t \right). \tag{3.27}$$

Note that fair prices are uniquely determined even in incomplete markets. Under the existence of a minimal equivalent martingale measure, see Föllmer & Schweizer (1991), fair prices have been shown to correspond to local risk minimizing prices, see Platen (2004c). Corollary 3.11 makes fair pricing via (3.27) very practicable since one can model and calibrate the GOP obtained by (3.22) from the market portfolio. This enables us to calculate the real world expectations in (3.27). It is clear from (3.27), (3.26) and (3.24) that fair pricing generalizes risk neutral pricing.

For the practically important case where a contingent claim is independent of the GOP, one obtains the following result by the fair pricing formula (3.27).

Corollary 3.15 *For a contingent claim H_T that is independent of the GOP value $S^{(\delta_*)}(T)$, the fair price $U_{H_T}(t)$ satisfies the* actuarial pricing formula

$$\begin{aligned} U_{H_T}(t) &= E\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(T)} \,\Big|\, \mathcal{A}_t\right) E\left(H_T \,\big|\, \mathcal{A}_t\right) \\ &= P(t,T)\, E\left(H_T \,\big|\, \mathcal{A}_t\right), \end{aligned} \tag{3.28}$$

where $P(t,T)$ denotes the fair price at time $t \in [0,T]$ of a zero coupon bond with maturity date T.

The formula (3.28) has been widely used in insurance and other areas of risk management, see, for instance, Bühlmann (1995) and Gerber (1990). One may regard (3.28) as a generalized actuarial pricing formula that is still valid when interest rates are stochastic.

4 Expectation of the Discounted GOP

It is important to have an idea about the typical dynamics of the GOP and future security prices. The SDE (2.24) for the GOP reveals a close link between its drift and diffusion coefficients. More precisely, the risk premium of the GOP equals the square of its volatility. To see this, let us rewrite (2.24) in the discounted form

$$d\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(t)\, |\theta(t)|\, (|\theta(t)|\, dt + dW_t), \tag{4.1}$$

where

$$dW_t = \frac{1}{|\theta(t)|} \sum_{k=1}^{d} \theta^k(t)\, dW_t^k \tag{4.2}$$

is the stochastic differential of a standard Wiener process W. This reveals a structural relationship between the drift and diffusion coefficients.

To benefit from this relationship let us reparameterize the GOP dynamics. The *discounted GOP drift*

$$\alpha(t) = \bar{S}^{(\delta_*)}(t)\, |\theta(t)|^2 \tag{4.3}$$

is the average change per unit of time of the discounted GOP. Using the parametrization (4.3), we get the total market price for risk in the form

$$|\theta(t)| = \sqrt{\frac{\alpha(t)}{\bar{S}^{(\delta_*)}(t)}}. \tag{4.4}$$

By substituting (4.3) and (4.4) into (4.1) we obtain the following SDE for the discounted GOP

$$d\bar{S}^{(\delta_*)}(t) = \alpha(t)\, dt + \sqrt{\bar{S}^{(\delta_*)}(t)\, \alpha(t)}\, dW_t \tag{4.5}$$

for $t \in [0,T]$. This is a time transformed squared Bessel process of dimension four, see Revuz & Yor (1999). Its *transformed time* $\varphi(t)$ at time t is given by the expression

$$\varphi(t) = \varphi(0) + \int_0^t \alpha(s)\, ds \tag{4.6}$$

with $\varphi(0) \geq 0$ as the possibly hidden random initial value.

Since one obtains the SDE

$$d\sqrt{\bar{S}^{(\delta_*)}(t)} = \frac{3\,\alpha(t)}{8\sqrt{\bar{S}^{(\delta_*)}(t)}}\, dt + \frac{1}{2}\sqrt{\alpha(t)}\, dW_t \tag{4.7}$$

from (4.5), by the Itô formula, the increase of the transformed time $\varphi(t)$ can be directly observed as

$$\varphi(t) - \varphi(0) = 4\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t \tag{4.8}$$

for $t \in [0,T]$. This emphasizes the fact that we are able to observe the drift of the discounted GOP. Note that it is easy to demonstrate that even under the simplest dynamics the estimation of a drift parameter for the market portfolio needs several hundred years of data to guarantee any reasonable level of confidence. Under the benchmark approach the above results resolve the problem of identifying the drift of the market portfolio.

For the analysis that follows, let us decompose the discounted GOP value at time $t \in [0,T]$ as

$$\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(0) + \varphi(t) - \varphi(0) + M(t). \tag{4.9}$$

Here $M = \{M(t),\, t \in [0,T]\}$ is the $(\underline{\mathcal{A}}, P)$-local martingale

$$M(t) = \int_0^t \bar{S}^{(\delta_*)}(s)\, |\theta(s)|\, dW_s \tag{4.10}$$

for $t \in [0,T]$. The quantity $\bar{S}^{(\delta_*)}(t)$ in (4.9) consists of a part $M(t)$, which reflects the trading uncertainty of the discounted GOP and a part $\varphi(t) - \varphi(0)$ that expresses the increase of its *underlying value*. This value can be interpreted as accumulated discounted wealth that has been generated in the economy. We have shown by (4.6) and (4.8) that it takes naturally the form of a transformed time. Remarkably, the dynamics of the discounted GOP turn out to be those of a very particular diffusion process when its underlying economic value is used as time scale.

The above relationships lead directly to the following result, which exploits equations (4.9) and (4.6) and a realistic martingale assumption for M.

Corollary 4.1 *If the local martingale M in (4.10) is a true $(\underline{\mathcal{A}}, P)$-martingale, then the expected change of the discounted GOP value over a given period equals the expected change of its underlying value. That is,*

$$E\left(\bar{S}^{(\delta_*)}(s) - \bar{S}^{(\delta_*)}(t) \,\middle|\, \mathcal{A}_t\right) = E\left(\varphi(s) - \varphi(t) \,\middle|\, \mathcal{A}_t\right) \tag{4.11}$$

for all $t \in [0, T]$ *and* $s \in [t, T]$.

By Corollary 4.1 and Theorem 3.8 it follows that if the present value of the discounted GOP is interpreted as its underlying value, then the expected future underlying value equals the expected future value of the discounted GOP. We emphasize that we have still not made any major assumptions about the particular dynamics of the GOP.

5 Dynamics of the GOP

In Platen (2004b) relation (4.11) has been illustrated for the US market. It appears in reality that the underlying value of the discounted GOP, which can be observed via (4.8), evolves relatively smoothly as a monotone increasing function of time. This matches the fact that only a limited quantity of discounted wealth is on average generated worldwide per unit of time which increases smoothly the underlying value of the discounted GOP. Based on this observation we make the following *smoothness assumption*.

Assumption 5.1 *The underlying value of the discounted GOP is twice differentiable with respect to time.*

Without loss of generality, the discounted GOP drift can then be expressed as

$$\alpha(t) = \alpha_0 \exp\left\{\int_0^t \eta(s)\, ds\right\} \tag{5.1}$$

for $t \in [0, T]$. The two parameters in (5.1) are a nonnegative, potentially random, constant $\alpha_0 > 0$ and an adapted process $\eta = \{\eta(t),\, t \in [0, T]\}$, called the *net growth rate*. This expression takes the typical growth nature of the market portfolio into account. According to (4.4) the parametrization (5.1) allows us to study the dynamics of the *normalized GOP*

$$Y(t) = \frac{\bar{S}^{(\delta_*)}(t)}{\alpha(t)} = \frac{1}{|\theta(t)|^2} \tag{5.2}$$

for $t \in [0, T]$. By application of the Itô formula and using (4.4), (4.11) and (4.5), we obtain the SDE

$$dY(t) = (1 - \eta(t)\, Y(t))\, dt + \sqrt{Y(t)}\, dW_t \tag{5.3}$$

for $t \in [0, T]$ with $Y(0) = \frac{\bar{S}^{(\delta_*)}(0)}{\alpha_0}$. It follows that the normalized GOP is a *square root process* with the inverse of the net growth rate $\frac{1}{\eta(t)}$ as reference level for its linear mean-reverting drift. The net growth rate $\eta(t)$ is then the speed of adjustment parameter for the mean-reversion. Note that besides initial values, the net growth rate is the only parameter process needed to characterize the dynamics of the normalized GOP and its stochastic volatility. Therefore, we end up with a parsimonious, realistic model for the GOP dynamics, namely

$$S^{(\delta_*)}(t) = Y(t)\,\alpha(t)\,S^{(0)}(t) \tag{5.4}$$

for $t \in [0, T]$. It only remains to specify $S^{(\delta_*)}(0)$, α_0 and the net growth rate process η as well as the short rate process r. The net growth rate for the world economy and its world stock portfolio, when denominated in units of a US-Dollar savings account, has been estimated for the entire last century in Dimson, Marsh & Staunton (2002) to be $\eta \approx 4.8\%$. This matches well the observed average squared volatility of the market portfolio of $E(|\theta(t)|^2) = E(\frac{1}{Y(\cdot)}) = \frac{1}{\eta} \approx 20.8\%$ observed over the last century. Under the benchmark approach one does not face any *risk premium puzzle*. The observed market portfolio risk premium matches perfectly the net growth of the world economy, which should be expected.

The resulting model for the GOP is the *minimal market model* (MMM), described in Platen (2001), which has been studied, for instance, in Platen (2002, 2004b) and Heath & Platen (2004). It is important to note that under the MMM the Radon-Nikodym derivative for the candidate risk neutral measure equals the inverse of a squared Bessel process of dimension four, which is a nonnegative, strict local martingale and thus a strict supermartingale, see Karatzas & Shreve (1991). This potentially explains the systematic decline in typically observed Radon-Nikodym derivatives, that is, benchmarked savings accounts. Obviously, under the MMM the APT is not applicable. However, the fair pricing formula (3.27) makes perfect sense for the competitive pricing of derivatives and can be directly applied by using the explicitly known transition density of the squared Bessel process of dimension four, see Revuz & Yor (1999).

The above analysis raises the question of whether the distribution of log-returns of the GOP, as above predicted, are actually observed. If the MMM is an accurate description of reality, then log-returns of the GOP, based on long time series data, should be Student t distributed with four degrees of freedom. This follows because the squared volatility of the GOP $|\theta(t)|^2 = \frac{1}{Y(t)}$ has a stationary inverse gamma density with four degrees of freedom, when assuming a constant net growth rate. For long time series of GOP log-returns, this inverse gamma density acts as mixing density for the resulting normal-mixture distribution, yielding the above mentioned Student t distribution.

This theoretical feature of the MMM is rather clear and testable. Importantly, it has already been documented in the literature as an empirical stylized fact for log-returns of large stock market indices. In an extensive Baysian estimation within a wide class of Pearson distributions, Markowitz & Usmen (1996) found that the Student t distribution with about 4.3 degrees of freedom matches well the daily S&P500 log-return data from 1962 until 1983. Independently, in Hurst & Platen (1997) it was found by maximum likelihood estimation within the rich class of symmetric generalized hyperbolic distributions that, not only for the S&P500 but also for most other regional stock market indices, daily log-returns for the period from 1982 until 1996 are likely to be Student t distributed with about four degrees of freedom. Another recent study by Breymann, Fergusson & Platen (2004) confirms, for the daily log-returns of the world stock market portfolio in 34 different currency denominations for the period from 1990 until 2003, that in all cases the Student t distribution provides the best fit in the class of symmetric generalized hyperbolic distributions. Furthermore, for 22 of the 34 currency denominations the Student t hypothesis cannot be rejected at the 99% confidence level. The average estimated number of degrees of freedom in this study is 3.94, which is very close to the theoretical value of 4.00 predicted by the MMM. Further empirical evidence from the derivatives area strongly supports the MMM, see Heath & Platen (2004). Forthcoming work will extend the above benchmark approach to models with event driven jumps and general semimartingale dynamics.

A Appendix

Proof of Theorem 3.3

To identify a discounted optimal portfolio we maximize the drift (2.16) subject to the constraint (3.1), locally in time, according to Definition 3.1. For this purpose we use the Lagrange multiplier λ and consider the function

$$G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\delta}, \dots, \psi^d_{\delta}) = \sum_{k=1}^{d} \psi^k_{\delta}\, \theta^k + \lambda \left((\gamma_{\tilde{\delta}})^2 - \sum_{k=1}^{d} \left(\psi^k_{\delta}\right)^2 \right). \tag{A.1}$$

by suppressing time dependence. For $\psi^1_{\delta}, \psi^2_{\delta}, \dots, \psi^d_{\delta}$ to provide a maximum for $G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\delta}, \dots, \psi^d_{\delta})$ it is necessary that the first-order conditions

$$\frac{\partial G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\delta}, \dots, \psi^d_{\delta})}{\partial \psi^k_{\delta}} = \theta^k - 2\,\lambda\, \psi^k_{\delta} = 0 \tag{A.2}$$

are satisfied for all $k \in \{1, 2, \dots, d\}$. Consequently, an optimal portfolio $S^{(\tilde{\delta})}$, which maximizes the drift, must have

$$\psi^k_{\tilde{\delta}} = \frac{\theta^k}{2\,\lambda} \tag{A.3}$$

for all $k \in \{1, 2, \dots, d\}$. We can now use the constraint (3.1) together with (2.19) and (3.3) to obtain from (A.3) the relation

$$(\gamma_{\tilde{\delta}})^2 = \sum_{k=1}^{d} \left(\psi^k_{\tilde{\delta}}\right)^2 = \left(\frac{|\theta|}{2\,\lambda}\right)^2. \tag{A.4}$$

If one admits the (excluded) case $|\theta(t)| = 0$ for some $t \in [0,T]$, it then follows from (A.4) that

$$\gamma_{\tilde{\delta}} = 0 \tag{A.5}$$

and the savings account is the only optimal portfolio, that is $S^{(\tilde{\delta})}(t) = S^{(\tilde{\delta})}(0)S^{(0)}(t)$. For the permitted case $|\theta(t)| > 0$, one obtains the equation

$$\psi_{\tilde{\delta}}^k(t) = \frac{|\gamma_{\tilde{\delta}}(t)|}{|\theta(t)|}\,\theta^k(t) \tag{A.6}$$

for $t \in [0,T]$ and $k \in \{1,2,\ldots,d\}$, from (A.3) and (A.4). Now, it follows from (2.11), (2.13), Assumption 2.1, (2.17), (A.6) and (2.23) that

$$\begin{aligned}
\pi_{\tilde{\delta}}^{(0)}(t) &= 1 - \sum_{j=1}^{d} \pi_{\tilde{\delta}}^{(j)}(t) \\
&= 1 - \frac{1}{\bar{S}^{(\tilde{\delta})}(t)} \sum_{k=1}^{d} \psi_{\tilde{\delta}}^k(t) \sum_{j=1}^{d} b^{-1\,j,k}(t) \\
&= 1 - \frac{|\gamma_{\tilde{\delta}}(t)|}{\bar{S}^{(\tilde{\delta})}(t)\,|\theta(t)|}\left(1 - \pi_{\delta_*}^{(0)}(t)\right)
\end{aligned} \tag{A.7}$$

for $t \in [0,T]$. For the special case where $\pi_{\delta_*}^{(0)}(t) = 1$, one obtains $\pi_{\tilde{\delta}}^{(0)}(t) = 1$ and in this case the optimal portfolio is once again the savings account.

In the case $\pi_{\delta_*}^{(0)}(t) \neq 1$ it follows by (A.7) that

$$|\gamma_{\tilde{\delta}}(t)| = \frac{1 - \pi_{\tilde{\delta}}^{(0)}(t)}{1 - \pi_{\delta_*}^{(0)}(t)}\,\bar{S}^{(\tilde{\delta})}(t)\,|\theta(t)| \tag{A.8}$$

for all $t \in [0,T]$. From (2.17), (A.6) and (A.8) we obtain the SDE (3.5). By using (2.16), (A.6) and (A.8) we then derive the jth optimal fraction

$$\pi_{\tilde{\delta}}^{(j)}(t) = \frac{\left(1 - \pi_{\tilde{\delta}}^{(0)}(t)\right)}{\left(1 - \pi_{\delta_*}^{(0)}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\, b^{-1\,j,k}(t) \tag{A.9}$$

for $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$. This yields the formula (3.6) by (2.23) and proves the theorem. $\square$

Conclusion

This paper demonstrates that the growth optimal portfolio (GOP) plays a major theoretical and practical role in various areas of finance, including portfolio optimization, derivative pricing and risk management, by acting as a benchmark.

It assumes that investors always prefer more for less. Under the additional assumptions that monetary authorities aim to maximize the long term growth of the total portfolio of their market participants or that the values of the savings accounts are small in the market portfolio and the GOP, it is shown that the market portfolio equals or approximates the GOP, thus establishing the CAPM. Even without such assumptions the paper identifies the GOP as a combination of the market portfolio and the savings accounts. The Markowitz efficient frontier and Sharpe ratio can also be derived naturally.

The discounted GOP can be realistically modeled as a time transformed squared Bessel process of dimension four. The transformed time can be interpreted as its underlying value. The increase in expected discounted GOP value is shown to equal that of its expected discounted underlying value. A particular dynamics of the normalized GOP is derived. It is that of a square root process of dimension four, which appears to provide realistic log-returns for the market portfolio.

For the pricing of contingent claims the GOP is nominated as numeraire for fair pricing, with expectations to be taken under the real world probability measure. In fact, for the minimal market model described here, no equivalent risk neutral martingale measure exists.

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