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On an Effective Solution of the Optimal Stopping Problem for Random Walks

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ON AN EFFECTIVE SOLUTION OF THE OPTIMAL STOPPING PROBLEM FOR RANDOM WALKS

We find a solution of the optimal stopping problem for the case when a reward function is an integer power function of a random walk on an infinite time interval. It is shown that an optimal stopping time is a first crossing time through a level defined as the largest root of Appell's polynomial associated with the maximum of the random walk. It is also shown that a value function of the optimal stopping problem on the finite interval $\{0, 1, ..., T\}$ converges with an exponential rate as $T \to \infty$ to the limit under the assumption that jumps of the random walk are exponentially bounded.

Keywords: optimal stopping, random walk, rate of convergence, Appell polynomials.

1. Introduction and the main result.

1. Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables defined on a probability space $(\Omega, F, \mathbf{P})$. Define a corresponding homogeneous Markov chain $X = (X_0, X_1, X_2, \ldots)$ such that

$$X_0 = x \in \mathbf{R}, \qquad X_k = x + S_k, \quad S_k = \sum_{i=1}^{k} \xi_i, \quad k \geq 1.$$

Let P_x denote a distribution function which corresponds to the sequence X. In other words, the system P_x, $x \in \mathbf{R}$, and X define a Markov family with respect to the flow of σ-algebras $(F_k)_{k \geq 0}$, $F_0 = \{\varnothing, \Omega\}$, $F_k = \sigma\{\xi_1, \ldots, \xi_k\}$.

For the random walk under consideration we discuss the optimal stopping problem which consists in finding the "value" function

$$V(x) = \sup_{\tau \in \mathrm{M}_0^\infty} E_x \, g(X_\tau) \, I\{\tau < \infty\}, \qquad x \in \mathbf{R}, \tag{1}$$

where $g(x)$ is a measurable function, $I\{\cdot\}$ is the indicator function and the supremum is taken over the class M_0^∞ of all stopping times τ with values in $[0, \infty]$ with respect to $(F_k)_{k \geq 0}$. We call a stopping time τ^* optimal if

$$E_x \, g(X_{\tau^*}) \, I\{\tau^* < \infty\} = V(x), \qquad x \in \mathbf{R}.$$

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We impose the following basic conditions on the random walk X and the function $g(x)$:

$$E\,\xi < 0, \quad g(x) = (x^+)^n = (\max(x,0))^n, \qquad n = 1, 2, \dots .$$

It is well known (see e.g., [4], [11]) that under very general assumptions on $g(x)$ the solution of the problem (1) can be characterised as follows: $V(x)$ is the least excessive majorant of the function $g(x)$, i.e. the smallest function $U = U(x)$ with the properties

$$U(x) \geq g(x), \qquad U(x) \geq TU(x),$$

where $TU(x) = E_x\,U(X_1) = EU(x+\xi)$.

It is also well known (see e.g. [11]) that if the stopping time

$$\tau^* = \inf\{k \geq 0 : V(X_k) = g(X_k)\}$$

is *finite* P_x-a.s. for any $x \in \mathbf{R}$ then under very general assumptions on $g(x)$ it is *optimal* in the class M_0^∞.

There are much less results in the literature about optimality of the stopping time

$$\tau^* = \inf\{k \geq 0 : X_k \in D^*\}$$

such that, generally speaking, $P_x\{\tau^* = \infty\} > 0$ for some $x \in \mathbf{R}$ (where $D^* = \{x \in \mathbf{R} : V(x) = g(x)\}$ is the "stopping region" and we assume always that $\inf\{\varnothing\} = \infty$).

It is the case under consideration where a stopping time *is not finite*, but nevertheless it is *optimal* (in the class M_0^∞). Note that the results described in [4], [11], present only qualitative features of the solution of the optimal stopping problem for Markov families but they do not present an effective method for finding $V(x)$ or for constructing the stopping region D^*. From this point of view it is always of interest to find both $V(x)$ and D^* in an explicit form. A list of papers with results of this type contains, in particular, the paper [6] where the existence of the optimal stopping time $\tau^* \in M_0^\infty$ was proved for the case $g(x) = x^+$ under the assumption that $E\xi < 0$ and it was shown that it has the threshold form:

$$\tau^* = \inf\{k \geq 0 : X_k \geq a^*\},$$

i.e. $D^* = \{x \in \mathbf{R} \ : x \geq a^*\}$. Here the value of the threshold a^* as well as the function $V(x)$ are completely determined by the distribution of the random variable

$$M = \sup_{k \geq 0} S_k, \qquad S_0 = 0$$

(we use everywhere, if possible, the same notation as in [6]).

2. The goal of this paper consists in the following: from one hand we generalise the result of [6] to the case $g(x) = (x^+)^n$ with $n = 2, 3, \dots$; on the other hand we suggest a different (compare to [6]) method for checking optimality of the corresponding stopping times (which are of the threshold form as in [6]).

To formulate the basic result we need the following definition.

Let η be a random variable such that $Ee^{\lambda|\eta|} < \infty$ for some $\lambda > 0$. Define the polynomials $Q_k(y) = Q_k(y;\eta)$, $k = 0,1,2,\dots$, using the decomposition

$$\frac{e^{uy}}{Ee^{u\eta}} = \sum_{k=0}^{\infty} \frac{u^k}{k!}\, Q_k(y). \tag{2}$$

The polynomials $Q_k(y)$, $k = 0,1,2,\dots$ (defined with help of the "generating" function $e^{uy}/Ee^{u\eta}$) are called *Appell polynomials* (also known as *Sheffer polynomials*, see e.g. [13]). The polynomials $Q_k(y)$ can be represented with help of the cumulants $\varkappa_1, \varkappa_2, \dots$ of the random variable η. For example,

$$Q_0(x) = 1, \quad Q_1(y) = y - \varkappa_1, \quad Q_2(y) = (y-\varkappa_1)^2 - \varkappa_2,$$
$$Q_3(x) = (y-\varkappa_1)^3 - 3\varkappa_2(y-\varkappa_1) - \varkappa_3.$$

In fact, to define the polynomials $Q_k(y)$, $k = 1,\dots,n$, uniquely it is sufficient to assume that $E|\eta|^n < \infty$. Under this assumption

$$\frac{d}{dy}\, Q_k(y) = kQ_{k-1}(y), \qquad k \le n \tag{3}$$

(this property is also used sometimes as the definition of Appell polynomials). Note also that decomposition (2) implies that for any $x \in \mathbf{R}$, $y \in \mathbf{R}$ and $k = 1,2,\dots$

$$Q_k(y;\eta + x) = Q_k(y - x;\eta).$$

We shall always use Appell polynomials generated by the random variable $M = \sup_{k\ge 0} S_k$, that is always

$$Q_k(y) = Q_k(y;M).$$

Theorem 1. Let $n = 1,2,\dots$ and be fixed. Assume

$$g(x) = (x^+)^n, \quad E\xi < 0, \quad E(\xi^+)^{n+1} < \infty.$$

Let a_n^* be the largest real root of the equation

$$Q_n(y) = 0 \tag{4}$$

and

$$\tau_n^* = \inf\{k \ge 0 : X_k \ge a_n^*\}.$$

Then the stopping time τ_n^* is optimal:

$$V_n(x) := \sup_{\tau\in M_0^\infty} E_x\,(X_\tau^+)^n I\{\tau < \infty\} = E_x\,(X_{\tau_n^*}^+)^n I\{\tau_n^* < \infty\}; \tag{5}$$

furthermore,

$$V_n(x) = E\, Q_n(M + x)\, I\{M + x \ge a_n^*\}. \tag{6}$$

Remark 1. For $n = 1$ and $n = 2$

$$a_1^* = EM, \quad a_2^* = EM + \sqrt{DM}.$$

Some cases when the distribution of M can be found in an explicit form are described in [2, §19]; see also some examples in [14] concerning explicit formulas for crossing times through a level for an upper-continuous random walk. Usually, to find $V_n(x)$ one needs, generally speaking, to know the distribution function of the random variable M. Numerical values of cumulants and the distribution function of M can be found with the help of Spitzer's identity (see e.g. [12]).

Remark 2. The methods used for the proof of Theorem 1 allows us to obtain also corresponding results for other reward functions $g(x)$. As an illustration we consider the case $g(x) = 1 - e^{-x^+}$ in Theorem 2 (see Section 4 below). Note that in [6] the case $g(x) = (e^x - 1)^+$ (with a discounting time factor) was studied too.

3. The idea of the proof of Theorem 1 is the following.

Together with $g(x) = (x^+)^n$, we consider the function $\widehat{g}(x) = x^n$ and solve for this case the optimal stopping problem

$$\widehat{V}_n(x) = \sup_{\tau \in \widehat{M}_0^\infty} E_x\, \widehat{g}(X_\tau)\, I\{\tau < \infty\}, \tag{7}$$

where $\widehat{M}_0^\infty$ is the class of special stopping times of the form: $\widehat{\tau} = \tau_a$, $a \geq 0$,

$$\tau_a = \inf\{k \geq 0 : X_k \geq a\}.$$

Since on the set $\{\tau_a < \infty\}$ the equality $\widehat{g}(X_{\tau_a}) = g(X_{\tau_a})$ holds then, obviously, $\widehat{V}_n(x) \leq V_n(x)$, as $V_n(x)$ (see (5)) is calculated over the class of stopping times $\widehat{M}_0^\infty$, which is larger than the corresponding class in (7).

Based on properties of Appell polynomials, it is possible to solve completely the problem (7). It turns out (see Section 2 and Section 3) that

$$\widehat{V}_n(x) = E\, Q_n(M + x)\, I\{M + x \geq a_n^*\} \tag{8}$$

and the optimal stopping time is $\widehat{\tau}_n = \tau_{a_n^*}$ (in the class $\widehat{M}_0^\infty$).

Further, using again properties of Appell polynomials, we manage to show that

$$\widehat{V}_n(x) \geq V_n(x), \quad x \in \mathbf{R}\ . \tag{9}$$

Thus, taking into account the inequality $\widehat{V}_n(x) \leq V_n(x)$, we get that $\widehat{V}_n(x) = V_n(x)$ and the stopping time $\widehat{\tau}_n = \tau_{a_n^*}$ is optimal in the class M_0^∞.

To carry out the indicated plan of proof for Theorem 1 we will have to list a number of auxiliary results for the maximum $M = \sup_{k\geq 0} S_k$ and some properties of Appell polynomials. That will be done in Section 2. In Section 3 the auxiliary optimal stopping problem (7) will be discussed and the detail of the proof for Theorem 1 will be given. In Section 4 we shall present a number of remarks and, in particular, formulate and prove Theorem 2 about optimal stopping for the reward function $g(x) = 1 - e^{-x^+}$. In the same section we will formulate and prove Theorem 3 about *a rate of convergence* as $T \to \infty$ for the value function of the optimal stopping problem for the reward functions $g(x) = (x^+)^n$, $n = 1, 2, \ldots$, and $g(x) = 1 - e^{-x^+}$ on the finite interval $\{0, 1, \ldots, T\}$.

2. Auxiliary results

We assume always below that $\xi, \xi_1, \xi_2, \dots$ is a sequence of independent identically distributed random variables such that

$$E\xi < 0, \quad S_k = \sum_{i=1}^{k} \xi_i, \qquad k \geq 1, \qquad S_0 = 0, \quad M = \max_{k \geq 0} S_k.$$

Lemma 1. The following properties hold:
(a) $P\{M < \infty\} = 1$, $P\{M = 0\} > 0$ and

$$M \stackrel{law}{=} (M + \xi)^+.$$

(b) Let $\sigma_a = \inf\{k \geq 0 : S_k \geq a\}$, $a \geq 0$, and $Ee^{\lambda M} < \infty$ for some $\lambda \in \mathbf{R}$. Then for all $u \leq \lambda$

$$E\, e^{\lambda(M - S_{\sigma_a})} e^{uS_{\sigma_a}} I\{\sigma_a < \infty\} = Ee^{\lambda M}\, E\, e^{uS_{\sigma_a}} I\{\sigma_a < \infty\}. \tag{10}$$

Proof. Property (a) is well known, see e.g. [5, Section 10.4, Theorem 4] and [2, §15].

The left and right sides of (10) are finite due to the assumption about finiteness of $Ee^{\lambda M}$. The equality (10) is implied by the fact that on the set $\{\sigma_a < \infty\} = \{M \geq a\}$ the inequalities $S_k < S_{\sigma_a}$ with $k < \sigma_a$ hold and, hence, the following equality holds

$$M - S_{\sigma_a} = \sup_{k \geq 0} (S_{k + \sigma_a} - S_{\sigma_a}).$$

Due to time homogeneity of sequence S_n this implies that $M - S_{\sigma_a} \stackrel{law}{=} M$. Besides, note that the random variable $M - S_{\sigma_a}$ does not dependent on events from σ-algebra $F_k = \sigma\{\xi_1, \dots, \xi_k\}$ on the set $\{\sigma_a = k\}$. This implies (10) due to the validity of the following equalities:

$$\begin{aligned} Ee^{\lambda(M - S_{\sigma_a})} e^{uS_{\sigma_a}} I\{\sigma_a < \infty\} &= E \sum_{k=0}^{\infty} E(e^{\lambda(M - S_k)} \mid F_k) e^{uS_{\sigma_a}} I\{\sigma_a = k\} \\ &= Ee^{\lambda M}\, Ee^{uS_{\sigma_a}} I\{\sigma_a < \infty\}. \end{aligned}$$

Lemma 2. *(a)* Let $Ee^{\lambda\xi} < 1$ for some $\lambda > 0$. Then for all $u \leq \lambda$

$$Ee^{uM} < \infty.$$

(b) For all $p > 0$

$$E(\xi^+)^{p+1} < \infty \implies E\, M^p < \infty.$$

Proof See the papers [8], [2], [6], and also [9] concerning upper bounds for $P\{M > x\}$ which also imply (a).

Lemma 3. Let $\tau_a = \inf\{k \geq 0 : X_k \geq a\}$, $a \geq 0$.
(a) If $E\, e^{\lambda\xi} < 1$ for some $\lambda \geq 0$, then for all $a \geq 0$ and $u \leq \lambda$

$$E_x I\{\tau_a < \infty\}\, e^{uX_{\tau_a}} = \frac{E\, I\{M + x \geq a\}\, e^{u(M+x)}}{E\, e^{uM}}. \tag{11}$$

(b) If $E(\xi^+)^{n+1} < \infty$ then for all $a \geq 0$

$$E_x I\{\tau_a < \infty\} X_{\tau_a}^n = E\, I\{M + x \geq a\}\, Q_n(M + x). \tag{12}$$

Proof (a) By Lemma 2 the condition $E\, e^{\lambda\xi} < 1$ implies finiteness of $E\, e^{uM}$ for $u \leq \lambda$.

If $x \geq a$ then $\tau_a = 0$ and relation (11) obviously holds. If $x < a$ then we can apply Lemma 1 (b) with $\lambda = u$:

$$\begin{aligned} E_x I\{\tau_a < \infty\}\, e^{uX_{\tau_a}} E e^{uM} &= E\, I\{\sigma_{a-x} < \infty\}\, e^{u(S_{\sigma_{a-x}} + x)} E e^{uM} \\ &= E\, I\{\sigma_{a-x} < \infty\}\, e^{u(M+x)} = E\, I\{M + x \geq a\}\, e^{u(M+x)}, \end{aligned}$$

and this is equivalent to (11).

(b) Let the condition $E\, e^{\lambda\xi} < 1$ hold for some $\lambda > 0$. Then both parts of (11) are, obviously, differentiable with respect to the parameter $u < \lambda$. Computing the n-th derivative at the point $u = 0$ and using definition (2) of the Appell polynomials, we get (12).

The fact that this relation holds also under the assumption $E(\xi^+)^{n+1} < \infty$, can be proved with the standard trick of "truncation" for jumps as follows.

Together with the original random walk S_k, $k \geq 0$, we consider a random walk $S_k^{(N)}$, $k \geq 0$ generated by random variables $\xi^{(N)} = \min(\xi, N)$, $N = 1, 2, \ldots$. Further we use index N for all functionals which are related to $S_k^{(N)}$, $k \geq 0$ exactly as they were used above for S_k, $k \geq 0$: $M^{(N)} = \sup_{k\geq 0} S_k^{(N)}$, $\tau_a^{(N)} = \inf\{k : X_k^{(N)} \geq a\}$ etc.

By Lemma 2 the maximum $M^{(N)}$ is an exponentially bounded random variable (that is $E \exp\{\lambda M^{(N)}\} < \infty$ for some $\lambda > 0$) and, hence, again by Lemma 2 equation (12) holds for the "truncated" random walk. Therefore, it is sufficient to check that the condition $E(\xi^+)^{n+1} < \infty$ implies as $N \to \infty$

$$M^{(N)} \xrightarrow{d} M, E\, (M^{(N)})^k \to E\, (M)^k, \qquad k = 1, \ldots, n,$$

and

$$E_x\, I\{\tau_a^{(N)} < \infty\}\, \big(X_{\tau_a^{(N)}}^{(N)}\big)^n \to E_x\, I\{\tau_a < \infty\}\, (X_{\tau_a})^n.$$

The validity of these relations is implied by the integrability and monotonicity of the sequences $\{M^{(N)}\}$ and $\{X_{\tau_a^{(N)}}^{(N)}\}$ as $N \to \infty$.

Lemma 4. Let $E\, (\xi^+)^{n+1} < \infty$. Then

$$E\, Q_n(M + x) = x^n. \tag{13}$$

Proof. At first assume that $E\, e^{\lambda\xi} < 1$ for some $\lambda > 0$. Then by Lemma 2 we have $E\, e^{\lambda M} < \infty$, and it follows from the definition of Appell polynomials that

$$e^{ux} = \frac{E e^{u(M+x)}}{E e^{uM}} = \sum_{k=0}^{\infty} \frac{u^k}{k!}\, E Q_k(M + x), \quad 0 \leq u < \lambda$$

which implies (13). The general case can be proved with the help of the technique of "truncated" jumps which has been described above.

Remark 3. The statement of Lemma 4 can be easily generalised to the case of Appell polynomials generated by an arbitrary random variable η such that $E\,|\eta|^n < \infty$:

$$E\,Q_n(\eta + x; \eta) = x^n.$$

Lemma 5. Let $n = 1, 2, \dots$. Then $Q_n(y)$ has a unique positive root a_n^* such that

$$Q_n(y) \le 0 \text{ for } 0 \le y < a_n^* \tag{14}$$

and $Q_n(y)$ is an increasing function for $y \ge a_n^*$.

Proof. For $n = 1$ the statement of this Lemma holds as $Q_1(y) = y - E\,M$, $a_1^* = E\,M > 0$. For $n \ge 2$, using the property (3), we proceed by induction . At first we need to show that $Q_n(0) \le 0$ for all $n = 1, 2, \dots$.

Let $\sigma_a = \inf\{k : S_k \ge a\}$, $a \ge 0$. Set

$$q(a, n) := E\,I\{\tau_a < \infty\} S^n_{\sigma_a}.$$

Obviously, we have $q(a, n) \ge 0$ for all $a \ge 0$. By Lemma 3(b) with $x = 0$

$$q(a, n) = E\,I\{M \ge a\}\,Q_n(M).$$

Since by Lemma 4 $E\,Q_n(M) = 0$, we have

$$q(a, n) = -E\,I\{M < a\}\,Q_n(M) = -P\{M < a\}\,Q_n(0) + E\,I\{M < a\}(Q_n(0) - Q_n(M)).$$

Using (3), we get

$$E\,I\{M < a\}\,|Q_n(M) - Q_n(0)| \le na \max_{0 \le x \le a} |Q_{n-1}(x)|\,P\{M < a\},$$

and so

$$q(a, n) = -P\{M < a\}\,Q_n(0) + o(a) \text{ as } a \to 0.$$

Since $q(a, n) \ge 0$ and $P\{M < a\} \ge P\{M = 0\} > 0$ (see Lemma 1) for all $a \ge 0$, we then have the required inequality $Q_n(0) \le 0$ for all $n = 1, 2, \dots$.

Now rewrite equation (3) as follows:

$$Q_n(y) = Q_n(0) + n \int_0^y Q_{n-1}(u)\,du.$$

Assume that for some $n > 1$ the inequalities $Q_{n-1}(y) \le 0$ with $y \in [0, a_{n-1}^*]$ and $Q_{n-1}(y) > 0$ with $y > a_{n-1}^* > 0$ hold. Then we get that the polynomial $Q_n(y)$ is negative and decreasing on the interval $(0, a_{n-1}^*)$. Obviously, it reaches its minimum at the point $y = a_{n-1}^*$. For $y \ge a_{n-1}^*$ the polynomial $Q_n(y)$ is increasing to infinity and, hence, there exists a root $a_n^* > a_{n-1}^* > 0$. By induction this implies that the statement of Lemma holds for all $n = 1, 2, \dots$.

Lemma 6. Let

$$f(x) = E\,I\{M + x \ge a^*\}\,G(M + x) < \infty,$$

where the function $G(x)$ is such that

$$G(y) \ge G(x) \ge G(a^*) = 0 \tag{15}$$

for all $y \geq x \geq a^* \geq 0$. Then for all x

$$f(x) \geq E\, f(\xi + x). \tag{16}$$

Proof. The inequality (16) is proved by the following chain of inequalities which exploit the property $M \stackrel{law}{=} (M+\xi)^+$ (see Lemma 1):

$$\begin{aligned} f(x) &= E\, I\{(M+\xi)^+ + x \geq a^*\}\, G((M+\xi)^+ + x) = E\, I\{x \geq a^*, M+\xi < 0\}\, G(x) \\ &\quad + E\, I\{M+\xi+x \geq a^*, M+\xi \geq 0\}\, G(M+\xi+x) \\ &\geq E\, I\{M+\xi+x \geq a^*,\ M+\xi < 0\}\, G(M+\xi+x) \\ &\quad + E\, I\{M+\xi+x \geq a^*,\ M+\xi \geq 0\}\, G(M+\xi+x) = E\, f(x+\xi). \end{aligned}$$

Lemma 7. Let $f(x)$ and $g(x)$ be nonnegative functions such that for all x

$$f(x) \geq g(x) \tag{17}$$

and

$$f(x) \geq E f(\xi + x). \tag{18}$$

Then for all x

$$f(x) \geq \sup_{\tau \in M_0^\infty} E\, I\{\tau < \infty\}\, g(S_\tau + x). \tag{19}$$

Proof. Conditions (17) and (18) imply the fact that the function $f(x)$ is an excessive majorant of $g(x)$. Now the required inequality is implied by Doob's theorem about preserving the supermartingale property under random stopping (see e.g. [1], [3], [9]).

3. Proof of Theorem 1

Let $g(x) = (x^+)^n$, $\widehat{g}(x) = x^n$ with the function $\widehat{V}_n(x)$ defined as in (7). At first we demonstrate the validity of (8) that is

$$\widehat{V}_n(x) = \sup_{a \geq 0} E_x\, I\{\tau_a < \infty\} X^n_{\tau_a} = E\, Q_n(M+x)\, I\{M+x \geq a_n^*\}. \tag{20}$$

To prove this equation note that by Lemma 3(b)

$$E_x\, I\{\tau_a < \infty\}\, X^n_{\tau_a} = E\, Q_n(M+x)\, I\{M+x \geq a\},$$

where $Q_n(M+x) \geq 0$ on the set $\{M+x \geq a\}$ for all $a \in [a_n^*, \infty)$. Hence, $EQ_n(M+x)\, I\{M+x \geq a\}$ is a decreasing function on the interval $[a_n^*, \infty)$.

Now let $a \in [0, a_n^*]$. From (13) it follows that

$$\begin{aligned} E\, Q_n(M+x)\, I\{M+x \geq a\} &= x^n - E\, Q_n(M+x)\, I\{M+x < 0\} \\ &\quad - E\, Q_n(M+x)\, I\{0 \leq M+x < a\}. \end{aligned}$$

Exploiting the fact that $Q_n(M+x)\, I\{0 \leq M+x \leq a\} \leq 0$ (see Lemma 5) and, hence, that $E\, Q_n(M+x)\, I\{0 \leq M+x < a\}$ is a decreasing function, we conclude that $E\, Q_n(M+x)\, I\{M+x \geq a\}$ is an increasing function on the interval $[0, a_n^*]$. As this function is also decreasing on $[a_n^*, \infty)$ (as was shown above) and is, obviously, continuous (by properties of Lebesgue integrals) for all a, then it achieves its maximum at the point $a = a_n^*$. Thus, (20) (as well as (8)) is proved.

To complete the proof we need only to check the inequality (9), that is show that $\widehat{V}_n(x) \geq V_n(x)$ (it has been already noted in Section 1 that the opposite inequality is valid). To demonstrate this we consider the function

$$f(x) = \widehat{V}_n(x) = E\, I\{M + x \geq a_n^*\}\, Q_n(M + x)$$

and apply Lemma 7 with $g(x) = (x^+)^n$. At first let us check condition (17) for $x \in (0, a_n^*)$ (otherwise, it is obvious). Note that for all $x \in (0, a_n^*)$ by Lemma 5

$$I\{M + x \geq a_n^*\}\, Q_n(M + x) = (Q_n(M + x))^+.$$

Hence, by Jensen's inequality and Lemma 4

$$f(x) = E\,(Q_n(M + x))^+ \geq (E\, Q_n(M + x))^+ = (x^+)^n = g(x).$$

Condition (18) in Lemma 7 holds with $G(y) = Q_n(y)$ by Lemma 6.

So, the function $f(x)$ is an excessive majorant of $g(x) = (x^+)^n$ and, hence, $f(x) \geq V_n(x)$. But $f(x) = \widehat{V}_n(x)$, and, hence, $\widehat{V}_n(x) \geq V_n(x)$.

Theorem 1 is proved.

4. Some remarks

Remark 4. The method of the proof of Theorem 1 may be used for other reward functions $g(x)$. To see an example, consider the following result.

Theorem 2. Let $E\,\xi < 0$ and $g(x) = 1 - e^{-x^+}$. Set

$$a^* = -\ln E e^{-M}.$$

The the stopping time

$$\tau_{a^*} = \inf\{k \geq 0 : X_k \geq a^*\}$$

is optimal:

$$V(x) = \sup_{\tau \in M_0^\infty} E_x\, g(X_\tau)\, I\{\tau < \infty\} = E_x\, g(X_{\tau_{a^*}})\, I\{\tau_{a^*} < \infty\};$$

furthermore,

$$V(x) = E\Big(1 - \frac{e^{-M-x}}{Ee^{-M}}\Big)^+.$$

Proof. We present here only a sketch of the proof as it is similar to the proof of Theorem 1 (and even simpler).

Let $g(x) = 1 - e^{-x^+}$ and

$$\widehat{V}(x) = \sup_{a \geq 0} E_x\, I\{\tau_a < \infty\}\, g(X_{\tau_a}^n). \tag{21}$$

At first let us show that

$$\widehat{V}(x) = E\Big(1 - \frac{e^{-M-x}}{Ee^{-M}}\Big)^+. \tag{22}$$

To see this, note that by Lemma 3(a) with $u = -1$ the following equation holds

$$E_x I\{\tau_a < \infty\}\, e^{-X_{\tau_a}} = \frac{E\, I\{M + x \ge a\}\, e^{-(M+x)}}{E\, e^{-M}},$$

and, hence,

$$q(a) := E_x I\{\tau_a < \infty\}\, g(X_{\tau_a}) = E\, I\{M + x \ge a\} \Big(1 - \frac{e^{-(M+x)}}{E\, e^{-M}}\Big).$$

Note also that the function $1 - e^{-a}/Ee^{-M}$ is monotone in the argument a with the unique root $a^* = -\ln Ee^{-M}$. The same considerations as were used in the proof of Theorem 1 demonstrate that $q(a)$ achieves its maximum at $a = a^*$ and

$$\widehat{V}(x) = E\, I\{M + x \ge a^*\}\Big(1 - \frac{e^{-(M+x)}}{E\, e^{-M}}\Big) = E\Big(1 - \frac{e^{-M-x}}{E\, e^{-M}}\Big)^+.$$

To complete the proof we need only check inequality (9), that is to show $\widehat{V}(x) \ge V(x)$. To do it consider the function

$$f(x) = \widehat{V}(x) = E\Big(1 - \frac{e^{-M-x}}{Ee^{-M}}\Big)^+.$$

Condition (17) holds due to Jensen inequality:

$$f(x) \ge \Big(1 - \frac{E\, e^{-M-x}}{E\, e^{-M}}\Big)^+ = (1 - e^{-x})^+ = g(x).$$

Condition (18) holds for $G(y) = (1 - e^{-y}/Ee^{-M})^+$ by Lemma 6. That completes the proof of Theorem 2.

Remark 5. A solution of the optimal stopping problem on the finite interval $\{0, 1, \ldots, T\}$, with the "value" function

$$V(x, T) = \sup_{\tau \in M_0^T} E_x\, g(X_\tau), \qquad x \in \mathbf{R},$$

where the supremum is taken over all stopping times τ, $\tau \le T < \infty$, can be obtained numerically using the well known method of "backward induction" (see details e.g. in [4], [11]). For large T a realisation of this method could require a huge amount of calculations even for simple distributions. Therefore, it is of interest to estimate a *rate of convergence* of $V(x, T)$ as $T \to \infty$ to the function $V(x)$, described in Theorems 1 and 2.

Theorem 3. Let $g(x) = (x^+)^n$, $n = 1, 2, \ldots$ or $g(x) = 1 - e^{-x^+}$, and let $Ee^{\lambda\xi} < 1$ for some $\lambda > 0$. Then there exist constants $C(x)$ and c, which do not depend on T and such that for each $x \in \mathbf{R}$ and all $T > 0$

$$0 \le V(x) - V(x, T) \le C(x)\, e^{-cT}. \tag{23}$$

Proof. Since the class M_0^∞ is larger than the class M_0^T, then $V(x) \ge V(x, T)$. To obtain the upper bound (23), note that

$$V(x, T) \ge E_x g(X_{\min(\tau_b, T)}) \ge E_x g(X_{\tau_b})\, I\{\tau_b \le T\},$$

where $\tau_b = \inf\{k \geq 0 : X_k \geq b\}$ is the optimal stopping time for the value function $V(x)$ with (by Theorem 1 or 2) the parameter b being equal to a_n^* or a^* accordingly to the form of the reward function $g(x)$. As $V(x) = E_x\, g(X_{\tau_b})\, I\{\tau_b < \infty\}$, we obtain

$$V(x) - V(x,T) \leq E_x\, g(X_{\tau_b})\, I\{T < \tau_b < \infty\}. \tag{24}$$

Below we shall show that

$$P_x\{T < \tau_b < \infty\} \leq C(x) e^{-cT}, \tag{25}$$

where $C(x)$ and c are some positive constants not depending on T. When the function $g(x)$ is bounded (as in Theorem 2), (25) and (24) imply immediately (23). For the case $g(x) = (x^+)^n$, $n = 1, 2, \ldots$ the same bound (25) in a combination with Lemma 2(b) and Holder's inequality implies (23) due to the inequality $X_{\tau_b} \leq M$.

Let $\psi(u)$ be defined by the following equation

$$Ee^{u\xi} = e^{\psi(u)} \qquad (u < \theta = \sup\{u \geq 0 : \psi(u) < \infty\}).$$

Note that $\psi(u)$ is a convex function and due to the conditions of Theorem 3 we have $\theta > 0$. These properties imply that the derivative $\psi'(0) = E\xi < 0$ and there exists positive $\lambda_0 \in (0, \theta)$ such that $\psi'(\lambda_0) = 0$ and $\psi(\lambda_0) < 0$ (see e.g. [2]). To prove (25), one can use the fact that the process

$$\exp\{\lambda(X_k - x) - k\psi(\lambda)\}, \qquad k \geq 0 \quad (\lambda \in (0, \theta))$$

is a nonnegative martingale with expectation

$$E \exp\{\lambda(X_k - x) - k\psi(\lambda)\} = 1.$$

Therefore, it can be used to construct a new measure $\widehat{P}_x(A)$, $A \in \sigma(\bigcup_{k \geq 0} F_k)$ such that for each $k \geq 0$ and a set $A \in F_k$

$$\widehat{P}_x(A) = E_x\, I\{A\}\, \exp\{\lambda_0(X_k - x) - k\psi(\lambda_0)\}. \tag{26}$$

(This is nothing else but the well-known Esscher transformation.) With the notation given above we get

$$\begin{aligned} &E_x\, I\{T < \tau_b < \infty\}\, \exp\{\lambda_0(X_{\tau_b} - x) - \tau_b \psi(\lambda_0)\} \\ &= \sum_{k=T+1}^{\infty} E_x\, I\{\tau_b = k\}\, \exp\{\lambda_0(X_k - x) - k\psi(\lambda_0)\} \\ &= \sum_{k=T+1}^{\infty} \widehat{P}_x\{\tau_b = k\} = \widehat{P}_x\{T < \tau_b < \infty\}. \end{aligned}$$

Since $X_{\tau_b} \geq b$, then

$$P_x\{T < \tau_b < \infty\} \leq e^{\lambda_0(x-b)} e^{\psi(\lambda_0)T}\, \widehat{P}_x\{T < \tau_b < \infty\}.$$

Taking into account that $\widehat{P}_x\{T < \tau_b < \infty\} \leq 1$, we get bound (25) with constants $c = -\psi(\lambda_0) > 0$, $C(x) = e^{\lambda_0(x-b)}$, $b = a_n^*$ or $b = a^*$ accordingly to the form of function $g(x)$.

Theorem 3 is proved.

Remark 6. The results of this paper can be generalised to the case of stochastic processes with continuous time parameter (that is for Levy processes instead of the random walk). This generalisation can be done by passage of limit from the discrete time case (similarly to the technique used in [10] for pricing of American options) or by use of the technique of pseudo-differential operators (described e.g. in the monograph [3] in the context of Levy processes).

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