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A Two-Factor Model for Low Interest Rate Regimes

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A Two-Factor Model for Low Interest Rate Regimes

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Abstract. This paper derives a two factor model for the term structure of interest rates that segments the yield curve in a natural way. The first factor involves modelling a non-negative short rate process that primarily determines the early part of the yield curve and is obtained as a truncated Gaussian short rate. The second factor mainly influences the later part of the yield curve via the market index. The market index proxies the growth optimal portfolio (GOP) and is modelled as a squared Bessel process of dimension four. Although this setup can be applied to any interest rate environment, this study focuses on the difficult but important case where the short rate stays close to zero for a prolonged period of time. For the proposed model, an equivalent risk neutral martingale measure is neither possible nor required. Hence we use the benchmark approach where the GOP is chosen as numeraire. Fair derivative prices are then calculated via conditional expectations under the real world probability measure. Using this methodology we derive pricing functions for zero coupon bonds and options on zero coupon bonds. The proposed model naturally generates yield curve shapes commonly observed in the market. More importantly, the model replicates the key features of the interest rate cap market for economies with low interest rate regimes. In particular, the implied volatility term structure displays a consistent downward slope from extremely high levels of volatility together with a distinct negative skew.

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1 Introduction

A wide range of literature exists on *interest rate term structure* (IRTS) modelling. For recent accounts of this area one can refer to Rebonato (1998), James & Webber (2000) and Brigo & Mercurio (2001b). This paper aims to provide a model for markets where one observes prolonged periods of low interest rates. This phenomenon was observed in Switzerland in the second half of the 1970s and during the past ten years in Japan. In the latter case, the Bank of Japan had to implement a Zero Interest Rate Policy, see Marumo, Nakayama, Nishioka & Yosida (2003) and Shiratsuka & Fujiki (2002). The *two factor term structure model* that we propose is an extension of the minimal market model derived in Platen (2002, 2004). It segments the IRTS naturally into the early section of the yield curve with dynamics according to a short rate model, and the medium to long end of the yield curve that depends on the discounted market index, as shown in Platen (2003). The discounted market index is completely determined by its volatility, which arises as the total market price for risk. Yield curve shapes typically observed are easily replicated by the proposed model using constant parameters.

We employ the fair pricing methodology developed in Platen (2002) under which the *growth optimal portfolio* (GOP) is used as the numeraire or *benchmark*. Derivative price processes denominated in units of the GOP are martingales under the real world probability measure. Thus, contingent claim prices can be calculated via conditional expectations using the *real world measure*. Note that an equivalent risk neutral martingale measure does not exist under the proposed model. Using this approach, the general specification of the proposed model can be applied to any interest rate environment. However, the focus of this study is the case where the short rate remains close to zero for a prolonged period of time, as mentioned above. Under such a low interest rate regime, the calculation of vanilla interest rate derivatives can be challenging and calibration to an entire market without the aid of time dependent parameters is extremely difficult. As an indication of the increase in complexity, the proximity of the short rate to zero results in a strictly positive probability for negative interest rates under many models. This can lead to substantial pricing errors for interest rate derivatives, as explained in Rogers (1996) for standard Gaussian models. One way to avoid this type of problem is to specify the market short rate as the non-negative part of a shadow short rate, as suggested in Black (1995) and Rogers (1995). Here the shadow short rate represents the economically optimal short rate that is allowed to take negative values. In this study we examine the use of Gaussian and square root processes to model the shadow short rate. We add a time dependent deterministic shift that is indirectly used to match the initial yield curve. The few remaining parameters are kept constant to form a class of parsimonious two-factor IRTS models. In each case we derive prices for options on zero coupon bonds and hence prices for interest rate caps and floors. For the truncated Gaussian version of the model we find that the stylised empirical features of liquid interest rate derivative markets are accurately replicated for standard and low interest rate regimes. More specifically, the model reproduces the key characteristics of the implied cap volatility term structure, namely the negative skew and downward slope from extremely high levels of volatility associated with low interest rate environments.

The paper is organised as follows. Section 2 summarises the benchmark approach for IRTS modelling. Section 3 presents the proposed model. Section 4 shows how to calculate European options on zero coupon bonds and documents properties of interest rate caps calculated under the model.

2 Benchmark Model

2.1 Growth Optimal Portfolio

Let us introduce a two-factor financial market model for the IRTS as a particular case of the *continuous, complete benchmark model* proposed in Platen (2002). In this framework uncertainty is modelled by two independent standard Wiener processes $W^k = \{W_t^k,\, t \in [0,T]\}$, $k \in \{1,2\}$. These are defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$, with finite time horizon $T \in (0,\infty)$ and filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ fulfilling the usual conditions, see Karatzas & Shreve (1991). Here $\mathcal{A}_t$ denotes the information available to the market at time $t \in [0,T]$.

Let us denote by B_t the value of the savings account at time $t \in [0,T]$, where we assume the dynamics

$$dB_t = B_t\, r_t\, dt \tag{2.1}$$

for $t \in [0,T]$ with initial value $B_0 = 1$. Here $r = \{r_t,\, t \in [0,T]\}$ is the adapted short rate process that will be specified below.

We select the GOP as benchmark or numeraire. It is well-known that the GOP value S_t at time t satisfies a stochastic differential equation (SDE) of the form

$$dS_t = S_t \left\{ r_t\, dt + \sum_{k=1}^{2} \theta_t^{(k)} \left(\theta_t^{(k)}\, dt + dW_t^k \right) \right\} \tag{2.2}$$

for $t \in [0,T]$ with finite initial value $S_0 > 0$, see Long (1990), Karatzas & Shreve (1998) or Platen (2002). Note that the kth volatility $\theta_t^{(k)}$ of the GOP is also the kth market price for risk, $k \in \{1,2\}$ and $t \in [0,T]$. In particular, $\theta_t^{(1)}$ models the market price for risk with respect to the Wiener process W^1, which is assumed to drive the stochastic fluctuations of the short rate. The market price for risk $\theta_t^{(2)}$ relates to the Wiener process W^2, which models the fluctuations of the GOP that are independent of short rate behaviour. From (2.2) one observes that the

GOP dynamics are completely characterized by the short rate and these market prices for risk.

In Figure 1 we plot the logarithm of the normalised *World Stock Accumulation Index* (WSI) denominated in Japanese Yen (JPY) from 1949 to 2003, as provided by Global Financial Data. According to Platen (2004), a well diversified global portfolio, such as the MSCI World Equity Accumulation Index, is a suitable proxy for the GOP. Thus the WSI shown in Figure 1 can also be interpreted as an approximation to the GOP.

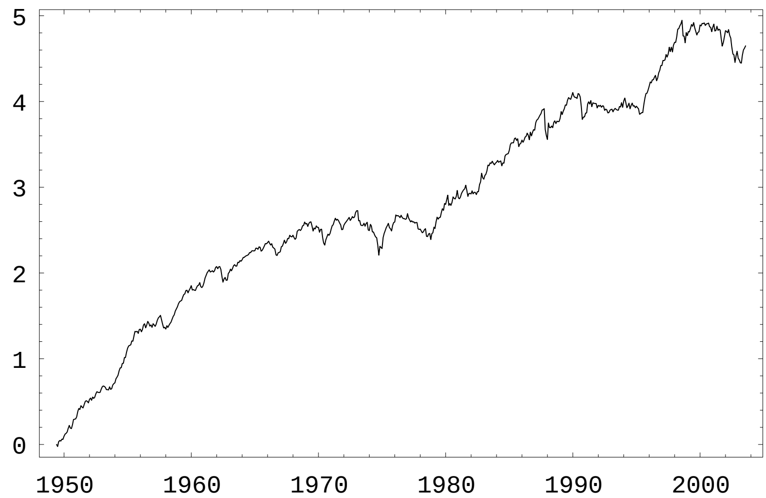

Figure 1: Logarithm of World Stock Accumulation Index denominated in JPY from 1949 to 2003.

To study the relationship between fluctuations in the GOP and the short rate, we examine the covariation $\langle \ln(S), r \rangle_t$, see Karatzas & Shreve (1991). The calculated covariation presented in Figure 2 is based on daily data for the JPY short rate from 1960 to 2003 and the corresponding data for the WSI from Figure 1. Notice that the values taken by the covariation process are extremely small. Therefore, it is reasonable to assume that the driving noise source of the GOP and that of the short rate are independent. To be precise, we make the following assumption.

Assumption 2.1 *We assume that the market price for short rate risk is zero, that is,* $\theta_t^{(1)} = 0$ *for all* $t \in [0, T]$.

Hence, we have set the *total market price for risk* $|\theta_t|$ at time t to

$$|\theta_t| = |\theta_t^{(2)}| = \theta_t^{(2)} \tag{2.3}$$

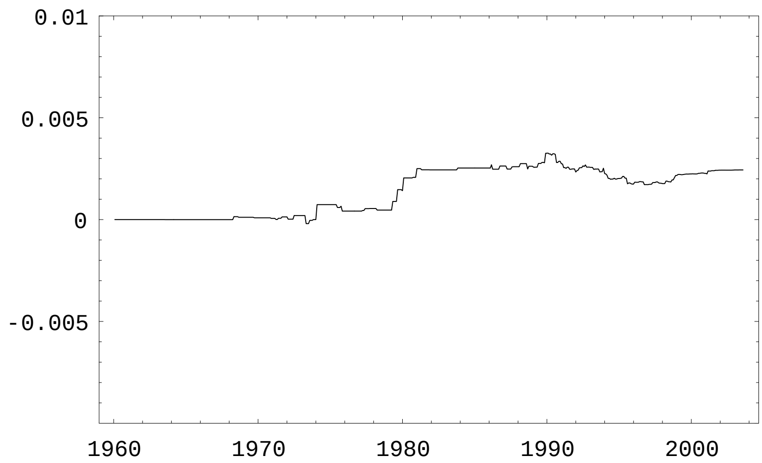

Figure 2: Covariation between the logarithm of World Stock Accumulation Index and the JPY short rate from 1960 to 2003.

for $t \in [0, T]$. It then follows by (2.1), (2.2) and (2.3) that the *discounted GOP*

$$\bar{S}_t = \frac{S_t}{B_t} \tag{2.4}$$

satisfies the SDE

$$d\bar{S}_t = \bar{S}_t \, |\theta_t| \left(|\theta_t| \, dt + dW_t^2 \right) \tag{2.5}$$

for $t \in [0, T]$. Thus, the short rate and its noise does not appear directly in the discounted GOP. By discounting the GOP we have separated the impact of the short rate from that of the total market price for risk.

2.2 Fair Pricing

The proposed two-factor term structure model cannot be accommodated under the standard risk neutral framework. Therefore, we use the *benchmark approach* outlined in Platen (2002, 2004). Under this approach the GOP is the numeraire or benchmark and prices denominated in units of the GOP are referred to as *benchmarked prices*. The choice of the GOP as numeraire allows us to price derivatives using conditional expectations with respect to the real world probability measure P, see Long (1990). This widens the range of models from which to choose since the existence of an equivalent risk neutral martingale measure is not required. In the given setup, any benchmarked self-financing portfolio satisfies a driftless SDE and is therefore an $(\underline{\mathcal{A}}, P)$-local martingale. Thus, all non-negative, self-financing benchmarked portfolio processes are $(\underline{\mathcal{A}}, P)$-supermartingales, see Revuz & Yor (1999). This ensures that the resulting price system of securities does not permit arbitrage as defined in Platen (2002).

Definition 2.2 *A price process* $U = \{U_t,\, t \in [0,T]\}$*, with* $E\left(\frac{|U_t|}{S_t}\right) < \infty$ *for* $t \in [0,T]$*, is called* fair *if the corresponding benchmarked price process* $\hat{U} = \{\hat{U}_t = \frac{U_t}{S_t},\, t \in [0,T]\}$ *forms an* $(\underline{\mathcal{A}}, P)$*-martingale.*

For a fair price process U, the last observed benchmarked value $\hat{U}_t$ is the best forecast for any future benchmarked values $\hat{U}_{\bar{T}}$, that is, $\hat{U}_t = E\left(\hat{U}_{\bar{T}} \,\middle|\, \mathcal{A}_t\right)$ for all $\bar{T} \in [0,T]$ and $t \in [0,\bar{T}]$.

Definition 2.3 *We define a* contingent claim $H_{\bar{T}}$ *that matures at a stopping time* $\bar{T} \in [0,T]$ *as an* $\mathcal{A}_{\bar{T}}$*-measurable, non-negative payoff with* $E\left(\frac{H_{\bar{T}}}{S_{\bar{T}}} \,\middle|\, \mathcal{A}_t\right) < \infty$ *for all* $t \in [0,\bar{T}]$*.*

As shown in Platen (2002), the minimal price process $U^{H_{\bar{T}}} = \{U_t^{H_{\bar{T}}},\, t \in [0,\bar{T}]\}$ that replicates the contingent claim $H_{\bar{T}}$, is given by the conditional expectation

$$U_t^{H_{\bar{T}}} = S_t\, E\left(\hat{H}_{\bar{T}} \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{S_t}{S_{\bar{T}}}\, H_{\bar{T}} \,\middle|\, \mathcal{A}_t\right) \tag{2.6}$$

for $t \in [0,\bar{T}]$. Subject to certain integrability conditions the process $\hat{U}^{H_{\bar{T}}} = \{\hat{U}_t^{H_{\bar{T}}} = \frac{U_t^{H_{\bar{T}}}}{S_t},\, t \in [0,\bar{T}]\}$ is an $(\underline{\mathcal{A}}, P)$-martingale and thus by Definition 2.2 $U^{H_{\bar{T}}}$ constitutes a fair price process. Note that there may be other self-financing prices processes that replicate the contingent claim $H_{\bar{T}}$. However, the fair price process is the minimal replicating price process. We call formula (2.6) the *fair pricing formula*. In the following subsection we show that the fair pricing concept generalises classical risk neutral pricing.

2.3 Risk Neutral Pricing

The Radon-Nikodym derivative process $\Lambda^\theta = \{\Lambda_t^\theta,\, t \in [0,T]\}$ for the candidate risk neutral measure P_θ with

$$\Lambda_t^\theta = \left.\frac{dP_\theta}{dP}\right|_{\mathcal{A}_t} = \frac{B_t}{S_t}\,\frac{S_0}{B_0} = \frac{\hat{B}_t}{\hat{B}_0} = \frac{\bar{S}_0}{\bar{S}_t} \tag{2.7}$$

for $t \in [0,T]$ equals, up to a constant factor, the benchmarked savings account or the inverse of the discounted GOP. Using the WSI shown in Figure 1 and the observed JPY short rate, we plot in Figure 3 the Radon-Nikodym derivative (2.7) denominated in JPY from 1949 to 2003.

Notice that the Radon-Nikodym derivative process seems to decline systematically, and is therefore unlikely to represent the trajectory of a martingale. The apparent systematic decline depicted in Figure 3 suggests that the Radon-Nikodym

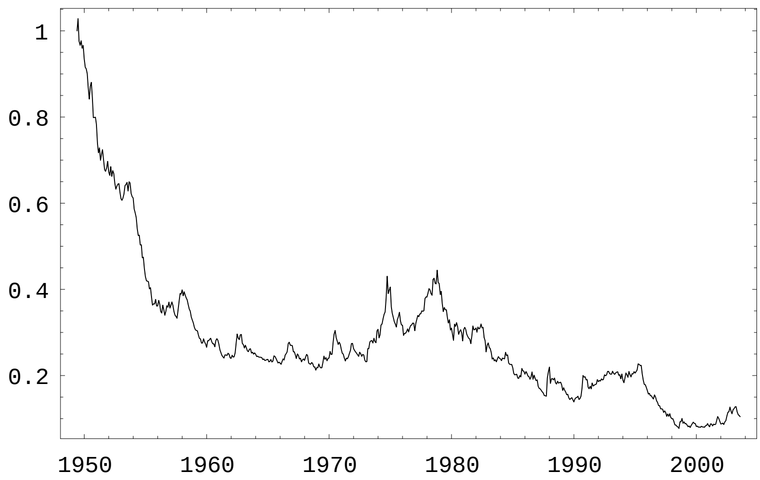

Figure 3: Radon-Nikodym derivative denominated in JPY from 1949 to 2003.

derivative process is likely to be a non-negative, strict local martingale and thus a strict *supermartingale*. Recall that the Radon-Nikodym derivative can be interpreted as the inverse of the discounted stock market index. Since average returns on long term investments in stock indices are consistently greater than the short rate, the decline in the Radon-Nikodym derivative in Figure 3 is not surprising. Moreover, if one examines the Radon-Nikodym derivative for the candidate risk neutral measure in other currencies, then the indicated supermartingale property is usually much more pronounced than in Figure 3, see Platen (2004). Therefore, the key condition of standard risk neutral pricing where one assumes that the Radon-Nikodym derivative process is an $(\underline{\mathcal{A}}, P)$-martingale, appears to be unrealistic.

However, for models where the risk neutral measure P_θ and the real world measure P are equivalent and the Radon-Nikodym derivative process Λ^θ is an $(\underline{\mathcal{A}}, P)$-martingale, the fair pricing formula (2.6) can be rewritten with (2.7) to recover the standard *risk neutral pricing formula*

$$U_t^{H_{\bar{T}}} = E\left(\frac{\Lambda_{\bar{T}}^\theta}{\Lambda_t^\theta}\,\frac{B_t}{B_{\bar{T}}}\,H_{\bar{T}}\,\middle|\,\mathcal{A}_t\right) = E_\theta\left(\frac{B_t}{B_{\bar{T}}}\,H_{\bar{T}}\,\middle|\,\mathcal{A}_t\right) \tag{2.8}$$

for $t \in [0, \bar{T}]$, see Karatzas & Shreve (1998). Here E_θ denotes expectation with respect to the candidate risk neutral measure P_θ.

3 Interest Rate Term Structure Model

3.1 Zero Coupon Bond Prices and Forward Rates

The fair price $P(t,\bar{T})$ of a *zero coupon bond* at time t is the value of one unit of domestic currency paid at the fixed maturity date $\bar{T} \in [0,T]$. Hence, from the fair pricing formula (2.6) we find that

$$P(t,\bar{T}) = E\left(\frac{S_t}{S_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right) = E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\,\frac{B_t}{B_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right) \tag{3.1}$$

for $t \in [0,\bar{T}]$. Recall Assumption 2.1 which implies that the Wiener processes driving the GOP and the short rate are independent. Hence the calculation of the fair zero coupon bond price in (3.1) simplifies to the product

$$P(t,\bar{T}) = M_{\bar{T}}(t,\bar{S}_t)\,G_{\bar{T}}(t,r_t) \tag{3.2}$$

for $t \in [0,\bar{T}]$. Here the *market price for risk contribution* to the bond price is defined by

$$M_{\bar{T}}(t,\bar{S}_t) = E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right) \tag{3.3}$$

and the *short rate contribution* to the bond price is given as

$$G_{\bar{T}}(t,r_t) = E\left(\frac{B_t}{B_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right) \tag{3.4}$$

for $t \in [0,\bar{T}]$.

The instantaneous *forward rate* $f(t,\bar{T})$ at time t for the maturity date $\bar{T} \in [0,T]$ is expressed as

$$f(t,\bar{T}) = -\frac{\partial}{\partial \bar{T}}\ln(P(t,\bar{T})) \tag{3.5}$$

for all $t \in [0,\bar{T}]$. Therefore, combining (3.5) with (3.2), the forward rate takes the form

$$f(t,\bar{T}) = m_{\bar{T}}(t,\bar{S}_t) + g_{\bar{T}}(t,r_t) \tag{3.6}$$

for all $t \in [0,\bar{T}]$. Here the *market price for risk contribution* to the forward rate is

$$m_{\bar{T}}(t,\bar{S}_t) = -\frac{\partial}{\partial \bar{T}}\ln\left(M_{\bar{T}}(t,\bar{S}_t)\right) \tag{3.7}$$

and the *short rate contribution* to the forward rate is

$$g_{\bar{T}}(t,r_t) = -\frac{\partial}{\partial \bar{T}}\ln\left(G_{\bar{T}}(t,r_t)\right) \tag{3.8}$$

for $t \in [0,\bar{T}]$. In the following we will specify our model in a natural way that allows us to calculate fair zero coupon bond prices and forward rates.

3.2 Discounted GOP Dynamics

We observe in (3.3) that the market price for risk contribution to the zero coupon bond price depends on the discounted GOP process $\bar{S} = \{\bar{S}_t,\, t \in [0,T]\}$. As in Platen (2004) we introduce the *discounted GOP drift*

$$\alpha_t = \bar{S}_t \, |\theta_t|^2 \tag{3.9}$$

for $t \in [0,T]$. The quantity α_t can be interpreted as the average increase in the discounted market index per unit of time. Therefore, the discounted GOP drift is likely to be a growth process. Using the parameter specification $\alpha = \{\alpha_t,\, t \in [0,T]\}$ given in (3.9) leads to a total market price for risk of the form

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t}}. \tag{3.10}$$

Therefore, combining (2.5), (3.9) and (3.10) we obtain for the discounted GOP the SDE

$$d\bar{S}_t = \alpha_t dt + \sqrt{\alpha_t \, \bar{S}_t} \, dW_t^2 \tag{3.11}$$

for $t \in [0,T]$. This is a time transformed squared Bessel process of dimension four, see Revuz & Yor (1999). For the Radon-Nikodym derivative Λ_t^θ we obtain from (2.7), (3.11) and the Itô formula the SDE

$$d\Lambda_t^\theta = -\Lambda_t^\theta \, |\theta_t| \, dW_t^2 \tag{3.12}$$

for $t \in [0,T]$. It is well known that the inverse of a squared Bessel process of dimension four is a *strict supermartingale*, see Revuz & Yor (1999). Thus, in this case the Radon-Nikodym derivative process is *not* a martingale, which is consistent with the observed sample path in Figure 3. Therefore, we emphasize that a model which assumes that the Radon-Nikodym derivative Λ_t^θ is a martingale, is in our view not very realistic.

The quadratic variation of the square root of the discounted GOP is

$$\left\langle \sqrt{\bar{S}} \right\rangle_t = \frac{1}{4} \int_0^t \alpha_s \, ds \tag{3.13}$$

for $t \in [0,T]$. In Figure 4 we plot the empirical quadratic variation of the square root of the discounted GOP denominated in JPY from 1949 to 2003. One observes that the slope of the empirical quadratic variation increases steadily over time. This feature motivates us to model the discounted GOP drift α_t as a deterministic function that increases over time in a similar manner. More precisely, we make the following assumption.

Assumption 3.1 *The discounted GOP drift α_t is assumed to equal*

$$\alpha_t = \alpha_0 \exp\{\eta\, t\} \tag{3.14}$$

for $t \in [0, T]$, where the constant η is called the net growth rate.

Using (3.13) and (3.14), it is easy to calculate the theoretical expression for the quadratic variation $\langle\sqrt{\bar{S}}\rangle_t$ in our model. We found that a good long term theoretical fit was obtained with the parameters $\alpha_0 = 0.02$ and $\eta = 0.05$, as shown by the dashed line in Figure 4. Note that while the initial value of the discounted GOP drift α_0 is important for calibration to historical equity data, the interest rate derivatives priced in this paper are not affected by this parameter.

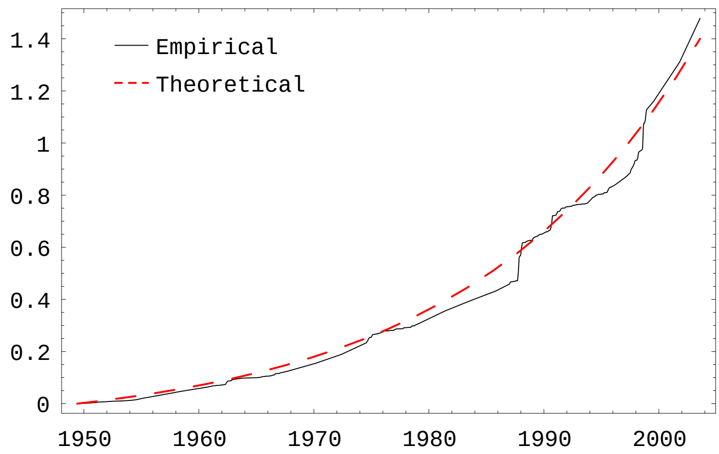

Figure 4: Empirical quadratic variation of the square root of the discounted GOP denominated in JPY from 1949 to 2003 and theoretical fit.

Under Assumption 3.1 the discounted GOP process, as a time transformed squared Bessel process of dimension four, can never reach zero under the real world probability measure P, see Revuz & Yor (1999). However, it will be absorbed at zero with strictly positive probability under the candidate risk neutral measure P_θ because under this measure the discounted GOP is a time transformed squared Bessel process of dimension zero. This reinforces the view that for a parsimonious, realistic model the candidate risk neutral measure P_θ may *not* be equivalent to the real world measure P.

3.3 Market Price for Risk Contribution

Using (3.3), (3.11) and the explicitly known transition density of a squared Bessel process of dimension four, one obtains the market price for risk contribution to the zero coupon bond price in the form

$$M_{\bar{T}}(t, \bar{S}_t) = 1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_t}{\alpha_{\bar{T}} - \alpha_t}\right\} \tag{3.15}$$

for $t \in [0, \bar{T}]$, see Platen (2003). We plot the market price for risk contribution to the zero coupon bond price in Figure 5, where we set $\alpha_0 = 0.02$, $\eta = 0.05$ and $|\theta_0| = 0.2$ to determine $\bar{S}_0$ by rearranging (3.10). With this realistic parameter set, we observe that $M_{\bar{T}}(0, \bar{S}_0) \approx M_0(0, \bar{S}_0)$ for maturities of up to about five years. Beyond this initial period, the market price for risk contribution starts to decrease markedly in value. Therefore, the market price for risk contribution does not impact the zero coupon bond price (3.2) until the mid to long end of the yield curve. This property of the fair zero coupon bond price, which is not observed under any risk neutral IRTS model, results from the fact that the discounted GOP is a strict $(\underline{\mathcal{A}}, P)$-local martingale.

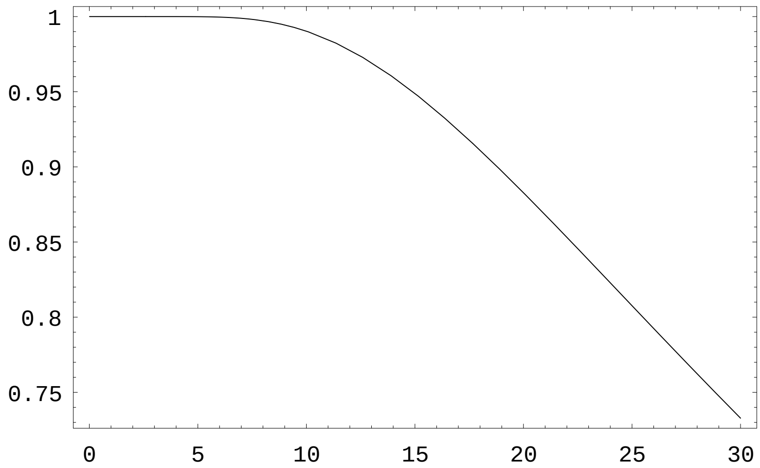

Figure 5: Market price for risk contribution to the zero coupon bond price.

The corresponding market price for risk contribution to the forward rate is calculated using (3.7) and is found to be

$$m_{\bar{T}}(t, \bar{S}_t) = \frac{2\,\eta^2\,\alpha_{\bar{T}}\,\bar{S}_t}{(\alpha_{\bar{T}} - \alpha_t)^2\left(\exp\left\{\frac{2\,\eta\,\bar{S}_t}{\alpha_{\bar{T}} - \alpha_t}\right\} - 1\right)} \tag{3.16}$$

for $t \in [0, \bar{T}]$. Note that the only random quantity in (3.16) is the discounted GOP value $\bar{S}_t$ at time t. In Figure 8 below the impact of the market price for risk contribution to the forward rate is shown

implicitly as the difference between the market forward rate and the implied short rate contribution. As should be expected, the market price for risk contribution only impacts the IRTS beyond maturities of five years, as seen in Figure 5.

3.4 The Shadow and Market Short Rates

To obtain results for the zero coupon bond price (3.2) one needs to identify the short rate contribution (3.4). This provides motivation to search for a suitable simple short rate model. The existing literature on short rate models is extensive. For recent surveys on short rate models we refer to Rebonato (1998), James & Webber (2000) and Brigo & Mercurio (2001b). The benchmark framework generalises the risk neutral approach and therefore admits any of the known short rate models. Most models focus on medium levels of the short rate, ranging typically between 2 and 15%. However, we seek a model that also has the potential to adequately describe the dynamics of low interest rate regimes where the short rate is close to zero for prolonged periods of time.

An important model for low interest rate environments was proposed in Black (1995) using the concept of interest rates as options. Black defines the *market short rate* r_t at time t as the positive part of a *shadow short rate* $\psi_t \in \Re$, thereby producing the call option payoff

$$r_t = \max(\psi_t, 0) \tag{3.17}$$

for all $t \in [0, T]$. The shadow short rate ψ_t can be interpreted as an economically optimal short rate. More precisely, the shadow short rate determines the market clearing equilibrium point between supply of and demand for instantaneous borrowing and investing. Hence it is reasonable to allow the shadow short rate to become negative. However, if the shadow short rate ψ_t is negative for a long period of time, then the Central Bank has no alternative other than to set the market short rate r_t either at or close to zero. Otherwise, investors would prefer to hold physical currency. This feature of real markets potentially explains short rate scenarios that remain close to zero for extended periods of time. As an example, prolonged deflation could cause a negative shadow short rate. One could argue that Japan provides a recent example of such an occurrence. Switzerland experienced a similar phenomenon in the late 1970s.

As in Black (1995) we model the market short rate r_t at time t by truncating the shadow short rate ψ_t to its non-negative part, as described in (3.17). In Figure 6 we illustrate a possible scenario for the hypothetical shadow short rate in conjunction with the Japanese market short rate for 1995 to 2003. The plotted hypothetical shadow short rate trajectory was derived from the observed short rate data using a truncated Fourier series expansion. Note that the fitted hypothetical shadow short rate is clearly negative in 2003. Of course, the authors do not necessarily believe this to be the precise estimation of the historical shadow short rate for Japan over the considered time period. Such a statement would require more advanced modelling techniques that combine the proposed model with detailed economic research. However, this is beyond the scope of this paper. Our more modest goal is to propose a simple class of IRTS models suitable for both standard and low interest rate environments.

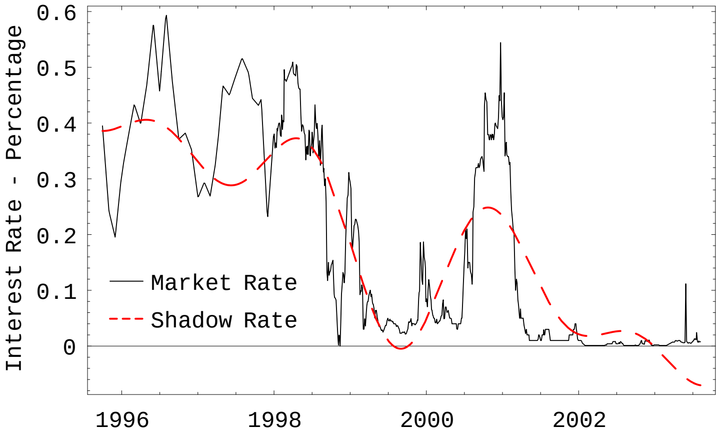

Figure 6: Market short rate and hypothetical shadow short rate for Japan from 1995 to 2003.

There is some debate as to whether or not *market interest rates* can become negative. Obviously, any implementation of the Black (1995) model precludes a negative market short rate, but allows for the possibility of a negative shadow short rate. Evidence on the likelihood of negative interest rates comes from Van Deventer & Imai (1997). When commenting on a period of very low interest rates in Japan, they reported that an interest rate floor on six-month Yen LIBOR with a three-year maturity and a strike price of zero was quoted at one basis point bid, three basis points offered. A positive price for a floor with a strike rate of zero indicates that the market perceived a strictly positive probability of negative interest rates. Furthermore, slightly negative market short rates were actually observed in Switzerland during 1979, see Kugler & Rich (2002), and more recently in Japan, as noted in James & Webber (2000). Therefore, we also mention that all of the results in this paper can easily be generalised to the case where the lower boundary of r_t in (3.17) is set at a level less than zero.

3.5 Short Rate Contribution

Combining the expression (3.4) with formula (3.17) we obtain the short rate contribution to the zero coupon bond price as

$$G_{\bar{T}}(t, \psi_t) = E\left(\exp\left\{-\int_t^{\bar{T}} \max(\psi_u, 0)\, du\right\} \Big| \mathcal{A}_t\right) \tag{3.18}$$

for $t \in [0, \bar{T}]$. The short rate contribution to the zero coupon bond price plays a pivotal role in calibrating the model to the initial term structure of interest rates. As follows from Figure 5, it almost exclusively determines zero coupon bond prices and forward rates for maturities of up to five years. Thus it is extremely important for the short end of the forward rate curve. This is due to the fact that the market price for risk contribution (3.15) is completely determined at time $t = 0$ by the initial values α_0, η and $\bar{S}_0$. It follows by rearranging the zero coupon bond pricing formula (3.2), that the short rate contribution can be calibrated by observing that

$$G_{\bar{T}}(0, \psi_0) = \frac{P(0, \bar{T})}{M_{\bar{T}}(0, \bar{S}_0)} \tag{3.19}$$

for time $t = 0$ and $\bar{T} \in [0, T]$. Therefore, in order to fit the initial IRTS, it is imperative that the short rate specification contains at least one time-dependent parameter. As an example, we plot in Figure 7 zero coupon bond prices $P(0, \bar{T})$ derived from the JPY swap curve, with dependence on maturity $\bar{T}$ observed as at 1-Jul-2004. The implied zero coupon bond *short rate contribution* (SRC) $G_{\bar{T}}(0, r_0)$ is also plotted in Figure 7. The parameters used to construct the implied SRC from (3.19) are identical to those used in Figure 5. We emphasize again that the difference that one observes between the two curves in Figure 7 is due to the impact of the market price for risk contribution, which only becomes significant at longer maturities, as previously illustrated in Figure 5. This is a direct consequence of the non-existence of the risk neutral martingale measure.

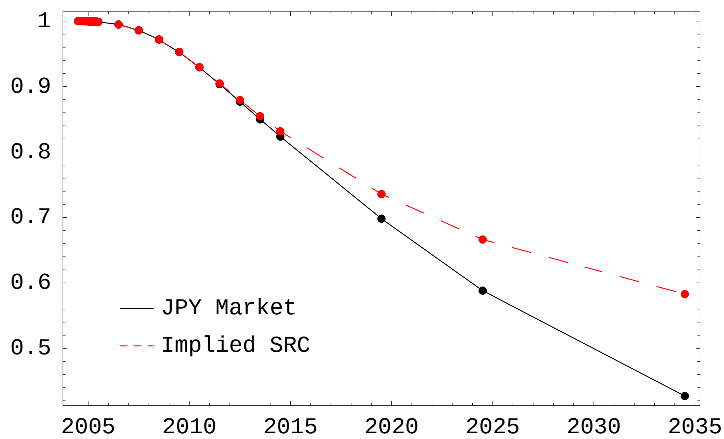

Figure 7: JPY zero bond prices $P(0, \bar{T})$ at 1-Jul-2004 with zero coupon bond short rate contribution $G_{\bar{T}}(0, r_0)$.

We specify a model for the shadow short rate with the addition of a time dependent parameter φ_t that is used to calibrate the initial yield curve. More precisely, we specify the shadow short rate ψ_t to be of the form

$$\psi_t = x_t + \varphi_t \tag{3.20}$$

for $t \in [0, T]$. Here the general Hull & White (1990) process $x = \{x_t, t \in [0, T]\}$

satisfies the SDE

$$dx_t = \gamma\,(\mu - x_t)\,dt + \sigma\,x_t^{\beta}\,dW_t^1 \tag{3.21}$$

for $t \in [0, T]$, with the initial value $x_0 \in \Re$, the long term average reference level $\mu \in \Re$, the scaling parameter $\sigma > 0$, the speed of adjustment $\gamma \geq 0$ and the elasticity exponent $\beta \geq 0$. Of course, restrictions on these parameters are required to ensure that a unique strong solution of (3.21) exists.

When the short rate is close to zero, one needs to focus on realistic modelling of the diffusion coefficient in (3.21), which is achieved via the exponent β. A Gaussian model, such as that proposed in Vasicek (1977) arises for $\beta = 0$ and seems to be quite realistic for most interest rate markets. Here the value of the diffusion coefficient is independent of the value of the factor x_t. In particular, this model appears to be consistent with empirical observations from economies experiencing low interest rate environments such as Japan, see Goldstein & Keirstead (1997) and Gorovoi & Linetsky (2001). Hence, Gaussian dynamics appear to be suitable for the shadow short rate. The positivity of the market short rate is then achieved via the option payoff feature described by formula (3.17). Square root processes, as in Cox, Ingersoll & Ross (1985) or Brigo & Mercurio (2001a), that arise for elasticity exponent $\beta = 0.5$, provide another interesting class of short rate models. These models possess the distinct advantage of retaining analytic tractability and positivity. Lognormal models, as suggested in Black & Karasinski (1991) follow for $\beta = 1$, are not well suited to low interest rate environments because the value of the diffusion coefficient decays too rapidly as the short rate approaches zero. For these reasons, we concentrate our investigation on Gaussian and square root processes as models for the shadow short rate, and do not consider models with higher elasticity exponents.

3.5.1 Shifted Gaussian Shadow Short Rate

The use of a Gaussian process to model the shadow short rate was probably first discussed in Black (1995). More recently, Gorovoi & Linetsky (2001) have extended this approach by providing analytic expressions for prices of zero coupon bonds and options on zero coupon bonds using an eigenfunction expansion technique. Related results can be found in Goldstein & Keirstead (1997) who use the same technique for the case when zero is defined as a reflecting boundary. The analysis of Gorovoi & Linetsky (2001) is restricted to one-dimensional diffusions with stationary densities. Therefore, their implementation does not include any time dependent parameters and hence the shadow short rate parameters were used to calibrate the model to a *Japanese Government Bond* (JGB) yield curve. However, this is not very robust for pricing derivatives. Instead, we suggest applying the deterministic shift extension method of Brigo & Mercurio (2001a). The deterministic shift frees the parameters of the Gaussian process, which can then be used in calibration to interest rate derivative markets.

There are many different numerical procedures available for one-factor Gaussian interest rate models that can be used to determine the short rate contribution. For example, a number of algorithms for Gaussian interest rate trees were suggested in Hull & White (1993, 1994). By comparing various methods, we found that an interest rate tree modified to replace all negative short rates with zero is far more efficient than the theoretically appealing eigenfunction expansion technique suggested in Gorovoi & Linetsky (2001). The latter is reasonably efficient for long dated securities but is least efficient for short dated securities.

The early section of the IRTS is where liquid interest rate derivative instruments are traded. Therefore, an interest rate tree or numerical PDE implementation for computing the impact of the short rate contribution appears to be adequate. Long dated interest rate caps require prices to be computed along the entire term structure, which can be achieved on a single interest rate tree. Hence, the results presented in this paper for the Gaussian shadow short rate process were calculated using interest rate trees. The implemented tree methodology is based on the earlier work in Hull & White (1993) rather than the later formulation of Hull & White (1994). This is due to the need to replace all negative values of the shadow short rate with zero.

Let us denote by $\Pi_{\bar{T}}(t, x_t) = \Pi_{\bar{T}}(t, \psi_t - \varphi_t)$ the price of a zero coupon bond subject to (3.17), where the shadow short rate is modelled using a time homogeneous Gaussian process. That is, with elasticity exponent $\beta = 0$ in SDE (3.21). By applying the deterministic shift extension of Brigo & Mercurio (2001a), we obtain the short rate contribution to the zero coupon bond price as

$$G_{\bar{T}}(t, \psi_t) = \exp\left\{ -\int_t^{\bar{T}} \varphi_u \, du \right\} \Pi_{\bar{T}}(t, \psi_t - \varphi_t) \tag{3.22}$$

for $t \in [0, \bar{T}]$. The resulting model allows an exact match to the initially observed IRTS by calibrating φ_u for $u \in [0, T]$. Furthermore, the long run reference level μ is absorbed into the deterministic shift and plays no role in the calibration to interest rate caps under this model, as shown in Brigo & Mercurio (2001a).

The corresponding short rate contribution to the forward rate can be calculated via (3.8) and (3.22) to obtain

$$g_{\bar{T}}(t, r_t) = \varphi_t - \frac{\partial}{\partial \bar{T}} \ln\left(\Pi_{\bar{T}}\left(t, \psi_t - \varphi_t\right)\right) \tag{3.23}$$

for $t \in [0, \bar{T}]$. Figure 8 illustrates four possible scenarios for the forward rate term structure that highlight the flexibility of the proposed model. The implied short rate contribution to the forward rate is plotted as a dashed line. Panel A shows a standard flat forward curve. The forward curve in Panel B is important because it is downward sloping, which is not possible under the model specified in Platen (2003). Panel C displays an upward sloping forward curve, while Panel D provides a inverse hump shaped forward rate curve. The latter is often observed in interest rate markets, yet is not obtainable under the majority of popular one-factor short rate models.

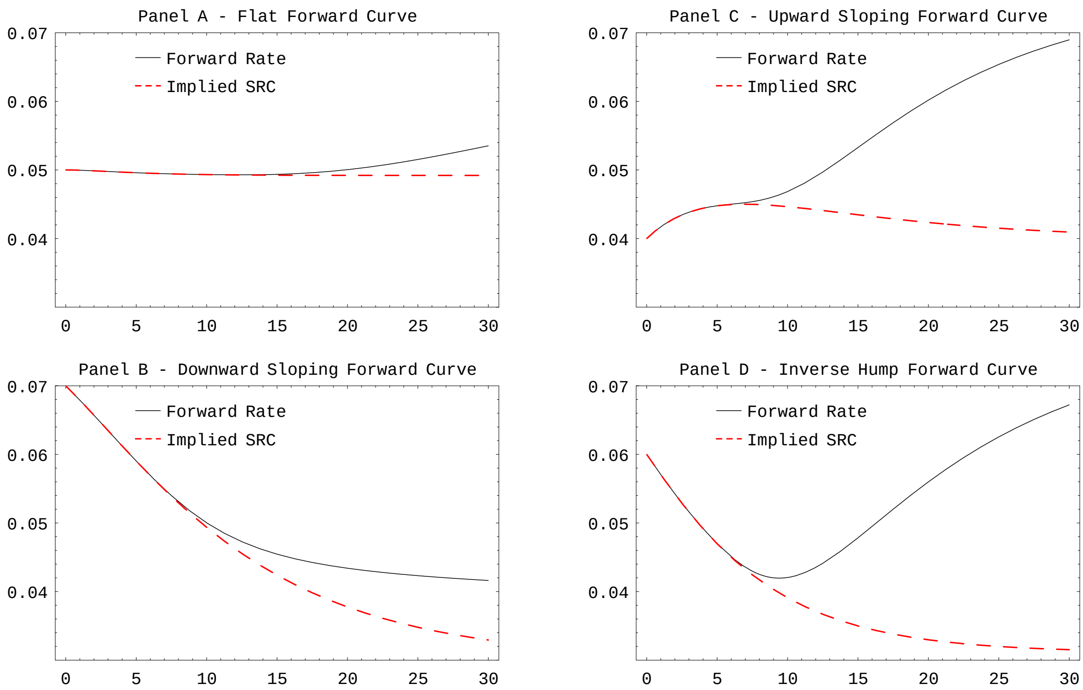

Figure 8: Possible forward rate curves and their implied short rate contributions.

3.5.2 Shifted Square Root Shadow Short Rate

The second model that we discuss for the shadow short rate is the CIR++ model of Brigo & Mercurio (2001a), under which the elasticity exponent is $\beta = \frac{1}{2}$. We consider this model because it allows the initial yield curve to be fitted precisely via the deterministic shift extension. Consequently, one can obtain analytic results for both zero coupon bond prices and options on zero coupon bonds with a strictly positive short rate. There is however, the restriction on the parameters $2\,\gamma\,\mu \geq \sigma^2$ that must be satisfied to guarantee strict positivity for a square root process. Furthermore, under the CIR++ model positive rates can only be guaranteed with the additional constraint that the deterministic shift extension must be strictly positive, that is $\varphi_t > 0$ for all $t \geq 0$. This can substantially affect the quality of calibration to the cap market, as discussed extensively in Brigo & Mercurio (2001a). However, the option feature (3.17) is not required when the shadow short rate parameterisation meets these conditions.

Under the square root shadow short rate model, the short rate contribution to the zero coupon bond price can be expressed analytically as

$$G_{\bar{T}}(t, r_t) = \exp\left\{A^*(t, \bar{T}) - B(t, \bar{T})\ (\psi_t - \varphi_t)\right\}, \tag{3.24}$$

where

$$
\begin{aligned}
A^*(t,\bar{T}) &= \ln\left(\frac{P(0,\bar{T})}{P(0,t)}\right) + A(t,\bar{T}) + A(0,t) - A(0,\bar{T}) \\
&\quad - \left[B(0,t) - B(0,\bar{T})\right](\psi_0 - \varphi_0), \\
A(t,\bar{T}) &= \frac{2\gamma\mu}{\sigma^2}\ln\left(\frac{2\kappa\exp\left\{(\gamma+\kappa)(\bar{T}-t)/2\right\}}{(\gamma+\kappa)\left(\exp\left\{\kappa(\bar{T}-t)\right\}-1\right)+2\kappa}\right), \\
B(t,\bar{T}) &= \frac{2\left(\exp\left\{\kappa(\bar{T}-t)\right\}-1\right)}{(\gamma+\kappa)\left(\exp\left\{\kappa(\bar{T}-t)\right\}-1\right)+2\kappa}, \\
\kappa &= \sqrt{\gamma^2+2\sigma^2}
\end{aligned}
\tag{3.25}
$$

for $t \in [0,\bar{T}]$. The corresponding short rate contribution to the forward rate is calculated using (3.8), (3.24) and (3.25) to be

$$
g_{\bar{T}}(t,r_t) = \frac{\partial}{\partial \bar{T}} A^*(t,\bar{T}) + \frac{\partial}{\partial \bar{T}} B(t,\bar{T})\,(r_t - \varphi_t) \tag{3.26}
$$

for $t \in [0,\bar{T}]$. Forward rate curves similar to those displayed in Figure 8 are easily obtained and are therefore omitted.

Note that regardless of the model chosen for the shadow short rate, there are two properties of the forward rate that generally hold. Firstly, the instantaneous forward rate equals the short rate, that is

$$
f(t,t) = g_t(t,r_t) = r_t \tag{3.27}
$$

for $t \in [0,\bar{T}]$. Secondly, the asymptotic *long forward rate* $f(0,\infty)$ can be represented as

$$
f(0,\infty) = \eta + \lambda_0, \tag{3.28}
$$

where λ_0 is the long term asymptotic limit of the short rate.

4 Interest Rate Derivatives

4.1 Zero Coupon Bond Options

We now focus on options on zero coupon bonds since they represent the fundamental building blocks for European style interest rate derivatives.

We introduce the sigma algebra $\mathcal{A}_t^{\bar{S}}$ generated by $\mathcal{A}_t$ and the path of the discounted GOP $\bar{S}$ until time T. That is, $\mathcal{A}_t^{\bar{S}} = \sigma\left\{\bar{S}_u,\, u \in [0,T]\right\} \cup \mathcal{A}_t$, noting that $\mathcal{A}_t \subseteq \mathcal{A}_t^{\bar{S}}$. This leads us to the following lemma, which is proved in the Appendix.

Lemma 4.1 *If the shadow short rate* ψ_t *is specified under the Gaussian model of* (3.20)−(3.21) *with* $\beta = 0$*, then the fair price at time* t *of a European call option* $\boldsymbol{zbc}_{\bar{T},T,K}(t, \bar{S}_t, \psi_t)$ *with strike price* K *and option expiry* $\bar{T}$ *on a zero coupon bond with maturity at time* $T \geq \bar{T}$ *is*

$$\boldsymbol{zbc}_{\bar{T},T,K}(t, \bar{S}_t, \psi_t) = E\left(\frac{\bar{S}_t^{(\delta_*)}}{\bar{S}_{\bar{T}}} \, M_T(\bar{T}, \bar{S}_{\bar{T}}) \, \Psi_{\bar{T},T}(t, \psi_t, K(\bar{S}_{\bar{T}})) \, \Big| \, \mathcal{A}_t \right), \tag{4.1}$$

where

$$\Psi_{\bar{T},T}(t, \psi_t, K(\bar{S}_{\bar{T}})) = E\left(\frac{B_t}{B_{\bar{T}}} \left(G_T(\bar{T}, \psi_{\bar{T}}) - K(\bar{S}_{\bar{T}}) \right)^+ \, \Big| \, \mathcal{A}_t^{\bar{S}} \right) \tag{4.2}$$

$$K(\bar{S}_{\bar{T}}) = K / M_T(\bar{T}, \bar{S}_{\bar{T}}) \tag{4.3}$$

for $0 \leq t \leq \bar{T} \leq T$.

Analogous to the computation of the short rate contribution to the zero coupon bond, one can calculate the expectation (4.2) using trees, numerical PDE methods or an eigenfunction expansion technique together with a deterministic shift extension. Note that $\Psi_{\bar{T},T}(t, \psi_t, K(\bar{S}_{\bar{T}}))$ is essentially the value of a call option on a zero coupon bond under a one-factor short rate model of the form (3.20)−(3.21), conditioned on the value of the discounted GOP. A key difference is that the conditional expectation is taken with respect to the real world probability measure P, whereas it is usually taken with respect to a risk neutral measure P^θ. The zero coupon bond call option price then follows from (4.1) by an outer conditional expectation under P. The price for a European put option on a zero coupon bond can be determined in the same way.

Computation of the zero coupon bond option price (4.1) can be performed accurately and efficiently using numerical integration. This is possible because the transition density for the discounted GOP is that of a time transformed squared Bessel process of dimension four, which is known explicitly.

Now let us denote the non-central chi-square distribution function by $\chi^2(\cdot, \nu, \lambda)$, where ν represents the degrees of freedom and λ is the non-centrality parameter, see Jamshidian (1995). For a shadow short rate that follows a square root process we obtain the following lemma, which is also proved in the Appendix.

Lemma 4.2 *If the shadow short rate* ψ_t *is strictly positive and specified under the square root model of* (3.20)−(3.21) *with* $\beta = \frac{1}{2}$*, then the fair price at time* t *of a European call option* $\boldsymbol{zbc}_{\bar{T},T,K}(t, \bar{S}_t, \psi_t)$ *with strike price* K *and option expiry* $\bar{T}$ *on a zero coupon bond with maturity at time* $T \geq \bar{T}$ *is*

$$\begin{aligned} \boldsymbol{zbc}_{\bar{T},T,K}(t, \bar{S}_t, \psi_t) = {} & G_T(t, \psi_t) \, \Upsilon_1(t, \bar{S}_t, \psi_t, d_1, e_2) \\ & - K \, G_{\bar{T}}(t, \psi_t) \, \Upsilon_2(t, \bar{S}_t, \psi_t, d_2, e_2), \end{aligned} \tag{4.4}$$

where

$$\Upsilon_1(t, \bar{S}_t, \psi_t, d_1, e_1) = E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}} \, M_T(\bar{T}, \bar{S}_{\bar{T}}) \; \chi^2\left(d_1, \frac{4\gamma\,\mu}{\sigma^2}, e_1\right) \,\bigg|\, \mathcal{A}_t \right) \tag{4.5}$$

$$\Upsilon_2(t, \bar{S}_t, \psi_t, d_2, e_2) = E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}} \; \chi^2\left(d_2, \frac{4\gamma\,\mu}{\sigma^2}, e_2\right) \,\bigg|\, \mathcal{A}_t \right) \tag{4.6}$$

with

$$\begin{aligned}
d_1 &= d_1(\bar{T}, \bar{S}_{\bar{T}}) = \left(2 + \frac{2(\zeta+\vartheta)}{B(\bar{T},T)}\right) \left[A(\bar{T},T) - A^*(\bar{T},T) - \ln\left(K(\bar{S}_{\bar{T}})\right)\right], \\
d_2 &= d_2(\bar{T}, \bar{S}_{\bar{T}}) = \left(\frac{2(\zeta+\vartheta)}{B(\bar{T},T)}\right) \left[A(\bar{T},T) - A^*(\bar{T},T) - \ln\left(K(\bar{S}_{\bar{T}})\right)\right], \\
e_1 &= e_1(t, \psi_t) = \frac{2\zeta(\psi_t - \varphi_t)\exp\{\kappa(\bar{T}-t)\}}{\zeta + \vartheta + B(\bar{T},T)}, \\
e_2 &= e_2(t, \psi_t) = \frac{2\zeta(\psi_t - \varphi_t)\exp\{\kappa(\bar{T}-t)\}}{\zeta + \vartheta}, \\
\zeta &= \frac{2\kappa}{\sigma^2(\exp\{\kappa(\bar{T}-t)\} - 1)}, \quad \vartheta = \frac{\gamma+\kappa}{\sigma^2} \quad \text{and} \quad \kappa = \sqrt{\gamma^2 + 2\sigma^2}
\end{aligned}$$

for $0 \le t \le \bar{T} \le T$*, using notation from* (3.25).

The conditional expectations (4.5) and (4.6) can be interpreted as special mathematical functions that can be evaluated numerically using the transition density of the discounted GOP.

4.2 Interest Rate Caps

An interest rate cap can be decomposed into a portfolio of caplets. Each caplet is equivalent to a put option on a zero coupon bond with adjustments to the strike rate and notional principal, see Brigo & Mercurio (2001b).

Let us denote a fair *interest rate cap* contract by $\mathbf{cap}_{\mathcal{T},K,N}(t, \bar{S}_t, r_t)$ at time $t \le T_0$ with strike rate K, notional principal N and the set of dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$. The ith individual *caplet* $\mathbf{cpl}_{T_{i-1},T_i,K}(t, \bar{S}_t, r_t)$ relates to the ith zero coupon bond put option $\mathbf{zbp}_{T_{i-1},T_i,K^*}(t, \bar{S}_t, r_t)$ for $i \in \{1, \ldots, n\}$ as

$$\begin{aligned}
\mathbf{cap}_{\mathcal{T},K,N}(t, \bar{S}_t, r_t) &= \sum_{i=1}^{n} \mathbf{cpl}_{T_{i-1},T_i,K,N}(t, \bar{S}_t, r_t) \\
&= \sum_{i=1}^{n} N_i^* \, \mathbf{zbp}_{T_{i-1},T_i,K_i^*}(t, \bar{S}_t, r_t),
\end{aligned} \tag{4.7}$$

where

$$N_i^* = N\,\left(1 + K\,(T_i - T_{i-1})\right) \quad \text{and} \quad K_i^* = \frac{1}{1 + K\,(T_i - T_{i-1})} \tag{4.8}$$

for $0 \leq t \leq T_0 \leq \ldots \leq T_n$.

Of particular interest are the implied cap volatilities from both a cross-sectional and term structure point of view. We price market caps on a semi-annual basis with the proposed model and then invert the market standard Black (1976) formula to obtain the implied volatility associated with each cap.

In Figure 9 we plot the at-the-money (ATM) implied *cap volatility term structure* (CVTS) using the arithmetic average of bid and ask volatilities for the Japanese cap market as at 1-Jul-2004. Note the consistent downward slope in the implied CVTS and the extremely high levels of volatility for short maturities. We fitted the proposed model under the truncated Gaussian shadow short rate specification of Section 3.5.1 and the CIR++ model of Section 3.5.2. The parameters used for the Gaussian model were $\gamma = 0.125$ and $\sigma = 0.013$. For the square root model we used $\gamma = 0.125$, $\mu = 0.036$ and $\sigma = 0.095$. For both implementations we set $\eta = 0.015$ and $|\theta_0| = 0.15$. The results for the Gaussian shadow short rate provide an excellent fit for maturities beyond one year. This is remarkable for a model that uses only a few constant parameters. We acknowledge that the model result for the first cap does not conform exactly to the market data, however it lies well within the bid-ask spread for this maturity. We also provide the best fit we could obtain for the CIR++ shadow short rate model to the Japanese ATM cap volatility data. The ability of the CIR++ model to fit the market is obviously less satisfactory than is the case for the Gaussian model. Furthermore, the numerical results and calibration to real data are significantly less stable under the CIR++ model. This is probably due to the fact that $\beta = 0$ is a more realistic choice for the elasticity exponent than $\beta = 0.5$ in low interest rate environments. It is further complicated by the restrictions that must be satisfied for the shadow short rate to remain strictly positive and thus avoid absorption at zero. For instance, if we increase the level of the short rate volatility parameter σ in the CIR++ model to match the market implied cap volatilities the positivity condition $2\,\gamma\,\sigma \geq \sigma^2$ appears to be violated. This can be corrected by increasing the speed of mean reversion γ and the long run reference level μ. However, in this case, we encounter significant stability problems and the implied CVTS that we obtain is too steep in comparison to the observed market data. A lognormal model with a constant volatility parameter provides even worse calibration results.

We emphasize that for the Gaussian shadow short rate model, truncation as suggested in Black (1995) is essential. Truncation not only ensures that the market short rate remains positive, it is also key in providing the correct implied cap volatility term structure. Furthermore, without truncation the robustness of calibrated parameters becomes a serious issue. Therefore, in the remainder of the paper we focus solely on the Gaussian shadow short rate model (3.20)−(3.21) with elasticity exponent $\beta = 0$ and truncation as per (3.17). Note that the good fit of the truncated Gaussian model has been achieved with very few parameters and it appears to be quite robust and flexible. For example, the proposed model can easily be fitted to other interest rate markets with a downward sloping CVTS

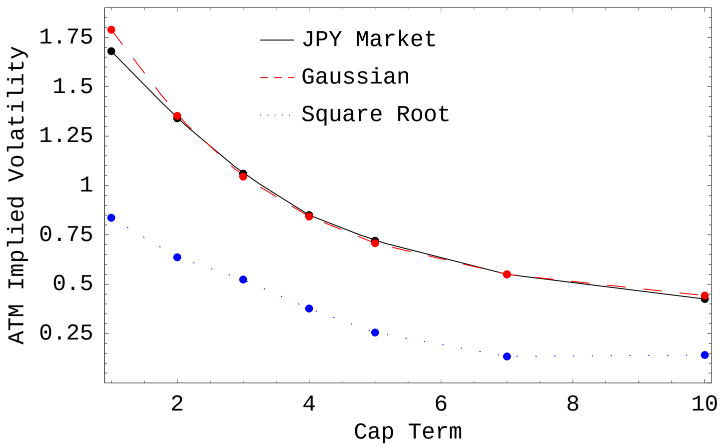

Figure 9: ATM implied cap volatilities for the JPY market, together with Gaussian and CIR++ implementations.

such as Switzerland and the United States. It also naturally produces the typical hump shaped ATM CVTS that arises under standard interest rate environments, such as those presently observed in Europe and Australia.

Figure 10 displays a cross-section of implied cap volatilities under the truncated Gaussian shadow short rate model, obtained as a function of the strike rate and term for the liquidly traded section of the cap market. Previous literature containing data on the Japanese cap market, such as Andersen & Andreasen (2000) or Long (2001), document a downward sloping CVTS and a consistent negative skew that flattens as the cap term increases. Both of these features are evident under the proposed model, as demonstrated in Figure 10. We use parameters obtained from the calibration to the ATM CVTS, that is, $\gamma = 0.125$, $\sigma = 0.013$, $\eta = 0.015$ and $|\theta_0| = 0.15$. Notice that the implied volatility surface covers a wide range of strike rates, from deep in-the-money (ITM) to deep out-of-the-money (OTM) caps.

Finally, in Figure 11 we illustrate the implications for implied cap volatilities that arise from the non-existence of an equivalent risk neutral martingale measure. That is, we plot the difference between implied cap volatilities calculated under the proposed two-factor model and those calculated using a one-factor truncated Gaussian short rate model without the market price for risk factor. Hence we are isolating the effect of the total market price for risk on the implied CVTS. Once again the differences only become evident after an initial period of at least five years and material differences are only apparent past ten years, which is beyond the maturity of most liquidly traded interest rate derivatives. Careful inspection of Figure 11 reveals that the addition of the total market price for risk factor leads to a relative increase in implied cap volatilities for longer maturities, which is similar to the observation made for stock market indices in Heath & Platen

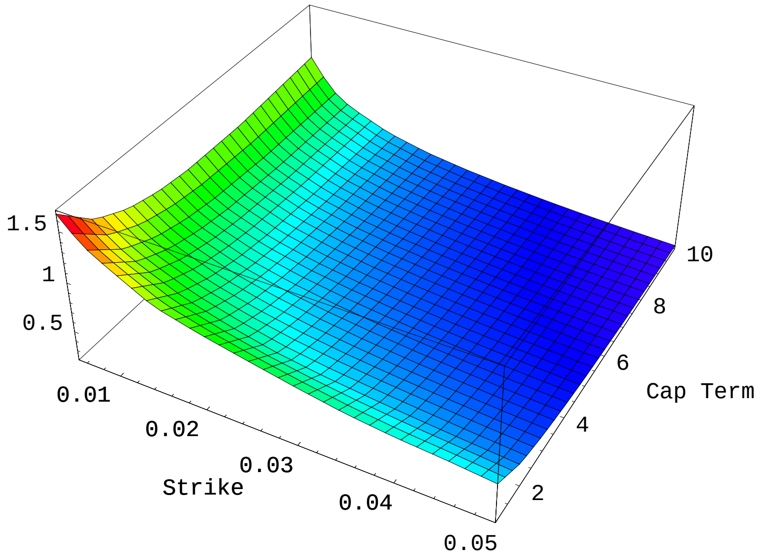

Figure 10: Implied cap volatilities for the truncated Gaussian model with terms up to 10 years.

(2004). Furthermore, one also observes that the market price for risk factor has a significant effect on the degree of skewness in the CVTS. In particular, the skewness of deep ITM caps increases as the cap moves further into the money. These theoretically predicted effects seem reasonable, given the nature of interest rate caps and the available evidence on the earlier sections of the CVTS. The proposed parsimonious model extends the minimal market model derived in Platen (2002, 2004) to the case of positive stochastic interest rates. Future research will demonstrate that under this model long dated interest rate derivatives can be hedged successfully on the basis of market data for major markets.

Conclusion

The paper proposes a two-factor truncated Gaussian interest rate term structure model for standard and low interest rate environments. The benchmark approach with the concept of fair pricing has been applied, where the growth optimal portfolio is used as numeraire or benchmark and contingent claim prices are calculated using conditional expectations under the real world probability measure. The proposed model is driven by a squared Bessel process of dimension four that models the discounted market index and a Gaussian shadow short rate that is truncated at zero to yield the market short rate. The Gaussian shadow short rate has a significant impact on the short end of the yield curve, whereas the discounted market index influences medium and long term interest rates. Using only a few natural constant parameters, the model reproduces the

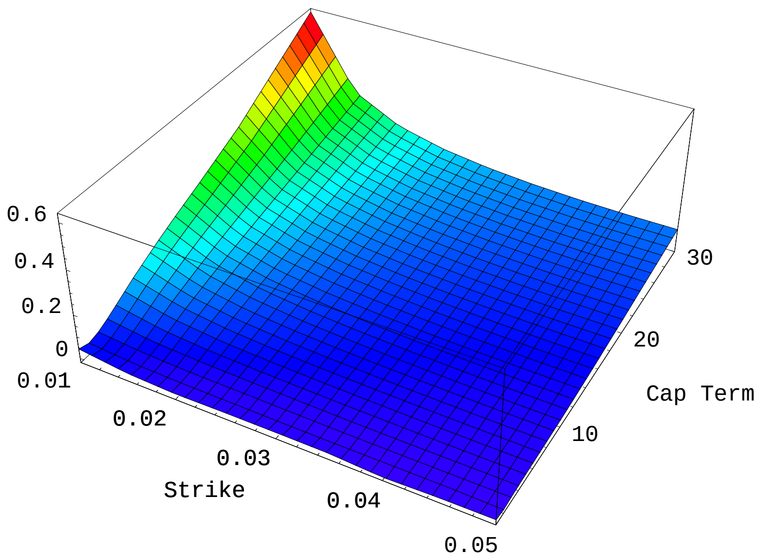

Figure 11: Differences in implied cap volatilities that result from non-existence of an equivalent risk neutral martingale measure.

most important stylized empirical features of the interest rate cap market under an economy experiencing low interest rates. These include a realistic range of forward rate curves, an implied cap volatility term structure that consistently slopes downwards from extremely high levels of volatility together with a distinct negative skew. Although the focus has been on low interest rate regimes, the model is readily applicable to medium level interest rate environments typically observed in developed markets. Forthcoming work will extend the above results in two ways. Firstly, we will broaden and generalise the family of forward rate volatility structures available by using a discounted GOP drift that is stochastic. Secondly, we will address the issue of simultaneous calibration to interest rate cap and swaption markets.

Appendix

Proof of Lemma 4.1: By application of the fair pricing formula (2.6) and the zero coupon bond representation (3.2) we find that

$$\begin{aligned}\mathbf{zbc}_{\bar{T},T,K}(t, \bar{S}_t, \psi_t) &= S_t\, E\left(\frac{(P(\bar{T},T) - K)^+}{S_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right) \\ &= E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\,\frac{B_t}{B_{\bar{T}}}\,\left(M_T(\bar{T},\bar{S}_{\bar{T}})\,G_T(\bar{T},\psi_{\bar{T}}) - K\right)^+\,\middle|\,\mathcal{A}_t\right). \qquad \text{(A.1)}\end{aligned}$$

We rewrite this conditional expectation as an iterated expectation conditioned on the sigma algebra $\mathcal{A}_t^{\bar{S}_T} = \sigma\left\{\bar{S}_u,\, u \in [0,T]\right\} \cup \mathcal{A}_t$ to obtain

$$\begin{aligned}
\mathbf{zbc}_{\bar{T},T,K}(t,\bar{S}_t,\psi_t) &= E\left(E\left(\frac{B_t}{B_{\bar{T}}} \left(G_T(\bar{T},\psi_{\bar{T}}) - \frac{K}{M_T(\bar{T},\bar{S}_{\bar{T}})} \right)^+ \,\middle|\, \mathcal{A}_t^{\bar{S}} \right) \right. \\
&\qquad \left. \cdot \frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\, M_T(\bar{T},\bar{S}_{\bar{T}}) \,\middle|\, \mathcal{A}_t \right) \\
&= E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\, M_T(\bar{T},\bar{S}_{\bar{T}})\, \Psi_{\bar{T},T}(t,\psi_t,K(\bar{S}_{\bar{T}})) \,\middle|\, \mathcal{A}_t \right) \qquad \text{(A.2)}
\end{aligned}$$

for $0 \le t \le \bar{T} \le T$. □

Proof of Lemma 4.2: The proof begins in the same way as the proof of Lemma 4.1. We separate the zero coupon bond price into its two components to the stage of (A.2). We can then determine the internal conditional expectation using the pricing function $u : [0,\bar{T}] \times \Re \to [0,\infty)$ as

$$u(t,\bar{S}_t,\psi_t) = E\left(\exp\left\{ -\int_t^{\bar{T}} \psi_u\, du \right\} H_{\bar{T}}(\bar{S}_{\bar{T}},\psi_{\bar{T}}) \,\middle|\, \mathcal{A}_t^{\bar{S}} \right) \qquad \text{(A.3)}$$

for $t \in [0,\bar{T}]$. Here we simplify our notation by introducing the payoff function

$$H_{\bar{T}}(\bar{S},\psi) = \left(\exp\{A^*(\bar{T},T) - B(\bar{T},T)\,(\psi - \varphi)\} - K(\bar{S})\right)^+. \qquad \text{(A.4)}$$

We recognise that the random variable $G_T(\bar{T},\psi_{\bar{T}})$ has a non-central chi-square distribution as in Jamshidian (1995). Hence it follows that

$$u(t,\bar{S}_t,\psi_t) = G_T(t,\psi_t)\,\chi^2\left(d_1, \frac{4\gamma\,\mu}{\sigma^2}, e_1\right) - K(\bar{S}_{\bar{T}})\,G_{\bar{T}}(t,\psi_t)\,\chi^2\left(d_2, \frac{4\gamma\,\mu}{\sigma^2}, e_2\right) \qquad \text{(A.5)}$$

for $0 \le t \le \bar{T} \le T$, where we use the notation in Lemma 4.2. Thus, using (A.2) and (A.5), the price of a zero coupon bond call option reduces to

$$\begin{aligned}
\mathbf{zbc}_{\bar{T},T,K}(t,\bar{S}_t,\psi_t) &= G_T(t,\psi_t)\, E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\, M_T(\bar{T},\bar{S}_{\bar{T}})\; \chi^2\left(d_1, \frac{4\gamma\,\mu}{\sigma^2}, e_1\right) \,\middle|\, \mathcal{A}_t \right) \\
&\quad - K\, G_{\bar{T}}(t,\psi_t)\, E\left(\frac{\bar{S}_t}{\bar{S}_{\bar{T}}}\; \chi^2\left(d_2, \frac{4\gamma\,\mu}{\sigma^2}, e_1\right) \,\middle|\, \mathcal{A}_t \right), \qquad \text{(A.6)}
\end{aligned}$$

which gives the call price (4.4) by (4.5) and (4.6). The put option value can be calculated using put-call parity. □

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