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A CLASS OF JUMP-DIFFUSION BOND PRICING MODELS WITHIN THE HJM FRAMEWORK

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Abstract. This paper considers a class of term structure models that is a parameterisation of the Shirakawa (1991) extension of the Heath, Jarrow & Morton (1992) model to the case of jump-diffusions. We consider specific forward rate volatility structures that incorporate state dependent Wiener volatility functions and time dependent Poisson volatility functions. Within this framework, we discuss the Markovianisation issue, and obtain the corresponding affine term structure of interest rates. As a result we are able to obtain a broad tractable class of jump-diffusion term structure models. We relate our approach to the existing class of jump-diffusion term structure models whose starting point is a jump-diffusion process for the spot rate. In particular we obtain natural jump-diffusion versions of the Hull & White (1990, 1994) one-factor and two-factor models and the Ritchken & Sankarasubramanian (1995) model within the HJM framework. We also give some numerical simulations to gauge the effect of the jump-component on yield curves and the implications of various volatility specifications for the spot rate distribution.

Keywords: Markovian HJM model, jump-diffusions, state dependent volatility.

1. Introduction

This paper considers a multi-factor jump-diffusion model of the term structure of interest rates under a specific volatility structure. The forward rate dynamics are driven by multi-dimensional Wiener and Poisson processes and the volatility structure is such that the Wiener volatility functions are state dependent whilst the Poisson volatility functions are time dependent. Working within the Heath, Jarrow & Morton (1992)(hereafter HJM) framework we obtain bond prices in an arbitrage free environment, even though the spot rate dynamics are non-Markovian. Imposing restrictions on the volatility structure, a Markovian multi-factor model is obtained. It turns out that the state variables of

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this model, can be expressed as functions of a finite number of benchmark forward rates or yields. The model that we thereby develop provides a fairly broad tractable class of jump-diffusion term structure models that would be suitable for both calibration and econometric estimation.

The literature on the incorporation of jump components into term structure models is not a very extensive one. For literature on jump-diffusion interest rate models that usually have as their starting point the spot rate dynamics, we would cite in particular Das (2002) and Chacko & Das (2002). Within the HJM term structure modelling framework, where the focus is on the forward rate dynamics, Shirakawa (1991) was the first attempt to incorporate discontinuous forward rate dynamics. Subsequently a very general framework for term structure modelling under marked point processes was developed by Björk, Kabanov & Runggaldier (1997). More recent work on jump-diffusion versions of the HJM framework include Glasserman & Kou (2003), who consider the market model, and Das (2000) who treats a discrete time version of the HJM model.

Here we use the Shirakawa (1991) framework, which assumes only a finite number of possible jump sizes and that there exists a sufficient number of traded bonds to hedge away all of the jump risks, in this way guaranteeing market completeness. We derive a Markovian representation of the stochastic dynamic system driving bond prices by considering certain specifications of the volatility functions of the instantaneous forward rate. Essentially we extend to the jump diffusion case the approach of the Markovianisation of HJM models developed by a number of authors. Early papers on the Markovianisation of HJM models under Wiener diffusions include Cheyette (1992), Carverhill (1994), Ritchken & Sankarasubramanian (1995) and Bhar & Chiarella (1997), where the conditions on the volatility structure for the spot rate process to be Markovian are examined for the one factor HJM models. Inui & Kijima (1998) and de Jong & Santa-Clara (1999) extend these conditions to multi factor HJM models. Duffie & Kan (1996) developed a square root volatility model. Further, Björk & Landèn (2002), Björk & Svensson (2001) and Chiarella & Kwon (2001b),(2003) generalise the above results in various directions by assuming more general forward rate volatility specifications. We extend some of these results to the Markovianisation of the jump-diffusion version of the HJM class of models. Using ideas from state space theory, Björk & Gombani (1999) allow forward rates to be driven by a multi dimensional Wiener process as well as by a marked point process and give the necessary and sufficient conditions on a deterministic volatility structure, for the existence of finite dimensional realizations. They also showed that the state variables constitute a minimal set of benchmark forward rates. Our model may be viewed as providing an extension to the framework of Björk & Gombani (1999) since we incorporate level dependent volatility structures with the Wiener process noise.

This paper makes two main contributions: Under the generalised HJM jump-diffusion framework and a specific formulation of level and time dependent volatility specifications, Markovian spot rate and bond price dynamics are obtained. In addition, finite dimensional affine realisations of the term structure in terms of forward rates and yields are obtained. Within this framework we develop some particular classes of jump-diffusion term structure models. In particular we develop what we believe is the natural extension of the Hull & White (1990), (1994) class of models and the Ritchken & Sankarasubramanian (1995) class of models to the jump-diffusion case.

The structure of this paper is as follows. In Section 2 we review the Shirakawa jump-diffusion term structure framework focusing on an economic interpretation of the underlying hedging argument. In Section 3 we assume a specific volatility structure, and obtain the corresponding Markovian representation of the spot rate and bond price dynamics in terms of a finite number of state variables that are driven by Markovian diffusion and jump processes. In Section 4, we express these state variables as finite dimensional affine realisations in terms of economic quantities observed in the market, such as forward rates and yields. In Section 5 we consider the case in which the Poisson volatilities are also state dependent and give some insight into the reason why a Markovian representation may not be possible in this case. We do however suggest a way in which an approximate Markovian representation may be developed, given the magnitude of the jump volatilities suggested by empirical studies. In Section 6 we develop some specific models, in particular a class of multi factor Hull & White (1990), (1994) and Ritchken & Sankarasubramanian (1995) jump-diffusion models and the equivalent forward rate curves. We also carry out a number of numerical simulations to gauge the implications of the various volatility specifications that generate these models. In Section 7 we conclude and discuss future directions for research.

2. The Model

In this section we review some fundamental relationships of the bond market and the main features of the Shirakawa (1991) model. Our exposition is in a less abstract setting than that of Shirakawa, as we wish to emphasize more the economic intuition of the underlying hedging argument.

Using $f(t,T)$ to denote the instantaneous forward interest rate at time t for instantaneous borrowing at time $T(\geqslant t)$, we define as $P(t,T)$, the price at time t of a default-free discount zero-coupon bond with maturity T, i.e.,

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \tag{1}$$

so that $P(T,T) = 1$.

Generalising the basic assumption of Shirakawa (1991), on the filtered probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$, [1] the stochastic differential equation for the instantaneous forward rate $f(t,T)$ driven by both Wiener and Poisson risk is given by

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n}\left(\sigma_i(t,T)dW_i(t) + \sum_{j=1}^{m_i}\beta_{ij}(t,T)[dQ_{ij}(t) - \lambda_{ij}dt]\right), \qquad (2)$$

where $\alpha : [0,T] \to R_+$ is the drift function, $W_i(t)$ are standard Wiener processes $(i = 1,2,\dots,n)$, $Q_{ij}(t)$ is a Poisson process with constant intensity λ_{ij} $(j = 1,2,\dots,m_i)$. The Poisson process Q_{ij} models the arrival time of the jump events. Recall that, by definition

$$dQ_{ij}(t) = \begin{cases} 1, & \text{if a jump occurs in the time interval } (t,\, t+dt) \text{ (with probability } \lambda_{ij}dt),\\ 0, & \text{otherwise (with probability } 1-\lambda_{ij}dt), \end{cases}$$

and $\mathbb{E}[dQ_{ij}(t) \mid \mathcal{F}_{t-}] = \lambda_{ij}dt$, $\mathbb{E}[dQ^2_{ij}(t) \mid \mathcal{F}_{t-}] = \lambda_{ij}dt$. At the Poisson jump times, the jump size is equal to $\beta_{ij}(t,T)$. Under these assumptions, the jump feature is modelled by a multivariate point process, allowing for a finite number of jumps. [2]

The volatility specifications allow for $\sigma_i : [0,T] \to R_+$, the volatility functions associated with the Wiener noise processes, which are positive valued, to be state dependent. Here we consider a specification of the general form

$$\sigma_i(t,T) = \sigma_i(t,T,\bar{f}(t)), \qquad \text{for } i = 1,\dots,n, \qquad (3)$$

where σ_i are well-defined functions that depend on time, maturity and $\bar{f}(t)$ is a vector of path dependent variables such as the instantaneous spot rate and/or instantaneous forward rates of different fixed maturities. By omitting this level dependence, we would obtain the special case of time deterministic Wiener volatility functions. The $\beta_{ij} : [0,T] \to R_+$, the volatility functions associated with the Poisson noise processes are assumed to be only time and maturity dependent. These volatility specifications generalise the Shirakawa framework by allowing the Wiener noise and Poisson noise to have separate volatility structures. Such a framework is appropriate if one believes that these different types of shocks impact differently across the forward curve. The empirical study of jump-diffusion interest rate models by Chiarella & Tô (2003), suggests that this may in fact be the case in some markets.

In stochastic integral equation form, equation (2) may be written

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{n}\left(\int_0^t \sigma_i(s,T,\bar{f}(s))dW_i(s) + \sum_{j=1}^{m_i}\int_0^t \beta_{ij}(s,T)[dQ_{ij}(s) - \lambda_{ij}ds]\right). \qquad (4)$$

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[1] In more formal notation we assume that $(\Omega,\, \mathcal{F},\, (\mathcal{F})_{0\le t\le T},\, \mathbb{P})$ is the probability space equipped with the natural filtration of a vector of standard Wiener processes $W_i(t)$ $(i = 1,2,\dots,n)$ and the Poisson processes $Q_{ij}(t)$ with intensity λ_{ij} $(j = 1,2,\dots,m_i)$, indexed on the time interval $[0,T]$.

[2] See Runggaldier (2003) for a good survey of jump-diffusion models.

Setting $T = t$ in equation (4), the stochastic integral equation for the instantaneous spot rate is given by

$$r(t) \equiv f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \sum_{i=1}^{n}\left(\int_0^t \sigma_i(s,t,\bar{f}(s))dW_i(s) + \sum_{j=1}^{m_i}\int_0^t \beta_{ij}(s,t)[dQ_{ij}(s) - \lambda_{ij}ds]\right). \tag{5}$$

With application of the jump-diffusion version of Ito's lemma, the dynamics for the bond price driven by Wiener and Poisson risk, may be expressed as[3]

$$\frac{dP(t,T)}{P(t^-,T)} = [r(t) + H(t,T,\bar{f}(t))]dt - \sum_{i=1}^{n}\left(\zeta_i(t,T,\bar{f}(t))dW_i(t) + \sum_{j=1}^{m_i}(1 - e^{-\xi_{ij}(t,T)})dQ_{ij}(t)\right), \tag{6}$$

where

$$\zeta_i(t,T,\bar{f}(t)) = \int_t^T \sigma_i(t,u,\bar{f}(t))du, \tag{7}$$

$$\xi_{ij}(t,T) = \int_t^T \beta_{ij}(t,u)du, \tag{8}$$

$$H(t,T,\bar{f}(t)) = -\int_t^T \alpha(t,u)du + \sum_{i=1}^{n}\left(\frac{1}{2}\zeta_i^2(t,T,\bar{f}(t)) + \sum_{j=1}^{m_i}\lambda_{ij}\xi_{ij}(t,T)\right). \tag{9}$$

In this economy we have $n + m_1 + m_2 + \ldots + m_n$ sources of risk, n due to the Wiener processes $W_i(t)$ $(i = 1, \cdots, n)$, and $m_1 + m_2 + \ldots + m_n$ due to the Poisson processes Q_{ij} $(j = 1, \cdots, m_i)$. Using the classical hedging portfolio argument of Vasicek (1977) that carries over to interest rate models the original Black-Scholes hedging approach, we thus place bonds of $n + \sum_{i=1}^{n} m_i + 1$ maturities in the hedging portfolio. [4] By taking an appropriate position in the $n + \sum_{i=1}^{n} m_i + 1$ bonds it is possible to eliminate both Wiener and Poisson risks and after some manipulations [5] to derive the forward rate drift restriction that extends the HJM forward rate drift restriction to now incorporate the

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[3] See Proposition 2.2 of Björk et al. (1997).

[4] The subtle issue in the hedging argument concerns whether or not the set of bonds in the hedging portfolio remains fixed over time. The Shirakawa analysis only established the existence of a set of bonds that would possibly change over time. Björk et al. (1997) established that the set of hedging bonds can in fact remain fixed over time.

[5] See Appendix 1 for full details of the hedging portfolio argument in the current context. The reader may refer to Björk et al. (1997), for the most general approach to deriving the arbitrage free dynamics for interest rate models under marked point processes.

jump feature, namely,

$$\begin{aligned}\alpha(t,T) =& \sum_{i=1}^{n} \sigma_i(t,T,\bar{f}(t))(-\phi_i(t) + \zeta_i(t,T,\bar{f}(t))) \\ & - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \beta_{ij}(t,T)(\psi_{ij}(t)e^{-\xi_{ij}(t,T)} - \lambda_{ij}).\end{aligned} \tag{10}$$

In equation (10) the ϕ_i are the market prices of diffusion risk associated with the Wiener process sources of uncertainty W_i, whilst the ψ_{ij} are the market prices of jump risk associated with the Poisson process sources of uncertainty Q_{ij}.

2.1. The Risk Neutral Dynamics under a General Volatility Specification.

By an application of Girsanov's theorem (Bremaud (1981)), for every fixed finite time horizon T, we can obtain a unique equivalent probability measure $\widetilde{\mathbb{P}}_t$ [6], under which the $\widetilde{W}_i(t) = -\int_0^t \phi_i(s)ds + W_i(t)$ are standard Wiener processes (for $i = 1, \dots, n$) and the Q_{ij} are Poisson processes (for $i = 1, \dots, n$ and $j = 1, \dots, m_i$) associated with intensity $\psi_{ij}(t)$ such that $\widetilde{W}_i$ and Q_{ij} are mutually independent.

Substitution of (10) into (9) reduces the stochastic differential equation for the bond price in the now arbitrage free economy to

$$\frac{dP(t,T)}{P(t^-,T)} = r(t)dt - \sum_{i=1}^{n}\left(\zeta_i(t,T,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{j=1}^{m_i}(1-e^{-\xi_{ij}(t,T)})[dQ_{ij}(t) - \psi_{ij}(t)dt]\right). \tag{11}$$

In addition, by obtaining the dynamics of the bond price measured in units of the money market account, the bond price can be expressed as

$$P(t,T) = \widetilde{\mathbb{E}}_t\left[\frac{B(t)}{B(T)} \mid \mathcal{F}_t\right] = \widetilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right], \tag{12}$$

where $\widetilde{\mathbb{E}}_t$ is expectation (given information at time t) with respect to the equivalent probability (risk neutral) measure $\widetilde{\mathbb{P}}_t$ and $B(t)$ is the accumulated money account

$$B(t) = \exp\left(\int_0^t r(s)ds\right).$$

Consequently, by substitution of the drift restriction (10) for $\alpha(s,t)$ into the equation (5), we obtain the dynamics of the spot interest rate $r(t)$ under the risk neutral measure

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[6] The Wiener processes $\widetilde{W}_i(t)$ $(i = 1, \cdots, n)$ and the Poisson processes $Q_{ij}(t)$ $(j = 1, \cdots, m_i)$ with intensity $\boldsymbol{\Psi}_i$ generate the $\widetilde{\mathbb{P}}_t$-augmentation of the filtration $\mathcal{F}_t$.

$\widetilde{\mathbb{P}}_t$, in the form

$$r(t) = f(0,t) + \sum_{i=1}^{n} \left(\int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds + \sum_{j=1}^{m_i} \int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1 - e^{-\xi_{ij}(s,t)}]ds \right)$$
$$+ \sum_{i=1}^{n} \left(\int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s) + \sum_{j=1}^{m_i} \int_0^t \beta_{ij}(s,t)[dQ_{ij}(s) - \psi_{ij}(s)ds] \right). \tag{13}$$

Under a general specification for $\sigma_i(s,t,\bar{f}(s))$ and $\beta_{ij}(s,t)$ the dynamics for $r(t)$ implied by (13) are non-Markovian due to the path dependency of some or all of the integral terms on the right-hand side of (13).

3. A Specific Volatility Structure

In order to generate specific term structure models and to be able to obtain Markovian representations of the spot rate dynamics (13), we shall consider more specific volatility structures. To make the discussion explicit, we shall also assume that $\bar{f}(t) = (r(t), f(t,T_1), f(t,T_2), \ldots, f(t,T_k))^\top$ where T_1, T_2, …, T_k are a set of fixed maturities.

Assumption 3.1. *For $i = 1, \ldots, n$, the state dependent Wiener volatility structure (3) is of the form*

$$\sigma_i(s,t,\bar{f}(s)) = \sigma_{0i}(s,\bar{f}(s))e^{-\int_s^t \kappa_{\sigma i}(u)du}, \tag{14}$$

and for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$, the time dependent Poisson volatility functions are of the form

$$\beta_{ij}(s,t) = \beta_{0ij}(s)e^{-\int_s^t \kappa_{\beta ij}(u)du}, \tag{15}$$

where $\kappa_{\sigma i}(t)$, $\kappa_{\beta ij}(t)$ and $\beta_{0ij}(t)$ are time deterministic functions and $\sigma_{0i}(t,\bar{f}(t))$ are time and state dependent functions.

We recall that in the no jump situation, the functional form (14) for the forward rate volatility derives, within the HJM framework, the extended Vasicek model of Hull-White (one-factor model) (see Baxter & Rennie (1996), Chiarella & El-Hassan (1996)) and the Hull-White two-factor and multi-factor models (see Chiarella & Kwon (2001a)). We shall now show that this case gives a Markovian representation of (13) that may be viewed as a generalisation of the Markovian multi-factor models to the jump-diffusion case.

The crucial property of the volatility functions (14) and (15) is that their derivatives with respect to the second argument (maturity) are given by

$$\frac{\partial}{\partial t}\sigma_i(s,t,\bar{f}(s)) = -\kappa_{\sigma i}(t)\,\sigma_i(s,t,\bar{f}(s)), \tag{16}$$

for $i = 1, \ldots, n$, and

$$\frac{\partial}{\partial t}\beta_{ij}(s,t) = -\kappa_{\beta ij}(t)\,\beta_{ij}(s,t), \tag{17}$$

for $i = 1, \ldots, n$, and $j = 1, \ldots, m_i$. This is a natural consequence of the functional forms (14) and (15), that allows the separation of the time dependent component from the maturity dependent component. As pointed out by Chiarella & Kwon (2003), this is in fact the key property of the volatility functions that leads to the Markovianisation of the model.

In many of the common models, the instantaneous spot rate itself is included in the set of state variables. Thus, in the following proposition, we derive the spot rate dynamics in both integral and differential form in terms of a number of stochastic factors and the spot rate.

Proposition 3.1. *Let $\sigma_i(s,t,\bar{f}(s))$ and $\beta_{ij}(s,t)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$, satisfy Assumption 3.1. Then the dynamics for the spot rate can be expressed as*

$$r(t) = f(0,t) + \sum_{i=1}^{n} \mathcal{D}_{\sigma i}(t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mathcal{D}_{\beta ij}(t), \tag{18}$$

in stochastic integral equation form, or,

$$\begin{aligned} dr(t) = {} & \left[D(t) + \sum_{i=1}^{n} \mathcal{E}_{\sigma i}(t) - \sum_{i=2}^{n} \hat{\kappa}_{\sigma i}(t)\mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \hat{\kappa}_{\beta ij}(t)\mathcal{D}_{\beta ij}(t)) - k_{\sigma 1}(t) r(t) \right] dt \\ & + \sum_{i=1}^{n} \left(\sigma_{0i}(t,\bar{f}(t)) d\widetilde{W}_i(t) + \sum_{j=1}^{m_i} \beta_{0ij}(t)\left[dQ_{ij}(t) - \psi_{ij}(t)dt\right] \right), \end{aligned} \tag{19}$$

in stochastic differential equation form, where

$$D(t) = \kappa_{\sigma 1}(t) f(0,t) + \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mathcal{E}_{\beta ij}(t), \tag{20}$$

$$\hat{\kappa}_{\sigma i}(t) = \kappa_{\sigma i}(t) - \kappa_{\sigma 1}(t), \tag{21}$$

$$\hat{\kappa}_{\beta ij}(t) = \kappa_{\beta ij}(t) - \kappa_{\sigma 1}(t), \tag{22}$$

and

$$\mathcal{E}_{\sigma i}(t) = \int_0^t \sigma_i^2(s,t,\bar{f}(s)) ds, \tag{23}$$

$$\mathcal{E}_{\beta ij}(t) = \int_0^t \psi_{ij}(s)\beta_{ij}^2(s,t) e^{-\xi_{ij}(s,t)} ds, \tag{24}$$

$$\mathcal{D}_{\sigma i}(t) = \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s)) ds + \int_0^t \sigma_i(s,t,\bar{f}(s)) d\widetilde{W}_i(s), \tag{25}$$

$$\mathcal{D}_{\beta ij}(t) = \int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1 - e^{-\xi_{ij}(s,t)}] ds + \int_0^t \beta_{ij}(s,t)(dQ_{ij}(s) - \psi_{ij}(s)ds). \tag{26}$$

Proof. Substitution of the stochastic quantities (25) and (26) into (13) derives (18). For the stochastic differential representation, take the stochastic differential of (18) and make use of properties (16) and (17), to obtain the stochastic differential equation for the instantaneous spot rate under the risk neutral measure, as,

$$\begin{aligned} dr(t) = \Bigg[& \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n} \left(\frac{\partial}{\partial t} \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds + \sum_{j=1}^{m_i} \frac{\partial}{\partial t}\Big(\int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds\Big) \right) \\ & - \sum_{i=1}^{n} \left(\kappa_{\sigma i}(t) \int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s) + \sum_{j=1}^{m_i} \kappa_{\beta ij}(t) \int_0^t \beta_{ij}(s,t)(dQ_{ij}(s) - \psi_{ij}(s)ds) \right) \Bigg] dt \qquad (27) \\ & + \sum_{i=1}^{n} \left(\sigma_{0i}(t,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{j=1}^{m_i} \beta_{0ij}(t)[dQ_{ij}(t) - \psi_{ij}(t)dt] \right), \end{aligned}$$

which, by using the results of Appendix 2, may be expressed as

$$\begin{aligned} dr(t) = \Bigg[& \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i^2(s,t,\bar{f}(s))ds - \sum_{i=1}^{n} \kappa_{\sigma i}(t) \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \left(\int_0^t \psi_{ij}(s)\beta_{ij}^2(s,t)e^{-\xi_{ij}(s,t)}ds - \kappa_{\beta ij}(t) \int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds \right) \\ & - \sum_{i=1}^{n} \kappa_{\sigma i}(t) \int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s) - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \kappa_{\beta ij}(t) \int_0^t \beta_{ij}(s,t)[dQ_{ij}(s) - \psi_{ij}(s)ds] \Bigg] dt \\ & + \sum_{i=1}^{n} \sigma_{0i}(t,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \beta_{0ij}(t)[dQ_{ij}(t) - \psi_{ij}(t)dt]. \qquad (28) \end{aligned}$$

Relation (18) allows one of the stochastic factors to be expressed in terms of the spot rate $r(t)$ and the remaining stochastic factors and here we take

$$\mathcal{D}_{\sigma 1}(t) = r(t) - f(0,t) - \sum_{i=2}^{n} \mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mathcal{D}_{\beta ij}(t). \qquad (29)$$

Use of expressions (23), (24), (25) and (26), and substitution of (29) into the stochastic differential equation (28) leads to the dynamics (19). □

The $\mathcal{E}_{\beta ij}(t)$ are deterministic time functions, whereas the $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$ are stochastic quantities depending on the path history up to time t. These stochastic quantities satisfy stochastic differential equations with drifts and diffusion terms that depend on themselves and the state variables $\bar{f}(t)$, as the next Proposition shows.

Proposition 3.2. *Given the forward rate volatility specifications of Assumption 3.1 and assuming that the market prices of jump risk are non-stochastic, the stochastic quantities* $\mathcal{E}_{\sigma i}(t)$*,* $\mathcal{D}_{\sigma i}(t)$ *and* $\mathcal{D}_{\beta ij}(t)$ *satisfy the stochastic differential equations, for* $i = 1, \cdots, n$, *and* $j = 1, 2, \cdots, m_i$,

$$d\mathcal{E}_{\sigma i}(t) = [\sigma_{0i}^2(t, \bar{f}(t)) - 2\kappa_{\sigma i}(t)\mathcal{E}_{\sigma i}(t)]dt, \tag{30}$$

$$d\mathcal{D}_{\sigma i}(t) = [\mathcal{E}_{\sigma i}(t) - \kappa_{\sigma i}(t)\mathcal{D}_{\sigma i}(t)]dt + \sigma_{0i}(t, \bar{f}(t))d\widetilde{W}_i(t), \tag{31}$$

and

$$d\mathcal{D}_{\beta ij}(t) = [\mathcal{E}_{\beta ij}(t) - \kappa_{\beta ij}(t)\mathcal{D}_{\beta ij}(t)]dt + \beta_{0ij}(t)\left[dQ_{ij}(t) - \psi_{ij}(t)dt\right]. \tag{32}$$

Proof. Taking the differential of the quantities (23), (24) and (25), the stated results follow. ☐

Section 4 shows how the $\bar{f}(t)$ can be expressed in terms of the stochastic quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$, $\mathcal{D}_{\beta ij}(t)$ (or vice versa). Thus, the instantaneous spot rate dynamics (19) are Markovian under the forward rate volatility specifications (14) and (15), since the stochastic quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$, $\mathcal{D}_{\beta ij}(t)$ display Markovian dynamics.[7]

We note that the drift term in (19) is a linear combination of $2n + m_1 + \ldots + m_n - 1$ stochastic variables, determined by (30), (31) and (32) and the spot rate. In the following section, an exponentially affine term structure of interest rates in terms of the these stochastic variables is obtained.

3.1. **Affine Term Structure of Interest Rates.** We obtain the multi-factor bond price formula in terms of the stochastic variables $\mathcal{E}_{\sigma i}(t), \mathcal{D}_{\sigma i}(t)$, and $\mathcal{D}_{\beta ij}(t)$, by using the Inui & Kijima (1998) approach. This consists of a direct substitution of the risk neutral forward rate dynamics and the volatility specifications (14) and (15) into the fundamental relationship between bond prices and forward rates in equation (1) and manipulating the resulting integrals.

Proposition 3.3. *Under Proposition 3.1 the bond price assumes the multi-factor exponential affine form given by*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ \mathcal{M}(t,T) - \frac{1}{2}\sum_{i=1}^{n} \mathcal{N}_{\sigma_i}^2(t,T)\mathcal{E}_{\sigma_i}(t) - \sum_{i=2}^{n}(\mathcal{N}_{\sigma i}(t,T) - \mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\sigma_i}(t) \right. \tag{33}$$

$$\left. - \sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathcal{N}_{\beta ij}(t,T) - \mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\beta ij}(t) - \mathcal{N}_{\sigma_1}(t,T)r(t) \right\},$$

---

[7] As stated in Proposition 3.2, the Markovianisation obtained depends on the assumption that the market prices of jump risk are non-stochastic. If one in fact wished to allow these to be stochastic (say for empirical studies) then one could still obtain a Markovian representation if the ψ_{ij} were assumed to follow some Markovian system of stochastic differential equations.

where,

$$\begin{aligned}\mathcal{M}(t,T) =&\mathcal{N}_{\sigma_1}(t,T)f(0,t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i}\int_0^t\int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1-e^{-\xi_{ij}(s,y)}]dyds \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{N}_{\beta ij}(t,T)\int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds,\end{aligned} \tag{34}$$

and

$$\mathcal{N}_x(t,T) \equiv \int_t^T e^{-\int_t^y \kappa_x(u)du}dy, \quad x \in \{\sigma_i, \beta_{ij}\}. \tag{35}$$

Proof. See Appendix 3 for details. □

The bond price formula (33) displays a finite dimensional affine structure in terms of a number of state variables ($n_s = 2n + m_1 + \ldots + m_n$ in our case) that are driven by diffusion processes and jump processes. In particular, the state variables $\mathcal{E}_{\sigma i}(t)$ are driven by jump-diffusion processes due to the dependency on the $\bar{f}(t)$, the state variables $\mathcal{D}_{\sigma i}(t)$ are driven by pure diffusion processes, whereas the state variables $\mathcal{D}_{\beta ij}(t)$ are driven by pure jump processes. These stochastic factors (namely $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$) have no easy economic interpretation. It would be very convenient and more intuitive for applications if we could express these stochastic factors in terms of economic quantities observed in the market, like forward rates, whose dynamics would be driven by combined jump-diffusion processes. In the next section, we will show that these stochastic factors can indeed be expressed in terms of benchmark forward rates with dynamics driven by jump-diffusion processes.

4. Finite Dimensional Affine Realisations in Terms of Forward Rates

We employ the basic ideas from Chiarella & Kwon (2003) and Björk & Svensson (2001) who show that, in a Markovian HJM framework with dynamics driven by diffusion processes, the state variables can be expressed as affine functions of a finite number of forward rates and yields. We introduce the jump component into their modelling framework and we assume state dependent Wiener volatility functions and time deterministic Poisson volatility functions. It seems that the inclusion of jumps makes it very hard or probably impossible to derive Markovianisation results under more general volatility specifications that allow the jump volatility functions to be stochastic. However in Section 5 we indicate how an approximate Markovianisation may be found in this case.

We use the exponential affine term structure of interest rates (33), where the bond price is a function of the instantaneous spot rate $r(t)$, and the stochastic quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\beta ij}(t)$, and $\mathcal{D}_{\sigma i}(t)$. We can then express the instantaneous forward rate as (from equation (1))

$$f(t,T)-f(0,T)+\frac{\partial\mathcal{M}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}r(t)=\sum_{i=1}^{n}\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}\mathcal{N}_{\sigma i}(t,T)\,\mathcal{E}_{\sigma i}(t) \tag{36}$$
$$+\sum_{i=2}^{n}\left(\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\mathcal{D}_{\sigma i}(t)+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(\frac{\partial\mathcal{N}_{\beta ij}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\mathcal{D}_{\beta ij}(t),$$

where the $\mathcal{N}_x(t,T)$ ($x\in\{\sigma_i,\beta_{ij}\}$) are defined in equation (35).

We now take a number of fixed forward rate maturities equal to the number of state variables remaining after excluding the instantaneous spot rate $r(t)$. We express these state variables in terms of forward rates with different fixed maturities. Thus, $\bar{n}_s(=n_s-1)$ forward rates of different fixed maturities T_h are required.

Proposition 4.1. *The forward rate of any maturity can be expressed in terms of the $\bar{n}_s$ benchmark forward rates and the instantaneous spot rate $r(t)$*[8] *as*

$$f(t,T)=f(0,T)+\bar{\mathcal{Q}}(t,T)+\sum_{h=1}^{\bar{n}_s}\bar{\mathcal{R}}_h(t,T)f(t,T_h)+\bar{\mathcal{S}}(t,T)r(t), \tag{37}$$

where, for $l=n+q-1$ and $k=2n+i-1$,

$$\bar{\mathcal{Q}}(t,T)=-\frac{\partial\mathcal{M}(t,T)}{\partial T}+\sum_{h=1}^{n_s}\left(\frac{\partial\mathcal{M}(t,T_h)}{\partial T_h}-f(0,T_h)\right)\left[\sum_{i=1}^{n}\bar{\varpi}_{ih}\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}\mathcal{N}_{\sigma i}(t,T)\right. \tag{38}$$
$$\left.+\sum_{q=2}^{n}\bar{\varpi}_{lh}\left(\frac{\partial\mathcal{N}_{\sigma q}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\bar{\varpi}_{kh}\left(\frac{\partial\mathcal{N}_{\beta ij}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\right],$$

$$\bar{\mathcal{R}}_h(t,T)=\sum_{i=1}^{n}\bar{\varpi}_{ih}\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}\mathcal{N}_{\sigma i}(t,T)+\sum_{q=2}^{n}\bar{\varpi}_{lh}\left(\frac{\partial\mathcal{N}_{\sigma_q}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\bar{\varpi}_{kh}\left(\frac{\partial\mathcal{N}_{\beta ij}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right), \tag{39}$$

and

$$\bar{\mathcal{S}}(t,T)=\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}-\sum_{h=1}^{n_s}\frac{\partial\mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h}\left(\sum_{i=1}^{n}\bar{\varpi}_{ih}\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}\mathcal{N}_{\sigma i}(t,T)\right. \tag{40}$$
$$\left.+\sum_{q=2}^{n}\bar{\varpi}_{lh}\left(\frac{\partial\mathcal{N}_{\sigma_q}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\bar{\varpi}_{kh}\left(\frac{\partial\mathcal{N}_{\beta ij}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\right),$$

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[8] Only up to time $t=\min_h T_h$. By reparameterising in terms of fixed time-to-maturity forward rates $f(t,t+T_h)$, we may allow for any $t\in\mathbb{R}_+$, a representation which would actually be more amenable to empirical estimation.

and $\bar{\varpi}_{\ell\hbar}$ *denotes the* $\ell\hbar^{th}$ *element of the matrix* $\bar{\mathbb{O}}^{-1}$*, the inverse of the square matrix* $\bar{\mathbb{O}}(t)$*, defined such that for* $i = 1, 2, \ldots, n$*,* $q = 2, \ldots, n$*, and* $j = 1, 2, \ldots, m_i$*,*

$$\bar{\mathbb{O}}(t) = \begin{bmatrix} \bar{\varphi}_1(t) & \bar{\varphi}_2(t) & \bar{\varphi}_3(t) \end{bmatrix},$$

where, $\bar{\varphi}_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma i}(t,T_h)}{\partial T_h} \mathcal{N}_{\sigma i}(t, T_h)\right]$ *is a* $\bar{n}_s \times n$ *matrix,*
$\bar{\varphi}_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma_q}(t,T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h}\right]$*, is a* $\bar{n}_s \times (n-1)$ *matrix, and*
$\bar{\varphi}_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_{ij}}(t,T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h}\right]$*, is a* $\bar{n}_s \times (m_1 + \ldots + m_n)$ *matrix.*
Assume that $\bar{\mathbb{O}}(t)$ *is invertible for all* $t \in \{t; t = \min_h T_h\}$*.*

Proof. Considering equation (36) for the maturities $T_1, T_2, \cdots, T_{\bar{n}_s}$ we obtain the system

$$\begin{bmatrix} f(t,T_1) - f(0,T_1) + \frac{\partial \mathcal{M}(t,T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_1)}{\partial T_1} r(t) \\ f(t,T_2) - f(0,T_2) + \frac{\partial \mathcal{M}(t,T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_2)}{\partial T_2} r(t) \\ \vdots \\ f(t,T_{\bar{n}_s}) - f(0,T_{\bar{n}_s}) + \frac{\partial \mathcal{M}(t,T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} r(t) \end{bmatrix} = \bar{\mathbb{O}}(t) \times \begin{bmatrix} \mathcal{E}_{\sigma 1}(t) \\ \vdots \\ \mathcal{E}_{\sigma n}(t) \\ \mathcal{D}_{\sigma 2}(t) \\ \vdots \\ \mathcal{D}_{\sigma n}(t) \\ \mathcal{D}_{\beta 11}(t) \\ \vdots \\ \mathcal{D}_{\beta n m_n}(t) \end{bmatrix}.$$

By inverting the matrix $\bar{\mathbb{O}}(t)$, the state variables $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$ are expressed in terms of forward rates of $\bar{n}_s$ distinct maturities as

$$\begin{bmatrix} \mathcal{E}_{\sigma 1}(t) \\ \vdots \\ \mathcal{E}_{\sigma n}(t) \\ \mathcal{D}_{\sigma 2}(t) \\ \vdots \\ \mathcal{D}_{\sigma n}(t) \\ \mathcal{D}_{\beta 11}(t) \\ \vdots \\ \mathcal{D}_{\beta n m_n}(t) \end{bmatrix} = \bar{\mathbb{O}}^{-1}(t) \times \begin{bmatrix} f(t,T_1) - f(0,T_1) + \frac{\partial \mathcal{M}(t,T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_1)}{\partial T_1} r(t) \\ f(t,T_2) - f(0,T_2) + \frac{\partial \mathcal{M}(t,T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_2}(t,T_1)}{\partial T_2} r(t) \\ \vdots \\ f(t,T_{\bar{n}_s}) - f(0,T_{\bar{n}_s}) + \frac{\partial \mathcal{M}(t,T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_{\bar{n}_s})}{\partial T_{\bar{n}_s}} r(t) \end{bmatrix}. \tag{41}$$

By substitution of expressions (41) for the state variables into the forward rate formula (36), one obtains (37) which expresses the forward rate of any maturity in terms of the $\bar{n}_s$ forward rates and the instantaneous spot rate $r(t)$. □

The following proposition displays the corresponding bond price formula.

Proposition 4.2. *The zero-coupon bond prices in terms of the* $\bar{n}_s$ *benchmark forward rates and the instantaneous spot rate* $r(t)$ *is given by the exponential affine form*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ \bar{\mathcal{Q}}^P(t,T) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^P(t,T) f(t,T_h) + \bar{\mathcal{S}}^P(t,T) r(t) \right\}, \tag{42}$$

where

$$\bar{\mathcal{Q}}^P(t,T) = -\int_t^T \bar{\mathcal{Q}}(t,s)ds, \;\; \bar{\mathcal{R}}_h^P(t,T) = -\int_t^T \bar{\mathcal{R}}_h(t,s)ds, \;\; \textit{and} \;\; \bar{\mathcal{S}}^P(t,T) = -\int_t^T \bar{\mathcal{S}}(t,s)ds.$$

Proof. By substitution of (37) into the fundamental relationship (1). □

The risk neutral dynamics for each benchmark forward rate $f(t,T_h)$ are given by

$$\begin{aligned} df(t,T_h) = &\sum_{i=1}^{n} \left(\sigma_i(t,T_h,\bar{f}(t))\zeta_i(t,T_h,\bar{f}(t)) - \sum_{j=1}^{m_i} \psi_{ij}(t)\beta_{ij}(t,T_h)e^{-\xi_{ij}(t,T_h)} \right) dt \\ &+ \sum_{i=1}^{n} \left(\sigma_i(t,T_h,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{j=1}^{m_i} \beta_{ij}(t,T_h)dQ_{ij}(t) \right), \end{aligned} \tag{43}$$

which are driven by both Wiener and Poisson processes. By using the system (41), the dynamics (19) of the spot rate $r(t)$ can be expressed in terms of the state vector (set $k = \bar{n}_s$)

$$\bar{f}(t) = (r(t), f(t,T_1), f(t,T_2), \ldots, f(t,T_{\bar{n}_s}))^\top,$$

as

$$\begin{aligned} dr(t) = &\left[D^f(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) f(t,T_h) - \bar{\mathcal{S}}^f(t) r(t) \right] dt \\ &+ \sum_{i=1}^{n} \left(\sigma_{0i}(t,\bar{f}(t))d\widetilde{W}_i(t) + \sum_{j=1}^{m_i} \beta_{0ij}(t)\left[dQ_{ij}(t) - \psi_{ij}(t)dt\right] \right), \end{aligned} \tag{44}$$

where, for $l = n + q - 1$ and $k = 2n + i - 1$,

$$\bar{\mathcal{R}}_h^f(t) = \sum_{i=1}^{n} \varpi_{ih} - \sum_{q=2}^{n} \hat{\kappa}_{\sigma q}(t)\varpi_{lh} - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \hat{\kappa}_{\beta ij}(t)\varpi_{kh}, \tag{45}$$

$$D^f(t) = D(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) \left(-f(0,T_h) + \frac{\partial \mathcal{M}(t,T_h)}{\partial T_h} \right), \tag{46}$$

and

$$\bar{\mathcal{S}}^f(t) = k_{\sigma 1}(t) + \sum_{h=1}^{\bar{n}_s} \bar{\mathcal{R}}_h^f(t) \frac{\partial \mathcal{N}_{\sigma_h}(t,T_1)}{\partial T_h}. \tag{47}$$

Thus a closed Markovian system for all the elements of the state vector has been obtained.

The advantage in obtaining the bond pricing formula (42) and the forward rate formula (37) is that they allow us to transfer the market information of a certain set of distinct forward rate curves and the instantaneous spot rate (in addition to the initial forward curves included in the terms $\frac{\partial \mathcal{M}(t,T_h)}{\partial T_h}$) into the bond price and the forward rate curve respectively.

Moreover, the yield to maturity which is defined as $R(t,T) \equiv -\ln P(t,T)/(T-t)$, becomes in terms of the forward rate

$$R(t,T) = -\frac{\int_t^T f(t,u)du}{T-t}, \tag{48}$$

and using expression (37) we could express the yield to maturity in terms of the same set of forward rates mentioned above. Applying similar invertibility arguments [9] we may express the forward curve in terms of a set of bonds or yields to maturity. Given that yields of different maturities are observed in the market, this model setup would prove to be very suitable for parameter estimation and model calibration.

Remark 4.1. Whether or not one includes the spot rate in the set of the state variables depends on the particular application. By doing so in the present paper allows us to relate the class of models developed here to the traditional models (e.g. Hull-White, Ritchken-Sankarasubramanian) that do take the instantaneous spot rate as the underlying state variable, as we shall show in Section 6. However the general framework developed does not tie us to such a choice, we may use any convenient set of interest rates as the state variables. Appendix 5 sets out the results of this section for the case when $r(t)$ is not one of the state variables.

5. State Dependent Poisson Volatility Structure

In previous sections, we considered the case in which only the Wiener volatility functions depend on a number of state variables. The case where both Wiener and Poisson volatilities are state dependent, poses some problems. We now indicate why, in the case that both Wiener and Poisson volatilities are state dependent, it seems impossible to obtain Markovian representation of the spot rate dynamics (13) and so we propose an approximate solution to the problem.

Assume that the Wiener volatilities follow the structure (14) for $i = 1, \dots, n$, and that for $j = 1, \dots, m_i$ the Poisson volatilities are expressed as

$$\beta_{ij}(s,t,\bar{f}(s)) = \beta_{0i}(s,\bar{f}(s))e^{-\int_s^t \kappa_{\beta i}(u)du}. \tag{49}$$

The derivative of the volatility functions (49) with respect to the second argument (maturity) still satisfies (17), so the dynamics of the spot rate (19) still follow. Given the state dependent volatility specifications (14) and (49) (assume that the market prices

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[9] See Corollary 2 of Chiarella & Kwon (2003).

of jump risk are non-stochastic), all the quantities $\mathcal{E}_{\sigma i}(t)$, $\mathcal{E}_{\beta ij}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$ are now stochastic.

The problem with this case arises from the process $\mathcal{E}_{\beta ij}(t)$. Recall that

$$\mathcal{E}_{\beta ij}(t) = \int_0^t \psi_{ij}(s)\beta_{ij}^2(s,t)e^{-\xi_{ij}(s,t)}ds, \tag{50}$$

and introduce for $n = 2, 3, \ldots$, the quantities

$$\mathcal{E}_{\beta ij}^{(n)}(t,\bar{f}) = \int_0^t \psi_{ij}(s)\beta_{ij}^n(s,t,\bar{f}(s))e^{-\xi_{ij}(s,t,\bar{f}(s))}ds.$$

We seek to obtain the stochastic differential equation for $\mathcal{E}_{\beta_{ij}}(t)$, which from (50) turns out to be

$$d\mathcal{E}_{\beta_{ij}} = (\psi_{ij}(t)\beta_{0ij}(t,\bar{f}(t)) - \kappa_{\beta_{ij}}(t)\mathcal{E}_{\beta_{ij}}(t,\bar{f}(t)) - \mathcal{E}_{\beta_{ij}}^{(2)}(t,\bar{f}(t)))dt.$$

The process $\mathcal{E}_{\beta ij}^{(2)}(t,\bar{f})$ in turn satisfies the stochastic differential equation

$$d\mathcal{E}_{\beta ij}^{(2)}(t,\bar{f}(t)) = \left(\psi_{ij}(t)\beta_{0ij}^2(t,\bar{f}(t)) - 2\kappa_{\beta ij}(t)\mathcal{E}_{\beta ij}^{(2)}(t,\bar{f}(t)) - \mathcal{E}_{\beta ij}^{(3)}(t,\bar{f}(t))\right)dt.$$

Thus, we are dealing with an infinite sequence of processes $\mathcal{E}_{\beta ij}^{(n)}(t,\bar{f})$, since for $n = 2, 3, \ldots$ we find that

$$d\mathcal{E}_{\beta ij}^{(n)}(t,\bar{f}(t)) = \left(\psi_{ij}(t)\beta_{0ij}^n(t,\bar{f}(t)) - n\kappa_{\beta ij}(t)\mathcal{E}_{\beta ij}^{(n)}(t,\bar{f}(t)) - \mathcal{E}_{\beta ij}^{(n+1)}(t,\bar{f}(t))\right)dt.$$

Therefore, when both Wiener and Poisson volatilities are state dependent, it seems that we cannot obtain a Markovian representation at least not by an approach similar to the one that led to the spot rate dynamics equation (19). To "close" this sequence will require some approximation. In practice, if would be the case that $\beta^n(t) \simeq 0$, for sufficiently large n (see the magnitudes of the jump component obtained by Chiarella & Tô (2003)) so in this way it is possible to achieve an approximate Markovianisation and have an approximate affine term structure.

6. Model Applications

In this section, we illustrate examples of jump-diffusion models that can be generated with the framework of this paper. We also relate our general model to known models and extend these models to incorporate jumps components. In particular, we consider examples of the Hull-White type models (hereafter HW) and the Ritchken & Sankarasubramanian (1995) class of models (hereafter RS) extended to the multi-factor jump-diffusion case and investigate their distributional profiles.

6.1. **Hull & White Type models.** One of the characteristic features of HW type models is that the underlying dynamics involve a mean reverting process for the instantaneous spot rate of interest as this is the underlying state variable[10] in this class of models and the volatility function is only time deterministic. So, to obtain HW type models under jump diffusions, we restrict our general model of state dependent Wiener volatilities to time deterministic volatilities. Thus the volatility specifications of Assumption 3.1 are simplified to

Assumption 6.1. *For $i = 1, \dots, n$, the time dependent Wiener volatility structure (3) is of the form*

$$\sigma_i(s,t) = \sigma_{0i}(s)e^{-\int_s^t \kappa_{\sigma i}(u)du}, \tag{51}$$

and for $i = 1, \dots, n$ and $j = 1, \dots, m_i$, the time dependent Poisson volatility functions are of the form (15) where $\kappa_{\sigma i}(t)$ and $\sigma_{0i}(t)$ are time deterministic functions.

This assumption will reduce the number of the stochastic state variables. To see this note that the state variables $\mathcal{E}_{\sigma i}(t)$ determined by (23) are now non stochastic as they assume the time deterministic form

$$\mathcal{E}_{\sigma i}(t) = \int_0^t \sigma_i^2(s,t)ds. \tag{52}$$

In addition, the stochastic state variables $\mathcal{D}_{\sigma i}(t)$ defined by (25) are of the form

$$\mathcal{D}_{\sigma i}(t) = \int_0^t \sigma_i(s,t)\zeta_i(s,t)ds + \int_0^t \sigma_i(s,t)d\widetilde{W}_i(s). \tag{53}$$

The dynamics (19), under Assumption 6.1, therefore become

$$dr(t) = [\Lambda(t) + \mathrm{S}(t) - k_{\sigma 1}(t)r(t)]\,dt + \sum_{i=1}^{n}\left(\sigma_{0i}(t)d\widetilde{W}_i(t) + \sum_{j=1}^{m_i}\beta_{0ij}(t)(dQ_{ij}(t) - \psi_{ij}(t)dt)\right), \tag{54}$$

where we set

$$\Lambda(t) = \frac{\partial}{\partial t}f(0,t) + \kappa_{\sigma 1}(t)f(0,t) + \sum_{i=1}^{n}\mathcal{E}_{\sigma i}(t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{E}_{\beta ij}(t), \tag{55}$$

the deterministic part of the drift term, and

$$\mathrm{S}(t) = -\sum_{i=2}^{n}[\kappa_{\sigma i}(t) - \kappa_{\sigma 1}(t)]\mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i}[\kappa_{\beta ij}(t) - \kappa_{\sigma 1}(t)]\mathcal{D}_{\beta ij}(t), \tag{56}$$

the stochastic part of the drift term. The spot rate dynamics (54) generalise the structure of the Hull & White (1994) two-factor model where the spot rate was driven only by one Wiener process. We recall that the basic idea of Hull & White (1994) was to

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[10] Of course, there is no reason why one could not define a class of HW or RS models where say some other rate e.g. the 6-month LIBOR rate serves as the underlying state variable. However it has become traditional for a wide class of models to use the instantaneous spot rate as the underlying state variable.

add to the drift term a stochastic factor driven by another Wiener process. In (54) we see that the drift contains the deterministic term $\Lambda(t)$, the stochastic term $\mathrm{S}(t)$ consisting of $n + m_1 + \ldots + m_n - 1$ stochastic factors and the mean reverting term for the instantaneous spot rate $r(t)$. It is also worth pointing out that as it is defined within the HJM framework, the spot rate process (54) is automatically calibrated to the currently observed yield curve through the $\Lambda(t)$ term. For these reasons we suggest that this representation is the natural extension of the HW model to the multi-factor jump-diffusion situation.

In Appendix 4,[11] by following similar arguments as in Section 4, the corresponding finite dimensional affine realisations, for the HW models, in terms of forward rates of $\hat{n}_s = n + m_1 + \ldots + m_n - 1$ different fixed maturities are derived.

In the following examples, the initial forward rate curve considered has the functional form $f(0,t) = (a_0 + a_1 t + a_2 t^2)\, e^{-vt}$ with parameters being estimated as $a_0 = 0.033287$, $a_1 = 0.014488$, $a_2 = -0.000117$, and $v = 0.0925$, which result in an upward sloping forward curve. The data used for interpolation are the US zero yields from the US zero curves up to 10 years including the spot US zero curve on July 20, 2001.

We will now consider the case of the one Wiener-two Poisson HW type of model.

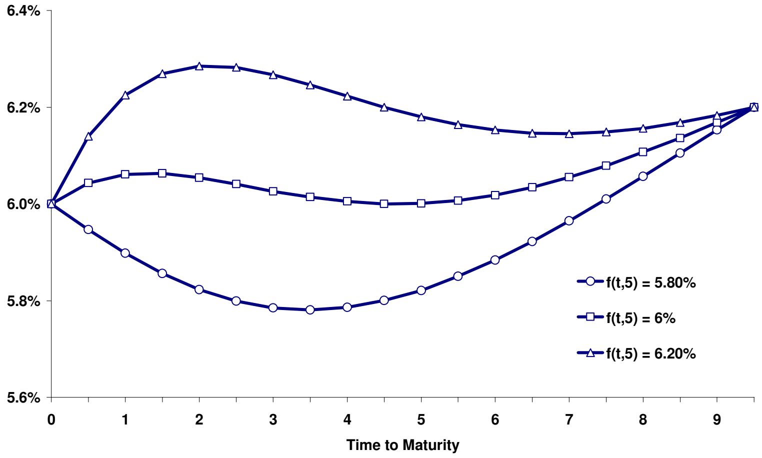

FIGURE 1. Forward rate curves at $t = 6$ months, for the one Wiener and two Poisson HW type model when $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\psi_1 = 1$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$ and $\psi_2 = 1.5$. The corresponding curves represent the $f(t,T)$ when $f(t,10) = 6.2\%$ and $f(t,5)$ takes the values of 5.8%, 6% and 6.2%.

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[11] The current case when the Wiener volatility function is independent of the state variables yields slightly different looking representations because the $\mathcal{E}_{\sigma i}(t)$ quantities became deterministic. This means that the elements $\bar{\varphi}_1(t)$ of the matrix $\bar{\mathbb{O}}(t)$ in Proposition 4.1 does not appear.

Thus, for $n = 1$ and $m_1 = 2$, consider the volatility functions

$$\sigma(t,T) = \sigma_0(t)e^{-\int_t^T \kappa_\sigma(u)du}, \tag{57}$$

and

$$\beta_i(t,T) = \beta_{0i}(t)e^{-\int_t^T \kappa_{\beta_i}(u)du}, \quad \text{with } i = 1,2. \tag{58}$$

The number of the stochastic state variables, in this case, is $3(= n + \sum_{i=1}^{m_i})$ including the $r(t)$, and using the results of Appendix 4 we may express these state variables in terms of 2 benchmark forward rates and the spot rate. In turn, the forward rate $f(t,T)$ and the bond prices $P(t,T)$ can be expressed in terms of the spot rate $r(t)$ and these benchmark forward rates. The state variables used here are the spot rate, the 5-year forward rate $f(t,5)$ and the 10-year forward rate $f(t,10)$.

Further the parameter values of the Wiener volatility are $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$. The parameter specifications for the jump volatility terms are $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$. The jump intensities are $\psi_1 = 1$ and $\psi_2 = 1.5$ respectively. The forward rate curves shown in Figure 1 are at 6 months time when it is assumed that $r = 6\%$, the 10-year forward rate $f(t,10)$ is 6.2% and the 5-year forward rate takes the values of 5.8%, 6% and 6.2%. For these volatility specifications, the spot rate volatility is 3.5% and the 10-year forward rate volatility is 16.5% of the spot rate volatility.

6.2. **Ritchken & Sankarasubramanian Type Models.** The RS class of models[12] considered in this example is characterised by state dependent Wiener volatility functions, so that

$$\sigma(t,T) = \sigma_1(T-t)\sigma_2(\bar{f}(t))e^{-\int_t^T \kappa_\sigma(u)du}. \tag{59}$$

We consider the case that $n = 1$ and $m_1 = 2$. The number of the state variable, in this case, is $4(= 2n + \sum_{i=1}^{m_i})$ including $r(t)$. Using the results from Proposition 4.1 and Proposition 4.2 we may express these state variables in terms of 3 benchmark forward rates and the spot rate. Thus we may set $\bar{f}(t) = (r(t), f(t,T_1), f(t,T_2), f(t,T_3))^\top$. In turn, the forward rate $f(t,T)$ and the bond prices $P(t,T)$ can be expressed in terms of the spot rate $r(t)$ and these benchmark forward rates. The state variables used now are the spot rate, the 2.5-year forward rate, the 5-year forward rate and the 10-year forward rate.

The initial forward rate curve and the volatility specifications considered here are the same as in Section 6.1. The forward rate curves shown in Figure 2 are in 6 months time

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[12] In the original Ritchken & Sankarasubramanian (1995) paper, the forward rate volatility functions considered are of the form $\sigma(r)e^{-\int_t^T k_\sigma(u)du}$. Subsequently, Ritchken & Chuang (1999) consider the forward rate volatility functions $(a_0+a_1(T-t))e^{-k(T-t)}$. The form (59) generalises this type of volatility structure.

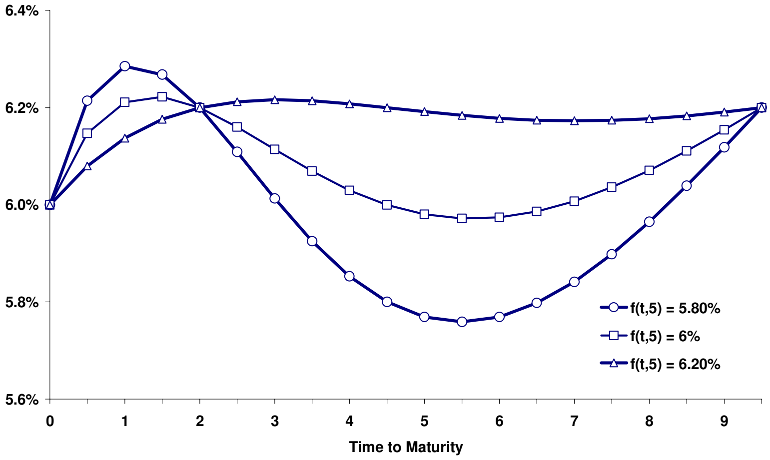

FIGURE 2. Forward rate curves at $t = 6$ months, for the One Wiener and Two Poisson RS type models when $\sigma_0 = 3.2\%$, $\kappa_\sigma = 0.18$, $\beta_{01} = 0.6\%$, $\kappa_{\beta_1} = 0.31$, $\psi_1 = 1$, $\beta_{02} = -1.28\%$, $\kappa_{\beta_2} = 0.17$ and $\psi_2 = 1.5$. The corresponding curves represent the $f(t,T)$ when $f(t,2.5) = f(t,10) = 6.2\%$ and $f(t,5)$ takes the values of 5.8%, 6% and 6.2%.

when $r = 6\%$, the 2.5-year forward rate and the 10-year forward rate is 6.2% and the 5-year forward rate takes the values of 5.8%, 6% and 6.2%.

In order to compare the different class of models examined, we select the model parameters so as to maintain, for all models, the spot rate volatility at 3.5% and the 10-year forward rate volatility at 16.5% of the spot rate volatility. To obtain these volatility levels, the set of the Wiener and Poisson volatility parameter values is the one used in each of the above examples.

Comparing Figure 1 and Figure 2 we see that the state dependent volatility models display forward rate curves with sharper curvature changes than the equivalent Hull White type models of Section 6.1. This is expected since the state dependent volatility models incorporate a larger number of state variables, which makes the model more flexible and able to capture more realistic forward rate behavior.

6.3. **Simulated Distributions.** In this section we perform simulations of the stochastic differential equation system under the risk neutral measure that results from the Markovianisation procedure. We examine and compare the simulated normalised distributions of $r(t)$ for the HW class of models and the RS class of models and in particular when one Wiener and two Poisson noise terms drive the forward rate dynamics. For all the simulation examples performed in this section, an "Euler-Maruyama approximation" is employed and we discretize the time interval $[0, 1]$ into $N = 400$ equal subintervals of length $\Delta t = 1/N$, and generate $100,000$ paths for $r(t)$. Furthermore, in order to compare the leptokurtosis levels of the two classes of models, the volatility parameters (Wiener and Poisson) have been selected as to provide the same variance of the simulated distributions, with variance being 0.0017 in all cases.

For the One Wiener and Two Poisson HW type of models, the volatility specifications considered, are $\sigma(s,t) = \sigma_0 e^{-k_\sigma(t-s)}$ and $\beta_i(s,t) = \beta_{0i} e^{-k_{\beta i}(t-s)}$ and constant ψ_i. We consider the discretised system of the instantaneous spot rate dynamics (54) with the two state variables $\mathcal{D}_{\beta i}(t)$ expressed in terms of the two benchmark forward rates $f(t,5)$ and $f(t,10)$, by making use of the system (86) in Appendix 4.

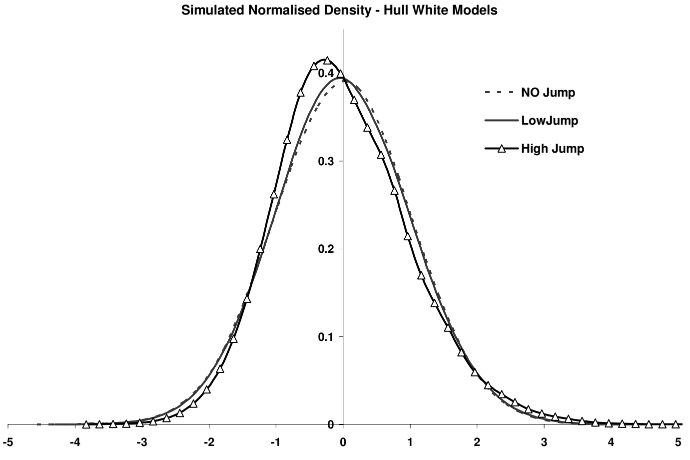

FIGURE 3. Simulated Normalised Density of the Instantaneous Spot Rate for the HW type of models at $t = 1$. The volatility magnitudes are; for the high jump volatility case $\sigma_0 = 0.9\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$; for the low jump volatility case $\sigma_0 = 3.8\%$, $\beta_{01} = 2\%$, $\beta_{02} = -1.2\%$; and for the no jump volatility case $\sigma_0 = 4.5\%$.

Figure 3 shows the simulated normalised distribution of $r(t)$ for the HW type of models at $t = 1$. The volatility parameter values used are $\kappa_\sigma = 0.18$, $\kappa_{\beta_1} = 0.31$, $\kappa_{\beta_2} = 0.17$, $\psi_1 = 1$ and $\psi_2 = 1.5$. We consider three sets of volatility magnitude parameters, one with high jump volatility, one with low jump volatility and one with no jump volatility which are respectively; a) $\sigma_0 = 0.9\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$, b) $\sigma_0 = 3.8\%$, $\beta_{01} = 2\%$, $\beta_{02} = -1.2\%$ and c) $\sigma_0 = 4.5\%$. We consider the no-jump volatility case in order to compare the distributional outcome with the Gaussian case. In fact, in the absence of jumps, the model reduces to the Gaussian case. Figure 3 shows that, compared to the normal distribution, with increasing jump magnitude, the distribution becomes asymmetric with long tail to the right. However the jump magnitude needs to be of a reasonable size for this effect to become pronounced.

For the RS type models, the Wiener volatilities are state dependent having the functional form (59). In particular, for the One Wiener and Two Poisson RS type models, we need four state variables to Markovianise the system and by considering the instantaneous spot rate $r(t)$ as one of the state variables, then $\bar{f}(t) = (r(t), f(t, T_1), f(t, T_2), f(t, T_3))^\top$. We further assume that $\sigma_1(T-t) = \sigma_0$ constant, and

$$\sigma_2(\bar{f}(t)) = \left[r(t) + \sum_{h=1}^{3} c_h f(t, T_h) \right]^\gamma ,$$

with $\gamma = 0.5$, so we considering a square root process for the Wiener volatilities.[13] For the Poisson volatility specifications, we consider $\beta_i(s,t) = \beta_{0i} e^{-k_{\beta i}(t-s)}$ and constant ψ_i.

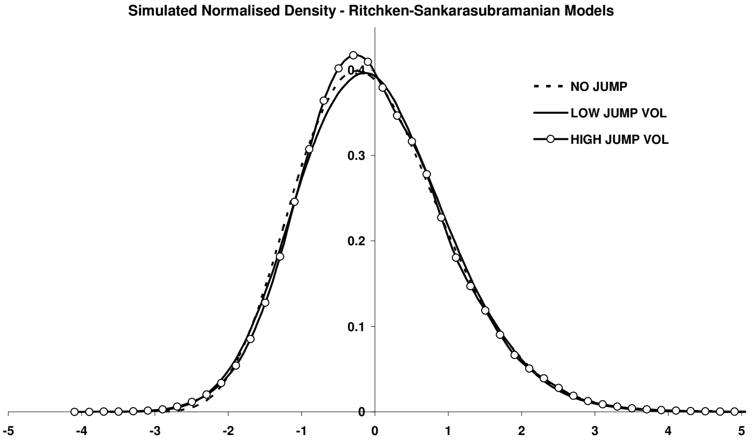

FIGURE 4. Simulated Normalised Density of the Instantaneous Spot Rate for the RS type models at $t = 1$. The volatility magnitude is set as $\sigma_0 = 1.2\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$ for the high jump volatility case; $\sigma_0 = 5.2\%$, $\beta_{01} = 2.4\%$, $\beta_{02} = -1.5\%$ for the low jump volatility case; and $\sigma_0 = 6.8\%$ for the no-jump volatility case.

We now consider the discretised system of the spot rate dynamics (44) which is the dynamics (19) with the state variables $\mathcal{E}_\sigma(t)$ and $\mathcal{D}_{\beta i}(t)$ expressed in terms of the three benchmark forward rates $f(t, 2.5)$, $f(t, 5)$, $f(t, 10)$ and the spot rate by using the system (41)). The simulated normalised distribution of $r(t)$ at $t = 1$ for the RS type of models is shown in Figure 4. The volatility parameter values used are $\kappa_\sigma = 0.18$, $\kappa_{\beta_1} = 0.31$, $\kappa_{\beta_2} = 0.17$, $\psi_1 = 1$ and $\psi_2 = 1.5$. We also set $c_1 = 2$, $c_2 = 1$, $c_3 = 2$. For the three cases of volatility magnitude, we consider $\sigma_0 = 1.2\%$, $\beta_{01} = 4\%$, $\beta_{02} = -2\%$ for the high jump volatility case, $\sigma_0 = 5.2\%$, $\beta_{01} = 2.4\%$, $\beta_{02} = -1.5\%$ for the low jump volatility case, and $\sigma_0 = 6.8\%$ for the no-jump volatility case. Considering the no-jump volatility case, in other words by relying on state dependent volatilities only, the skewness obtained is relatively large. Adding jumps does not change the order of the magnitude of the skewness. However, in the HW models the jump magnitude significantly change the order of magnitude of the skewness (see Table 1 and Table 2).

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[13] There is a positive probability that this type of dynamics may drive interest rates to negative values. Thus more general state dependent volatility functions may be employed, that are well defined for negative values as it has been shown in Nikitopoulos (2004).

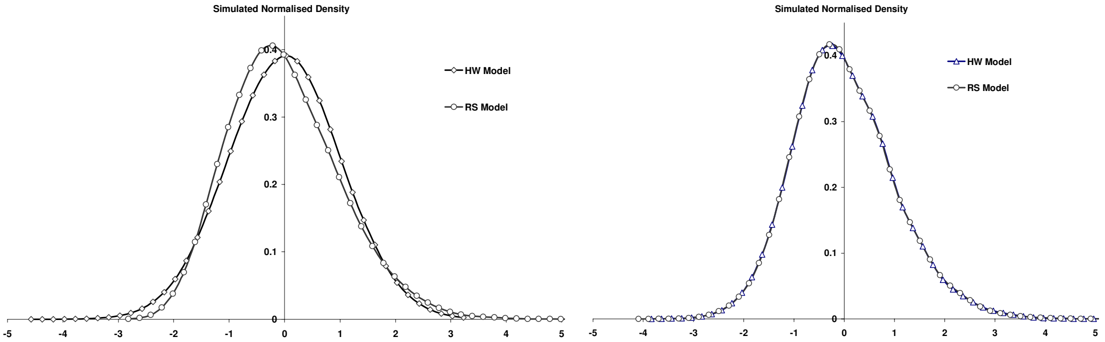

Figure 5. Comparison of Simulated Normalised Density of the Instantaneous Spot Rate for the HW and RS type of models at $t = 1$ when a) no jump and b) large jump volatility is considered.

Figure 5 compares the simulated normalised distribution of $r(t)$ for the HW and RS type of models at $t = 1$ for the cases considered earlier. The no jump cases are to the left and the large jump volatility cases to the right. In the large volatility cases similar distributions are obtained, however the two models show differences when we compare the statistical properties of the spot rate changes as Table 2 illustrates. Further, in order to gauge the effect of the jump parameters and the state dependent volatility on the simulated normalised distributions, we compare in Table 1 and Table 2 the statistical properties of the simulated distributions of the spot rate and the spot rate changes (recall that variance of the spot rate is 0.17% in all cases and expressed in percentage terms).

Statistical Information on $r(t)$						
	no-jump		low jump		high jump	
	HW	RS	HW	RS	HW	RS
Mean	6.59	6.60	6.62	6.61	6.61	6.60
Variance	0.17	0.17	0.17	0.17	0.17	0.17
Skewness	0.0045	0.5004	0.0396	0.3463	0.4313	0.4494
Kurtosis	3.0081	3.3555	3.0438	3.224	3.4875	3.5451

Table 1. The statistical measures of the spot rate from simulated distributions for different jump magnitudes under the HW and RS models.

We observe that when the jump volatilities are low the HW model is very close to a Gaussian one, although the RS model exhibits a variation from the Gaussian model with high skewness and kurtosis. Also, the state dependent models (RS) with or without jumps certainly display higher kurtosis and higher skewness of the spot rate compared to the equivalent (with respect to the jump size) deterministic volatility models (HW). This indicates that state dependent volatilities may capture more efficiently the asymmetric feature of the empirical spot rate distribution.

Statistical Information on $dr(t)$						
	no-jump		low jump		high jump	
	HW	RS	HW	RS	HW	RS
Mean	0.0009	0.0019	0.0006	0.0012	-0.0011	-0.0015
Variance	0.0005	0.0005	0.0004	0.0003	0.00001	0.00002
Skewness	-0.0019	-0.0025	0.0016	-0.0139	0.0018	-0.0634
Kurtosis	2.9820	3.3040	2.9899	3.3037	3.0033	3.2354

TABLE 2. The statistical measures of the spot rate changes from simulated distributions for different jump magnitudes under the HW and RS models.

By increasing the jump volatilities, both models exhibit asymmetric normalised distributions with a long tail to the right as we made the choice of the positive jump size to dominate the negative jump size. However, the state dependent RS model without jump, has high kurtosis of the spot rate changes but not particularly high negative skewness. When the state dependent model is combined with the jump diffusion model, such as RS with jumps, then both high kurtosis and sufficiently negative skewness of the spot rate changes are obtained. Thus, jumps on one hand and state dependent volatility on the other hand, yield models that capture better the stylised empirical facts of interest rate movements. However, the combination of both state dependent volatilities and jumps succeeds in accommodating most of the empirical distributional behavior of the spot rate and the spot rate changes. This feature should help to produce derivative security pricing models with improved valuation accuracy.

The feature that has made it tractable and possible to quantify these characteristics is the ability to obtain Markovian structures for the interest rate dynamics. This Markovian class of models that incorporates the more realistic jump-diffusion processes combined with stochastic volatilities may be employed for more accurate derivative pricing and hedging and also in empirical studies of interest rate markets (see Chiarella & Tô (2004)).

7. CONCLUSION

In this paper we have developed a class of jump diffusion term strucutre models within the framework of Shirakawa (1991). By an appropriate choice of a state dependent and time dependent forward rate volatility functions, we obtained Markovian representations of the spot rate dynamics and derived the corresponding exponential affine bond pricing formulas. Furthermore, the state variables of the model have been expressed in terms of a set of benchmark forward rates and yields, a fact that makes the model suitable for both calibration and parameter estimation. Thus for state dependent Wiener volatilities and time deterministic Poisson volatilities, we have been able to extend the results concerning finite dimensional affine realisations of HJM models in terms of forward rates discussed in Chiarella & Kwon (2003) to the jump diffusion case. We have provided some numerical examples to demonstrate the nature of the Hull-White and Ritchken & Sankarasubramanian class of models when they are extended to incorporate jumps. Summarising these results, the combination of state dependent volatilities and jumps succeeds in accommodating most of the empirical distributional behavior of the spot rate and the spot rate changes which will in turn provide more accurate derivative security pricing models.

In the case of state dependent Poisson volatility specifications, we have given some insight into why it becomes difficult to obtain a Markovian representation of the system, and we have proposed an "approximate" Markovian structure.

Further developments of this work would include estimation and model calibration. Incorporation of the jump processes into the HJM model as well as state dependent volatility structures allows a more efficient fit to market information. Additionally and more importantly, the tractability of the Markovian structures obtained provides an efficient and more accurate basis for Monte-Carlo simulations, that may be employed for derivative pricing and hedging purposes. Furthermore the framework developed here may be extended to credit risk models, as shown in Chiarella, Schlögl & Nikitopoulos (2003). This seems a natural extension as the fundamental processes used in credit risk models are jump-diffusion processes.

Appendix 1. The No-Arbitrage Condition in the Bond Market

Setting $n_H = n + \sum_{i=1}^{n} m_i$, consider a hedging portfolio containing bonds of maturities T_1, T_2, $\cdots$, T_{n_H+1} in proportions w_1, w_2, $\cdots$, w_{n_H+1} with $w_1 + w_2 + \cdots + w_{n_H+1} = 1$. We denote with $P_h(t) = P(t, T_h)$ $(h = 1, 2, \ldots, (n_H + 1))$ the value of these $n_H + 1$ zero-coupon bonds, and for simplicity of notation we write the stochastic differential equation for P in the general form

$$\frac{dP_h(t)}{P_h(t)} = \mu_{P_h}(t)dt + \sum_{i=1}^{n} \left(\nu_{P_{h,i}}(t)dW_i(t) + \sum_{j=1}^{m_i} \chi_{P_{h,ij}}(t)dQ_{ij}(t) \right),$$

where

$$\begin{aligned} \mu_{P_h}(t) &= r(t) + H(t, T_h, \bar{f}(t)) \\ \nu_{P_{h,i}}(t) &= -\zeta_i(t, T_h, \bar{f}(t)), \qquad \text{and} \\ \chi_{P_{h,ij}}(t) &= e^{-\xi_{ij}(t,T_h)} - 1. \end{aligned}$$

Let V be the value of the hedging portfolio, then the return on the portfolio is given by

$$\begin{aligned}\frac{dV}{V} &= w_1\frac{dP_1}{P_1} + w_2\frac{dP_2}{P_2} + \cdots + w_{n_H+1}\frac{dP_{n_H+1}}{P_{n_H+1}}\\ &= \sum_{h=1}^{n_H+1} w_h\mu_{P_h}dt + \sum_{h=1}^{n_H+1} w_h\sum_{i=1}^{n}\left(\nu_{P_{h,i}}dW_i(t) + \sum_{j=1}^{m_i}\chi_{P_{h,ij}}dQ_{ij}(t)\right).\end{aligned}$$

In order to eliminate both Wiener and Poisson risks we need to choose w_1, w_2, $\cdots$, w_{n_H+1} so that for every $i = 1, 2, \ldots, n$

$$\sum_{h=1}^{n_H+1} w_h\nu_{P_{h,i}} = 0, \tag{60}$$

and for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m_i$,

$$\sum_{h=1}^{n_H+1} w_h\chi_{P_{h,ij}} = 0. \tag{61}$$

The hedging portfolio then becomes riskless, thus, it should earn the risk-free rate of interest $r(t)$, i.e.,

$$\frac{dV}{V} = \sum_{h=1}^{n_H+1} w_h\mu_{P_h}dt = r(t)dt.$$

From the last equality and the fact that $w_1 + w_2 + \cdots + w_{n_H+1} = 1$, we have

$$\sum_{h=1}^{n_H+1} w_h(\mu_{P_h} - r(t)) = 0. \tag{62}$$

Equations (60), (61) and (62) form a system of $n_H + 1$ equations with $n_H + 1$ unknowns w_1, w_2, $\cdots$, w_{n_H+1}. This system can only have a non-zero solution if

$$\begin{vmatrix} \nu_{P_{1,1}}(t) & \nu_{P_{2,1}}(t) & \cdots & \nu_{P_{n_H+1,1}}(t)\\ \vdots & \vdots & \vdots & \vdots\\ \nu_{P_{1,n}}(t) & \nu_{P_{2,n}}(t) & \cdots & \nu_{P_{n_H+1,n}}(t)\\ \chi_{P_{1,11}}(t) & \chi_{P_{2,11}}(t) & \cdots & \chi_{P_{n_H+1,11}}(t)\\ \vdots & \vdots & \vdots & \vdots\\ \chi_{P_{1,nm_n}}(t) & \chi_{P_{2,nm_n}}(t) & \cdots & \chi_{P_{n_H+1,nm_n}}(t)\\ \mu_{P_1} - r(t) & \mu_{P_2} - r(t) & \cdots & \mu_{P_{n_H+1}} - r(t)) \end{vmatrix} = 0.$$

This implies that for $h = 1, 2, \ldots, (n_H+1)$ there exist $\phi_1(t)$, $\phi_2(t)$, $\ldots$,$\phi_n(t)$ and ($\psi_{11}(t)$, $\ldots$, $\psi_{1m_1}(t)$), ($\psi_{21}(t)$, $\ldots$, $\psi_{2m_2}(t)$), $\ldots$, ($\psi_{n1}(t)$, $\ldots$, $\psi_{nm_n}(t)$), such that

$$\mu_{P_h} - r(t) = -\sum_{i=1}^{n}\left(\phi_i(t)\nu_{P_{h,i}}(t) + \sum_{j=1}^{m_i}\psi_{ij}(t)\chi_{P_{h,ij}}(t)\right).$$

Since the bond maturities are arbitrary, for bonds of any maturity T we must have that

$$\mu_P - r(t) = -\sum_{i=1}^{n}\left(\phi_i(t)\nu_{P_i}(t) + \sum_{j=1}^{m_i}\psi_{ij}(t)\chi_{P_{ij}}(t)\right). \tag{63}$$

The economic interpretation of condition (63) is that the excess return of each bond above the risk free rate is equal to the total risk premium required as compensation for bearing the risk associated with the Wiener processes and the Poisson processes. Consequently, we may interpret $\mathbf{\Psi}_i$ as the vectors of the market prices of Poisson jump risks (one associated with each possible jump size) and $\mathbf{\Phi}$ as the vector of the market price of the Wiener diffusion risks. By recalling that $\mu_P(t) = r(t) + H(t,T,\bar{f}(t))$ and substituting the expressions for $\nu_{P_i}(t)$, with $i = 1,\dots,n$, and $\chi_{P_i}(t)$, with $j = 1,\dots,m_i$, we obtain

$$\begin{aligned} H(t,T) &\equiv -\int_t^T \alpha(t,u)du + \sum_{i=1}^{n}\frac{1}{2}\zeta_i^2(t,T,\bar{f}(t)) + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\lambda_{ij}\beta_{ij}(t,T) \\ &= \sum_{i=1}^{n}\left(\phi_i(t)\zeta_i(t,T,\bar{f}(t)) - \sum_{j=1}^{m_i}\psi_{ij}(t)(e^{-\xi_{ij}(t,T)} - 1)\right). \end{aligned} \tag{64}$$

By taking the derivative of (64) with respect to T and manipulating appropriately we derive the forward rate drift restriction that extends the HJM forward rate drift restriction to now incorporate the jump feature, i.e.,

$$\alpha(t,T) = \sum_{i=1}^{n}\left(\sigma_i(t,T,\bar{f}(t))(-\phi_i(t) + \zeta_i(t,T,\bar{f}(t))) - \sum_{j=1}^{m_i}\beta_{ij}(t,T)(\psi_{ij}(t)e^{-\xi_{ij}(t,T)} - \lambda_{ij})\right). \tag{65}$$

Appendix 2. Simplification of Terms Used in Equation (27)

Let

$$\begin{aligned} S(s,t,\bar{f}(s)) &= \sigma(s,t,\bar{f}(s))\int_s^t \sigma(s,u,\bar{f}(s))du \\ &= \sigma_0^2(s,\bar{f}(s))e^{-\int_s^t \kappa_\sigma(v)dv}\int_s^t e^{-\int_s^u \kappa_\sigma(v)dv}du. \end{aligned}$$

Then the derivative of $S(s,t,\bar{f}(s))$ with respect to the second argument is given by

$$\frac{\partial S(s,t,\bar{f}(s))}{\partial t} = \sigma_0^2(s,\bar{f}(s))e^{-2\int_s^t \kappa_\sigma(v)dv} - \kappa_\sigma(t)S(s,t,\bar{f}(s)).$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t \left(\sigma(s,t,\bar{f}(s))\int_s^t \sigma(s,u,\bar{f}(s))du\right) ds\\
&\quad= \int_0^t \frac{\partial}{\partial t} S(s,t,\bar{f}(s))ds + S(t,t,\bar{f}(t))\\
&\quad= \int_0^t \left[-\kappa_\sigma(t)S(s,t,\bar{f}(s)) + \sigma_0^2(s,\bar{f}(s))e^{-2\int_s^t \kappa_\sigma(v)dv}\right] ds\\
&\quad= \int_0^t \sigma^2(s,t,\bar{f}(s))ds - \kappa_\sigma(t)\int_0^t S(s,t,\bar{f}(s))ds.
\end{aligned}$$

Now consider the corresponding term in equation (27), with the Poisson volatility functions, and let

$$F(s,t) = \beta(s,t)[1 - e^{-\int_s^t \beta(s,u)du}] = \beta_0(s)e^{-\int_s^t \kappa_\beta(v)dv}\left[1 - e^{-\int_s^t \beta_0(s)e^{-\int_s^u \kappa_\beta(v)dv}du}\right].$$

Then

$$\frac{\partial F(s,t)}{\partial t} = \beta^2(s,t)e^{-\int_s^t \beta(s,u)du} - \kappa_\beta(t)F(s,t),$$

and

$$\frac{\partial}{\partial t}\int_0^t \psi(s)F(s,t)ds = \int_0^t \psi(s)\beta^2(s,t)e^{-\int_s^t \beta(s,u)du}ds - \kappa_\beta(t)\int_0^t \psi(s)F(s,t)ds.$$

Appendix 3. Derivation of the Bond Price Formula

We derive the bond price formula using the Inui & Kijima (1998) approach. The forward rate dynamics under the risk neutral measure are

$$\begin{aligned}
f(t,T) = f(0,T) &+ \sum_{i=1}^n \int_0^t \sigma_i(s,T,\bar{f}(s))\zeta_i(s,T,\bar{f}(s))ds + \sum_{i=1}^n \int_0^t \sigma_i(s,T,\bar{f}(s))d\widetilde{W}_i(s) \qquad (66)\\
&+ \sum_{i=1}^n\sum_{j=1}^{m_i} \int_0^t \psi_{ij}(s)\beta_{ij}(s,T)[1 - e^{-\xi_{ij}(s,T)}]ds + \sum_{i=1}^n\sum_{j=1}^{m_i} \int_0^t \beta_{ij}(s,T)[dQ_{ij}(s) - \psi_{ij}(s)ds].
\end{aligned}$$

Using the fundamental relationship $P(t,T) = \exp\left(-\int_t^T f(t,y)dy\right)$, we may write[14]

$$\begin{aligned}
P(t,T) &=\\
&= \exp\Bigg(-\int_t^T f(0,y)dy - \sum_{i=1}^n \int_0^t\int_t^T \sigma_i(s,y,\bar{f}(s))\zeta_i(s,y,\bar{f}(s))dyds - \sum_{i=1}^n \int_0^t\int_t^T \sigma_i(s,y,\bar{f}(s))dyd\widetilde{W}_i(s)\\
&\quad - \sum_{i=1}^n\sum_{j=1}^{m_i} \int_0^t\int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1 - e^{-\xi_{ij}(s,y)}]dyds - \sum_{i=1}^n\sum_{j=1}^{m_i} \int_0^t\int_t^T \beta_{ij}(s,y)dy[dQ_{ij}(s) - \psi_{ij}(s)ds]\Bigg). \qquad (67)
\end{aligned}$$

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[14] We assume that the conditions for application of stochastic Fubini theorem are satisfied.

Further, incorporate the volatility specifications (14) and (15) and functions (35), to derive[15]

$$\int_t^T \sigma_i(s,y,\bar{f}(s))dy = \sigma_i(s,t,\bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma i}(u)du} dy = \sigma_i(s,t,\bar{f}(s))\mathcal{N}_{\sigma i}(t,T), \tag{68}$$

and similarly

$$\int_t^T \beta_{ij}(s,y)dy = \beta_{ij}(s,t) \int_t^T e^{-\int_t^y \kappa_{\beta ij}(u)du} dy = \beta_{ij}(s,t)\mathcal{N}_{\beta ij}(t,T). \tag{69}$$

Therefore, by integrating from 0 to t and for $i = 1, \dots, n$

$$\int_0^t \int_t^T \sigma_i(s,y,\bar{f}(s))dy d\widetilde{W}_i(s) = \mathcal{N}_{\sigma i}(t,T) \int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s), \tag{70}$$

and for $j = 1, \dots, m_i$

$$\int_0^t \int_t^T \beta_{ij}(s,y)dy[dQ_{ij}(s) - \psi_{ij}(s)ds] = \mathcal{N}_{\beta ij}(t,T) \int_0^t \beta_{ij}(s,t)[dQ_{ij}(s) - \psi_{ij}(s)ds]. \tag{71}$$

Similarly, for $i = 1, \dots, n$, we manipulate the term

$$\begin{aligned}
&\int_t^T \sigma_i(s,y,\bar{f}(s))\zeta_i(s,y,\bar{f}(s))dy = \int_t^T \sigma_i(s,y,\bar{f}(s)) \int_s^y \sigma_i(s,v,\bar{f}(s))dvdy \\
&= \sigma_i(s,t,\bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma i}(u)du} dy \int_s^t \sigma_i(s,v,\bar{f}(s))dv + \sigma_i^2(s,t,\bar{f}(s)) \int_t^T e^{-\int_t^y \kappa_{\sigma i}(u)du} \int_t^y e^{-\int_t^v \kappa_{\sigma i}(u)du} dvdy \\
&= \sigma_i(s,t,\bar{f}(s))\mathcal{N}_{\sigma i}(t,T)\zeta_i(s,t,\bar{f}(s)) + \frac{1}{2}\sigma_i^2(s,t,\bar{f}(s))\mathcal{N}_{\sigma i}^2(t,T),
\end{aligned} \tag{72}$$

since

$$\begin{aligned}
&\int_t^T e^{-\int_t^y \kappa_{\sigma i}(u)du} \int_t^y e^{-\int_t^v \kappa_{\sigma i}(u)du} dvdy = \int_t^T d\left(\frac{1}{2}[\int_t^y e^{-\int_t^v \kappa_{\sigma i}(u)du} dv]^2\right) \\
&= \frac{1}{2}\left(\int_t^T e^{-\int_t^y \kappa_{\sigma i}(u)du} dy\right)^2 = \frac{1}{2}\mathcal{N}_{\sigma i}^2(t,T).
\end{aligned} \tag{73}$$

Therefore integrating equation (72) from 0 to t we obtain (for $i = 1, \dots, n$)

$$\begin{aligned}
&\int_0^t \int_t^T \sigma_i(s,y,\bar{f}(s))\zeta_i(s,y,\bar{f}(s))dyds = \\
&\mathcal{N}_{\sigma i}(t,T) \int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds + \frac{1}{2}\mathcal{N}_{\sigma i}^2(t,T) \int_0^t \sigma_i^2(s,t,\bar{f}(s))ds.
\end{aligned} \tag{74}$$

Substitute the results[16] (70), (71) and (74) into equation (67), and collect like terms and the bond price formula will simplify to

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[15] Note that

$\sigma_i(s,y,\bar{f}(s)) = \sigma_{0i}(s,\bar{f}(s))e^{-\int_s^y \kappa_{\sigma i}(u)du} = \sigma_{0i}(s,\bar{f}(s))e^{-\int_s^t \kappa_{\sigma i}(u)du}e^{-\int_t^y \kappa_{\sigma i}(u)du} = \sigma_i(s,t,\bar{f}(s))e^{-\int_t^y \kappa_{\sigma i}(u)du}$.

[16] The results (70) and (74) have been already proven in Inui & Kijima (1998).

$$\begin{aligned}P(t,T) = \exp\Bigg(&-\int_t^T f(0,y)dy - \sum_{i=1}^{n}\mathcal{N}_{\sigma i}(t,T)\int_0^t \sigma_i(s,t,\bar{f}(s))\zeta_i(s,t,\bar{f}(s))ds\\ &-\frac{1}{2}\sum_{i=1}^{n}\mathcal{N}_{\sigma i}^2(t,T)\int_0^t \sigma_i^2(s,t,\bar{f}(s))ds - \sum_{i=1}^{n}\mathcal{N}_{\sigma i}(t,T)\int_0^t \sigma_i(s,t,\bar{f}(s))d\widetilde{W}_i(s)\\ &-\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{N}_{\beta ij}(t,T)\int_0^t \beta_{ij}(s,t)[dQ_{ij}(s)-\psi_{ij}(s)ds]\\ &-\sum_{i=1}^{n}\sum_{j=1}^{m_i}\int_0^t\int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1-e^{-\xi_{ij}(s,y)}]dyds\Bigg). \end{aligned} \tag{75}$$

By using the definitions (23),(25) and (26), equation (75) simplifies further to

$$\begin{aligned}P(t,T) = \exp\Bigg(&-\int_t^T f(0,y)dy - \frac{1}{2}\sum_{i=1}^{n}\mathcal{N}_{\sigma i}^2(t,T)\mathcal{E}_{\sigma i}(t) - \sum_{i=1}^{n}\mathcal{N}_{\sigma i}(t,T)\mathcal{D}_{\sigma i}(t)\\ &-\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{N}_{\beta ij}(t,T)\left\{\mathcal{D}_{\beta ij}(t) - \int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds\right\}\\ &-\sum_{i=1}^{n}\sum_{j=1}^{m_i}\int_0^t\int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1-e^{-\xi_{ij}(s,y)}]dyds\Bigg). \end{aligned} \tag{76}$$

Thus the bond price formula, where the bond price is a function of the state variables $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\beta ij}(t)$ and $\mathcal{D}_{\sigma i}(t)$ with $i = 1, \cdots, n$ and $j = 1, \cdots, m_i$, may be expressed as,

$$\begin{aligned}P(t,T) =& \frac{P(0,T)}{P(0,t)}\exp\Bigg\{\bar{\mathcal{M}}(t,T) - \frac{1}{2}\sum_{i=1}^{n}\mathcal{N}_{\sigma i}^2(t,T)\mathcal{E}_{\sigma i}(t)\\ &-\sum_{i=1}^{n}\mathcal{N}_{\sigma i}(t,T)\mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{N}_{\beta ij}(t,T)\mathcal{D}_{\beta ij}(t)\Bigg\}, \end{aligned} \tag{77}$$

where,

$$\begin{aligned}\bar{\mathcal{M}}(t,T) =& -\sum_{i=1}^{n}\sum_{j=1}^{m_i}\int_0^t\int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1-e^{-\xi_{ij}(s,y)}]dyds\\ &+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathcal{N}_{\beta ij}(t,T)\left\{\int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds\right\}. \end{aligned} \tag{78}$$

3.1. **The case where the spot rate is one of the state variables.** To derive the corresponding term structure of interest rates, substitute the expression (29) for the $\mathcal{D}_{\sigma 1}(t)$ into the bond price formula (77) to obtain the multi factor affine term structure of interest rates in the form

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\Bigg\{\mathcal{M}(t,T)-\mathcal{N}_{\sigma 1}(t,T)r(t)-\frac{1}{2}\sum_{i=1}^{n}\mathcal{N}^2_{\sigma i}(t,T)\mathcal{E}_{\sigma i}(t) \tag{79}$$
$$-\sum_{i=2}^{n}(\mathcal{N}_{\sigma i}(t,T)-\mathcal{N}_{\sigma 1}(t,T))\mathcal{D}_{\sigma i}(t)-\sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathcal{N}_{\beta ij}(t,T)-\mathcal{N}_{\sigma 1}(t,T))\mathcal{D}_{\beta ij}(t)\Bigg\},$$

where

$$\mathcal{M}(t,T)=\mathcal{N}_{\sigma 1}(t,T)f(0,t)+\bar{\mathcal{M}}(t,T),$$

with $\mathcal{N}_x(t,T)$ $(x\in\{\sigma_i,\beta_{ij}\})$ defined as in equation (35).

Appendix 4. Finite Dimensional Affine Realisations in Terms of Forward Rates for HW Models

Using the Inui & Kijima (1998) approach and under the volatility specifications of Proposition 6.1, the spot rate dynamics (54) lead us to the multi factor bond price formula in terms of the state variables $r(t),\mathcal{D}_{\sigma i}(t)$, and $\mathcal{D}_{\beta ij}(t)$. The multi-factor affine term structure of interest rates is

$$P(r(t),t,T)=\exp\Bigg\{\hat{\mathcal{M}}(t,T)-\mathcal{N}_{\sigma_1}(t,T)r(t)-\sum_{i=2}^{n}(\mathcal{N}_{\sigma i}(t,T)-\mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\sigma_i}(t) \tag{80}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathcal{N}_{\beta ij}(t,T)-\mathcal{N}_{\sigma_1}(t,T))\mathcal{D}_{\beta ij}(t)\Bigg\},$$

where

$$\hat{\mathcal{M}}(t,T)=\mathcal{M}(t,T)-\frac{1}{2}\sum_{i=1}^{n}\mathcal{N}^2_{\sigma_i}(t,T)\mathcal{E}_{\sigma_i}(t), \tag{81}$$

with $\mathcal{M}(t,T)$ and $\mathcal{N}_x(t,T)$ $(x\in\{\sigma_i,\beta_{ij}\})$ are defined in equation (34) and (35) respectively. To derive the result (80), we perform similar manipulations as in Appendix 3. The bond price formula (80) generalises the affine term structure of interest rates of multi factor Hull & White type of models, to the case of jump diffusions.

Use of the exponential affine term structure of interest rates (80), where the bond price is a function of the instantaneous spot rate $r(t)$, and the stochastic quantities $\mathcal{D}_{\beta ij}(t)$ and $\mathcal{D}_{\sigma i}(t)$ lead us to express the instantaneous forward rate as (from equation (1))

$$f(t,T)-f(0,T)+\frac{\partial\hat{\mathcal{M}}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}r(t)= \tag{82}$$
$$\sum_{i=2}^{n}\left(\frac{\partial\mathcal{N}_{\sigma i}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\mathcal{D}_{\sigma i}(t)+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(\frac{\partial\mathcal{N}_{\beta ij}(t,T)}{\partial T}-\frac{\partial\mathcal{N}_{\sigma_1}(t,T)}{\partial T}\right)\mathcal{D}_{\beta ij}(t).$$

Take a number of fixed forward rate maturities equal to the number of state variables $\hat{n}_s (= n + m_1 + \ldots + m_n - 1)$ excluding the spot rate. Then these state variables can be expressed in terms of forward rates of $\hat{n}_s$ different fixed maturities as the following proposition shows.

PropositionA5.1. *The forward rate of any maturity can be expressed in terms of the $\hat{n}_s$ benchmark forward rates and the instantaneous spot rate $r(t)$ as*

$$f(t,T) = f(0,T) + \mathcal{Q}(t,T) + \sum_{h=1}^{\hat{n}_s} \mathcal{R}_h(t,T) f(t,T_h) + \mathcal{S}(t,T) r(t), \tag{83}$$

where, for $l = q-1$ and $k = n+i-1$,

$$\mathcal{Q}(t,T) = -\frac{\partial \hat{\mathcal{M}}(t,T)}{\partial T} + \sum_{h=1}^{\hat{n}_s} \left(\frac{\partial \hat{\mathcal{M}}(t,T_h)}{\partial T_h} - f(0,T_h) \right) \left[\sum_{q=2}^{n} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma q}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right) \right. \tag{84}$$

$$\left. + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right) \right],$$

$$\mathcal{R}_h(t,T) = \sum_{q=2}^{n} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right),$$

and

$$\mathcal{S}(t,T) = \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} - \sum_{h=1}^{\hat{n}_s} \frac{\partial \mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h} \left(\sum_{q=2}^{n} \varpi_{lh} \left(\frac{\partial \mathcal{N}_{\sigma_q}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right) \right. \tag{85}$$

$$\left. + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varpi_{kh} \left(\frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T)}{\partial T} \right) \right),$$

and $\varpi_{\ell\hbar}$ denotes the $\ell\hbar^{th}$ element of the matrix $\mathbb{O}^{-1}$, the inverse of the square matrix $\mathbb{O}(t)$, defined such that for $i = 1, 2, \ldots, n$, $q = 2, \ldots, n$, and $j = 1, 2, \ldots, m_i$,

$$\mathbb{O}(t) = \begin{bmatrix} \varphi_2(t) & \varphi_3(t) \end{bmatrix},$$

where, $\varphi_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma_q}(t,T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h} \right]$, is a $\hat{n}_s \times (n-1)$ matrix, and $\varphi_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_{ij}}(t,T_h)}{\partial T_h} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_h)}{\partial T_h} \right]$, is a $\hat{n}_s \times (m_1 + \ldots + m_n)$ matrix. Assume that $\mathbb{O}(t)$ is invertible for all $t \in \{t; t = \min_h T_h\}$.

Proof. Considering equation (82) for the maturities $T_1, T_2, \cdots, T_{\hat{n}_s}$ we obtain the system

$$\begin{bmatrix} f(t,T_1) - f(0,T_1) + \frac{\partial \hat{\mathcal{M}}(t,T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_1)}{\partial T_1} r(t) \\ f(t,T_2) - f(0,T_2) + \frac{\partial \hat{\mathcal{M}}(t,T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_2)}{\partial T_2} r(t) \\ \vdots \\ f(t,T_{\bar{n}_s}) - f(0,T_{\bar{n}_s}) + \frac{\partial \hat{\mathcal{M}}(t,T_{\bar{n}_s})}{\partial T_{\hat{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_{\hat{n}_s})}{\partial T_{\hat{n}_s}} r(t) \end{bmatrix} = \mathbb{O}(t) \times \begin{bmatrix} \mathcal{D}_{\sigma 2}(t) \\ \vdots \\ \mathcal{D}_{\sigma n}(t) \\ \mathcal{D}_{\beta 11}(t) \\ \vdots \\ \mathcal{D}_{\beta n m_n}(t) \end{bmatrix}.$$

By inverting the matrix $\mathbb{O}(t)$, the state variables $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$ are expressed in terms of forward rates of $\hat{n}_s$ distinct maturities as

$$\begin{bmatrix} \mathcal{D}_{\sigma 2}(t) \\ \vdots \\ \mathcal{D}_{\sigma n}(t) \\ \mathcal{D}_{\beta 11}(t) \\ \vdots \\ \mathcal{D}_{\beta n m_n}(t) \end{bmatrix} = \mathbb{O}^{-1}(t) \times \begin{bmatrix} f(t,T_1) - f(0,T_1) + \frac{\partial \hat{\mathcal{M}}(t,T_1)}{\partial T_1} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_1)}{\partial T_1} r(t) \\ f(t,T_2) - f(0,T_2) + \frac{\partial \hat{\mathcal{M}}(t,T_2)}{\partial T_2} - \frac{\partial \mathcal{N}_{\sigma_2}(t,T_1)}{\partial T_2} r(t) \\ \vdots \\ f(t,T_{\hat{n}_s}) - f(0,T_{\hat{n}_s}) + \frac{\partial \hat{\mathcal{M}}(t,T_{\hat{n}_s})}{\partial T_{\hat{n}_s}} - \frac{\partial \mathcal{N}_{\sigma_1}(t,T_{\hat{n}_s})}{\partial T_{\hat{n}_s}} r(t) \end{bmatrix}, \tag{86}$$

By substitution of expressions (86) for the state variables into the forward rate formula (82), one obtains (83) which expresses the forward rate of any maturity in terms of the forward rates of $\hat{n}_s$ fixed maturities and the instantaneous spot rate $r(t)$. □

Appendix 5. Benchmark Forward Rates as Sole State Variables

By substituting (23), (24), (25) and (26) into the stochastic differential equation (28) we obtain the dynamics for the spot rate in terms of the stochastic factors $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$, as

$$\begin{aligned} dr(t) = {} & \left[\hat{D}(t) + \sum_{i=1}^{n} \mathcal{E}_{\sigma i}(t) - \sum_{i=1}^{n} k_{\sigma i}(t) \mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n} \sum_{j=1}^{m_i} k_{\beta ij}(t) \mathcal{D}_{\beta ij}(t)) \right] dt \\ & + \sum_{i=1}^{n} \left(\sigma_{0i}(t, \bar{f}(t)) d\widetilde{W}_i(t) + \sum_{j=1}^{m_i} \beta_{0ij}(t) \left[dQ_{ij}(t) - \psi_{ij}(t) dt \right] \right), \end{aligned} \tag{87}$$

where

$$\hat{D}(t) = \frac{\partial}{\partial t} f(0,t) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \mathcal{E}_{\beta ij}(t). \tag{88}$$

The corresponding affine term structure of interest rates (see Appendix 3 for details) is given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ \hat{\mathcal{M}}(t,T) - \frac{1}{2}\sum_{i=1}^{n} \mathcal{N}_{\sigma i}^2(t,T)\mathcal{E}_{\sigma i}(t) \right. \tag{89}$$
$$\left. - \sum_{i=1}^{n} \mathcal{N}_{\sigma i}(t,T)\mathcal{D}_{\sigma i}(t) - \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mathcal{N}_{\beta ij}(t,T)\mathcal{D}_{\beta ij}(t) \right\},$$

where,

$$\begin{aligned} \hat{\mathcal{M}}(t,T) = &- \sum_{i=1}^{n}\sum_{j=1}^{m_i} \int_0^t \int_t^T \psi_{ij}(s)\beta_{ij}(s,y)[1-e^{-\xi_{ij}(s,y)}]dyds \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mathcal{N}_{\beta ij}(t,T) \int_0^t \psi_{ij}(s)\beta_{ij}(s,t)[1-e^{-\xi_{ij}(s,t)}]ds. \end{aligned} \tag{90}$$

The bond price (89) is a function of the state variables $\mathcal{E}_{\sigma i}(t)$, $\mathcal{D}_{\sigma i}(t)$ and $\mathcal{D}_{\beta ij}(t)$ with $i = 1, \cdots, n$ and $j = 1, \cdots, m_i$. From equation (1), we can express the relation between the instantaneous forward rate curve and the state variables as

$$\begin{aligned} f(t,T) - f(0,T) + f(0,t) + \frac{\partial \mathcal{M}(t,T)}{\partial T} &= \sum_{i=1}^{n} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T} \mathcal{N}_{\sigma i}(t,T)\,\mathcal{E}_{\sigma i}(t) \\ + \sum_{i=1}^{n} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T}\,\mathcal{D}_{\sigma i}(t) &+ \sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T}\,\mathcal{D}_{\beta ij}(t), \end{aligned} \tag{91}$$

where $\mathcal{N}_x(t,T)$ $(x \in \{\sigma_i, \beta_{ij}\})$ are defined as in equation (35).

Taking a number of fixed maturity forward rates equal to the number of the state variables, it becomes possible to express the state variables in terms of forward rates with different fixed maturities. Thus, we consider forward rates of $n_s (= 2n + m_1 + \ldots + m_n)$ different fixed maturities T_h, as shown in the following proposition.

PropositionA4.1. *The forward rate of any maturity can be expressed in terms of the* n_s *benchmark forward rates* $f(t,T_h)$, $(h = 1, \ldots, n_s)$ *as*

$$f(t,T) = f(0,T) - f(0,t) + \mathcal{Q}(t,T) + \sum_{h=1}^{n_s} \mathcal{R}_h(t,T) f(t,T_h), \tag{92}$$

where, for $l = n + i$ *and* $k = 2n + i$,

$$\mathcal{R}_h(t,T) = \sum_{i=1}^{n} \left(\varpi_{ih} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T} \mathcal{N}_{\sigma i}(t,T) + \varpi_{lh} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T} + \sum_{j=1}^{m_i} \varpi_{kh} \frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T} \right), \tag{93}$$

and

$$\mathcal{Q}(t,T) = \frac{\partial \mathcal{M}(t,T)}{\partial T} - \sum_{h=1}^{n_s} \left(\frac{\partial \mathcal{M}(t,T_h)}{\partial T_h} - f(0,T_h) + f(0,t) \right) \left[\sum_{i=1}^{n} \left(\varpi_{ih} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T} \mathcal{N}_{\sigma i}(t,T) \right.\right. \tag{94}$$

$$\left.\left. + \varpi_{lh} \frac{\partial \mathcal{N}_{\sigma i}(t,T)}{\partial T} + \sum_{j=1}^{m_i} \varpi_{kh} \frac{\partial \mathcal{N}_{\beta ij}(t,T)}{\partial T} \right) \right].$$

Denote as $\varpi_{\ell\hbar}$ *the* $\ell\hbar^{th}$ *element of matrix* $\mathbb{O}^{-1}(t)$*, the inverse of the square matrix* $\mathbb{O}(t)$*, such that, for* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m_i$*,*

$$\mathbb{O}(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) & \varphi_3(t) \end{bmatrix},$$

where, $\varphi_1(t) = \left[\frac{\partial \mathcal{N}_{\sigma i}(t,T_h)}{\partial T_h} \mathcal{N}_{\sigma i}(t,T_h) \right]$ *is an* $n_s \times n$ *matrix,*
$\varphi_2(t) = \left[\frac{\partial \mathcal{N}_{\sigma i}(t,T_h)}{\partial T_h} \right]$ *is an* $n_s \times n$ *matrix, and*
$\varphi_3(t) = \left[\frac{\partial \mathcal{N}_{\beta_{ij}}(t,T_h)}{\partial T_h} \right]$ *is an* $n_s \times (m_1 + \ldots + m_n)$ *matrix.*
Assume that $\mathbb{O}(t)$ *is invertible for all* $t \in \{t; t = \min_h T_h\}$*.*

Proof. Similar manipulations as Proposition 4.1. □

PropositionA4.2. *The zero-coupon bond prices in terms of the benchmark forward rates* $f(t,T_h)$ *is given by*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(\mathcal{Q}^P(t,T) + \sum_{h=1}^{n_s} \mathcal{R}_h^P(t,T) f(t,T_h) \right), \tag{95}$$

where

$$\mathcal{Q}^P(t,T) = -\int_t^T \mathcal{Q}(t,s)ds, \quad \text{and} \quad \mathcal{R}_h^P(t,T) = -\int_t^T \mathcal{R}_h(t,s)ds. \tag{96}$$

Proof. By substitution of (92) into the fundamental relationship (1). □

Thus by setting the set of the state dependent variables $\bar{f}(t)$ of the forward rate volatility functions considered in Assumption 3.1 as the set of the benchmark forward rates, i.e.,

$$\bar{f}(t) = (f(t,T_1), f(t,T_2), \ldots, f(t,T_{n_s}))^\top,$$

we have a closed Markovian system.

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References

Baxter, M. & Rennie, A. (1996), *Financial Calculus An Introduction to Derivative Pricing*, Cambridge University Press.

Bhar, R. & Chiarella, C. (1997), 'Transformation of Heath-Jarrow-Morton Models to Markovian Systems', *European Journal of Finance* **3**, 1–26.

Björk, T. & Gombani, A. (1999), 'Minimal Realisations for Forward Rates', *Finance and Stochastics* **3**(4), 413–432.

Björk, T., Kabanov, Y. & Runggaldier, W. (1997), 'Bond Market Structure in the Presence of Marked Point Processes', *Mathematical Finance* **7**(2), 211–239.

Björk, T. & Landèn, C. (2002), 'On the Construction of Finite Dimensional Realisations for Nonlinear Forward Rate Models', *Finance and Stochastics* **6**(3), 303–329.

Björk, T. & Svensson, L. (2001), 'On the Existence of Finite Dimensional Realisations for Nonlinear Forward Rate Models', *Mathematical Finance* **11**(2), 205–243.

Bremaud, P. (1981), *Point Processes and Queues: Martingale Dynamics*, Springer-Verlag.

Carverhill, A. (1994), 'When is the Short Rate Markovian', *Mathematical Finance* **4**, 305–312.

Chacko, G. & Das, S. (2002), 'Pricing Interest Rate Derivatives: A General Approach', *The Review of Financial Studies* **15**(1), 195–241.

Cheyette, O. (1992), 'Term Structure Dynamics and Mortgage Valuation', *Journal of Fixed Income* **1**, 28–41.

Chiarella, C. & El-Hassan, N. (1996), 'A Preference Free Partial Differential Equation for the Term Structure of Interest Rates', *Financial Engineering and the Japanese Markets* **3**, 217–238.

Chiarella, C. & Kwon, O. (2001a), 'Forward Rate Dependent Markovian Transformations of the Heath-Jarrow-Morton Term Structure Model', *Finance and Stochastics* **5**(2), 237–257.

Chiarella, C. & Kwon, O. (2001b), 'Formulation of Popular Interest Models under the HJM Framework', *Asia Pacific Financial Markets* **8**(1), 1–22.

Chiarella, C. & Kwon, O. (2003), 'Finite Dimensional Affine Realisations of HJM Models in Terms of Forward Rates and Yields', *Review of Derivatives Research* **6**(2), 129–155.

Chiarella, C., Schlögl, E. & Nikitopoulos, C. S. (2003), 'A Defaultable Term Structure Model with State Dependent Volatilities', *Scool of Finance and Economics, UTS* . Working Paper.

Chiarella, C. & Tô, T. (2003), 'The Jump Component of the Volatility Structure of Interest Rate Futures Markets: An International Comparison', *Journal of Futures Markets* **23**, 1125–1158.

Chiarella, C. & Tô, T. (2004), 'Estimation of Affine Term Structure Models within the HJM Framework', *Scool of Finance and Economics, UTS* . Working Paper.

Das, S. (2002), 'The Surprise Element: Jumps in the Interest Rates', *Journal of Econometrics* **106**, 27–65.

Das, S. R. (2000), *Interest Rate Modelling with Jump-Diffusion Processes.* Chapter 6 in Advanced Fixed-Income Valuation Tools by N. Jegadeesh and B. Tuckman, John Wiley & Sons.

de Jong, F. & Santa-Clara, P. (1999), 'The Dynamics of the Forward Interest Rate Curve: A Formulation with State Variables', *Journal of Financial and Quantitative Analysis* **34**(1), 131–140.

Duffie, D. & Kan, R. (1996), 'A Yield-Factor Model of Interest Rates', *Mathematical Finance* **6**(4), 379–406.

Glasserman, P. & Kou, S. G. (2003), 'The Term Structure of Simple Forward Rates with Jump Risk', *Mathematical Finance* **13**(3), 383–410.

Heath, D., Jarrow, R. A. & Morton, A. (1992), ‘Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation’, *Econometrica* **60**, 77–105.

Hull, J. & White, A. (1990), ‘Pricing Interest Rate Derivative Securities’, *The Review of Financial Studies* **3**(4), 573–592.

Hull, J. & White, A. (1994), ‘Numerical Procedures for Implementing Term Structure Models II: Two-Facors Models’, *Journal of Derivatives* **Winter**, 37–49.

Inui, K. & Kijima, M. (1998), ‘A Markovian Framework in Multi-Factor Heath-Jarrow-Morton Models’, *Journal of Financial and Quantitative Analysis* **33**(3), 423–440.

Nikitopoulos, C. S. (2004), ‘A Class of Markovian Models for the Term Structure of Interest Rates Under Jump-Diffusions’, *School of Finance and Economics, University of Technology, Sydney* . PhD Thesis.

Ritchken, P. & Chuang, Y. (1999), ‘Interest Rate Option Pricing with Volatility Humps’, *Review of Derivatives Research* **3**, 237–262.

Ritchken, P. & Sankarasubramanian, L. (1995), ‘Volatilities Structures of Forward Rates and the Dynamics of Term Structure’, *Mathematical Finance* **5**(1), 55–73.

Runggaldier, W. J. (2003), ‘Jump-Diffusion Models’. Chapter 5 in Handbook Heavy Tailed Distributions in Finance, ed. S. T. Rachev, Elsevier, North-Holland, 169-209.

Shirakawa, H. (1991), ‘Interest Rate Option Pricing with Poisson-Gaussian Forward Rate Curve Processes’, *Mathematical Finance* **1**(4), 77–94.

Vasicek, O. (1977), ‘An Equilibrium Characterisation of the Term Structure’, *Journal of Financial Economics* **5**, 177–188.