CLASSES OF INTEREST RATE MODELS UNDER THE HJM FRAMEWORK

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Abstract. Although the HJM term structure model is widely accepted as the most general, and perhaps the most consistent, framework under which to study interest rate derivatives, the earlier models of Vasicek, Cox-Ingersoll-Ross, Hull-White, and Black-Karasinski remain popular among both academics and practitioners. It is often stated that these models are special cases of the HJM framework, but the precise links have not been fully established in the literature. By beginning with certain forward rate volatility processes, it is possible to obtain classes of interest models under the HJM framework that closely resemble the traditional models listed above. Further, greater insight into the dynamics of the interest rate process emerges as a result of natural links being established between the model parameters and market observed variables.

Introduction

Many of the early interest rate models extended in various ways Vasicek's model [Vas77] in which the spot rate was assumed to follow a mean reverting process with constant volatility and constant mean reversion level. The common tool used in these models was the no-arbitrage arguments of Black-Scholes and Merton, which produced the pricing partial differential equation for the bond, and bond option, prices in a systematic manner. Well developed techniques from the theory of partial differential equations were then applied to solve, either analytically or numerically, these pricing equations.

The quantity driving this class of models was the instantaneous spot rate of interest, and, since the spot rate is a *non-traded* quantity, these models usually involved the market price of interest rate risk. And as the market price of risk is an unobservable quantity, assumptions then had to be made, often based on mathematical convenience rather than economic considerations, so as to obtain a pricing PDE that enabled the application of various solution techniques.

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A major departure from this general theme came with the model introduced by Heath-Jarrow-Morton [HJM92], who took as the quantities driving the model the continuum of instantaneous forward rates, which are directly related to the prices of traded bonds. They used techniques from stochastic calculus to construct a very general framework for the evolution of interest rates that had the useful feature that the model is naturally calibrated to the currently observed yield curve.

Although the HJM model is widely accepted as the most general and consistent framework under which to study interest rate derivatives, the added complexity and the absence of efficient numerical techniques under the general HJM framework saw the earlier models retain their popularity, particularly among practitioners. However, with the rapid advances in computer technology, HJM models are becoming increasingly practical, and various forms of the model are currently being adopted by practitioners for the pricing and hedging of interest rate derivatives.

The main inputs into the HJM framework are the forward rate volatility processes, and it was shown in [HJM92] that the Cox-Ingersoll-Ross model was a special case of the general 1-factor HJM framework, corresponding to a particular choice of the volatility process. However, it has remained unclear whether other popular models could be derived within the HJM framework, and whether there exists a systematic approach for generating the various interest rate models. It would not be unfair to say that the earlier interest rate models were developed under somewhat ad hoc assumptions, particularly in respect to the market price of risk, and lack the coherence and consistency of a well developed theory. This paper, may, then be regarded as a first step towards the resolution of this problem, where classes of interest rate models resembling the traditional models are derived from the HJM framework, with the ultimate goal being the development of a unifying framework, or technique, capable of generating other models in a systematic manner.

The structure of the remainder of this paper is as follows. A brief review of the HJM framework is given in §1, and the special case of exponentially decaying volatility processes is considered in §2. The corresponding models turn out to be Markovian, and in §3, it is shown that the state variables for the model are expressible in terms of a finite number of fixed tenor forward rates. In §4, §5, and §6, it is shown that models resembling multifactor generalisations of the Vasicek, the Hull-White two factor, and the Cox-Ingersoll-Ross models can all be obtained from the HJM framework. After presenting a generalisation of the HJM framework in §7, it is shown in §8 that models resembling a multifactor generalisation of Black-Karasinski model can also be obtained from the the HJM framework.

1. HEATH-JARROW-MORTON FRAMEWORK

In this section, a brief overview of the general Heath-Jarrow-Morton framework is given. For further details, the reader is referred to [HJM92], [MR97], or [Bjö96].

In the risk-neutral n-dimensional HJM framework, market activities take place in the finite time interval $[0, \tau]$, $0 < \tau \in \mathbb{R}$, and the arrival of market information is captured by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le \tau}, \mathbb{P})$, generated by n independent Wiener processes $W_i(t)$, $1 \le i \le n$.

Instantaneous forward rate process $f(t, T, \omega)$, representing the time T instantaneous rate of return as seen at time t, is assumed to satisfy the stochastic integral equation[1]

$$f(t, T, \omega) = f(0, T) + \sum_{i=1}^{n} \int_0^t \sigma_i^*(s, T, \omega)\, ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, T, \omega)\, dW_i(s), \tag{1.1}$$

where $\sigma_i^*(s, T, \omega) = \sigma_i(s, T, \omega) \int_s^T \sigma(s, u, \omega)\, du$, and $0 \le t \le T \le \tau$. Relatively mild regularity assumptions are imposed on $\sigma_i(t, T, \omega)$ so that the integrals are well defined, and required manipulations are valid.

The *spot rate* process $r(t, \omega)$, representing the instantaneous rate of return at time t, is defined by the equation $r(t, \omega) = f(t, t, \omega)$. From (1.1),

$$r(t, \omega) = f(0, t) + \sum_{i=1}^{n} \int_0^t \sigma_i^*(s, t, \omega)\, ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, t, \omega)\, dW_i(s). \tag{1.2}$$

The *money market account* $B(t, \omega)$, representing the time t value of unit investment made at time 0, is given by the equation

$$B(t, \omega) = e^{\int_0^t r(u, \omega)\, du}. \tag{1.3}$$

Finally, the time t price of a T maturity *zero coupon bond*, denoted $P(t, T, \omega)$, is defined as

$$P(t, T, \omega) = e^{-\int_t^T f(t, u, \omega)\, du}, \tag{1.4}$$

and (1.4) provides the important link between $f(t, T, \omega)$, the quantity driving dynamics of the HJM framework, and $P(t, T, \omega)$, a traded financial asset.

The differential forms of (1.1) and (1.2) are given by

$$\begin{aligned} df(t, T, \omega) &= \sum_{i=1}^{n} \sigma_i^*(t, T, \omega)\, dt + \sum_{i=1}^{n} \sigma_i(t, T, \omega)\, dW_t^i, \text{ and} && (1.5) \\ dr(t, \omega) &= \left[\frac{\partial f(0, t)}{\partial t} + \sum_{i=1}^{n} \int_0^t \frac{\partial \sigma_i^*(s, t, \omega)}{\partial t}\, ds \right. \\ &\quad \left. + \sum_{i=1}^{n} \int_0^t \frac{\partial \sigma_i(s, t, \omega)}{\partial t}\, dW_s^i \right] dt && (1.6) \\ &\quad + \sum_{i=1}^{n} \sigma_i(t, t, \omega)\, dW_t^i. \end{aligned}$$

Using stochastic Fubini's Theorem and Itô's Lemma, it was shown in [HJM92] that

$$\frac{dP(t, T, \omega)}{P(t, T, \omega)} = r(t, \omega) dt + \sum_{i=1}^{n} a_i(t, T, \omega)\, dW_i(t), \tag{1.7}$$

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[1] In expressions such as $f(t, T, \omega)$, the argument ω represents path-dependent parameters for f. For example, ω may represent the spot rate $r(t, \omega)$, or the forward rate itself $f(t, T, \omega)$. In the first case, we may write $f(t, T, \omega) = f(t, T, r(t))$.

where

$$a_i(t,T,\omega) = -\int_t^T \sigma_i(t,u,\omega)\,du, \quad 1 \le i \le n. \tag{1.8}$$

It follows from a simple application of Itô's lemma that the discounted bond price process $Z(t,T) = P(t,T)/B(t)$ satisfies

$$dZ(t,T,\omega) = Z(t,T,\omega)\sum_{i=1}^{n} a_i(t,T,\omega)\,dW_i(t), \tag{1.9}$$

and is consequently an $\{\mathcal{F}_t\}$-martingale under $\mathbb{P}$. This implies

$$Z(t,T,\omega) = \mathbb{E}\left[Z(T,T)\right|\mathcal{F}_t](\omega) \tag{1.10}$$

for the discounted bond price, and

$$P(t,T,\omega) = \mathbb{E}\left[e^{-\int_t^T r(u)\,du}\middle|\,\mathcal{F}_t\right](\omega) \tag{1.11}$$

for the bond price itself. More generally, if $C(t,T_C,\omega)$ is the price process for a T_C-expiry option on $P(t,T,\omega)$, with $T_C \le T$ and payoff $g(T_C,\omega)$, then

$$C(t,T_C,\omega) = \mathbb{E}\left[e^{-\int_t^{T_C} r(u)\,du} g(T_C)\middle|\mathcal{F}_t\right](\omega). \tag{1.12}$$

2. HJM Models with Exponentially Decaying Volatility

This section examines in detail the n-factor HJM models that correspond to volatility processes of the form

$$\sigma_i(s,t,\omega) = \varsigma_i(s, r(s,\omega))e^{-\int_s^t \kappa_i(v)\,dv}, \tag{2.1}$$

where ς_i and κ_i are deterministic functions for $1 \le i \le n$. By taking suitable specialisations of ς_i and κ_i, it will be shown that models closely resembling multi-factor generalisations of the extended Vasicek, the two factor Hull-White, and the Cox-Ingersoll-Ross models can be obtained from the HJM framework.

For notational convenience, the argument ω representing path-dependence will be omitted from all processes. So, for example, the i-th volatility process will be written $\sigma_i(s,t)$ rather than $\sigma_i(s,t,\omega)$.

2.1. **Spot Rate Dynamics.**

The n-factor HJM model of §1 with volatility processes given by (2.1) were studied by Carverhill [Car94], Ritchken and Sankarasubramanian [RS95], Inui and Kijima [IK98], and Bhar and Chiarella [BC97] in their attempt to obtain Markovian transformations of the HJM framework. The property of the volatility process (2.1) which plays a crucial role in the analysis is

$$\frac{\partial \sigma_i(s,t)}{\partial t} = -\kappa_i(t)\sigma_i(s,t), \tag{2.2}$$

which enables computation of certain stochastic integrals which would otherwise not be possible. This simple identity will also be used frequently in the analysis of this section.

For $1 \leq i \leq n$, define the state variables $\xi_i(t)$ and $\zeta_i(t)$ by

$$\begin{aligned}
\xi_i(t) &= \int_0^t \sigma_i^2(s,t)\,ds, && (2.3)\\
\zeta_i(t) &= \int_0^t \sigma_i^*(s,t)\,ds + \int_0^t \sigma_i(s,t)\,dW_i(s). && (2.4)
\end{aligned}$$

Note that $\zeta_i(t)$ is path-dependent in general, while $\xi_i(t)$ is path-dependent if and only if $\sigma_i(s,t)$ is path-dependent. The stochastic differential equations for $\xi_i(t)$ and $\zeta_i(t)$ are given by

$$\begin{aligned}
d\xi_i(t) &= \left[\varsigma_i^2(t,r(t)) - 2\kappa_i(t)\xi_i(t)\right] dt, && (2.5)\\
d\zeta_i(t) &= \left[\xi_i(t) - \kappa_i(t)\zeta_i(t)\right] dt + \varsigma_i(t,r(t))\,dW_i(t). && (2.6)
\end{aligned}$$

Now, with volatility processes of the form (2.1), the SDE (1.6) for the spot rate $r(t)$ can be written

$$\begin{aligned}
dr(t) = {}& \left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t}\int_0^t \sigma_i(s,t)\int_s^t \sigma_i(s,u)\,du\,ds \right.\\
&\left. + \sum_{i=1}^n \int_0^t \frac{\partial \sigma_i(s,t)}{\partial t}\,dW_i(s)\right] dt + \sum_{i=1}^n \sigma_i(t,t)\,dW_i(t)\\
= {}& \left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^n \left(\int_0^t \sigma_i^2(s,t)\,ds - \kappa_i(t)\int_0^t \sigma_i^*(s,t)\,ds\right)\right.\\
&\left. - \sum_{i=1}^n \kappa_i(t)\int_0^t \sigma_i(s,t)\,dW_i(s)\right] dt + \sum_{i=1}^n \varsigma_i(t,r(t))\,dW_i(t),
\end{aligned}$$

which, in turn, can be written

$$dr(t) = \left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^n \left(\xi_i(t) - \kappa_i(t)\zeta_i(t)\right)\right] dt + \sum_{i=1}^n \varsigma_i(t,r(t))\,dW_i(t). \qquad (2.7)$$

Using (1.2), it is possible to express one of the $\zeta_i(t)$ in terms of the spot rate and the remaining $\zeta_i(t)$. Consequently, one of the $\zeta_i(t)$ can be eliminated from (2.7), and $r(t)$ introduced in its place, and this is precisely the procedure adopted in [RS95] and [IK98]. This also implies that the processes $r(t)$, $\xi_i(t)$, and $\zeta_i(t)$, $1 \leq i \leq n$, form a $2n$-dimensional Markovian system with dynamics determined by (2.7), (2.5), and (2.6). Furthermore, the spot rate process is mean-reverting in these models.

It will be shown in §3 that, in fact, the state variables $\xi_i(t)$ and $\zeta_i(t)$ can be expressed in terms of a finite set of fixed tenor forward rates, and it is shown below that the the bond price for these models takes the exponential affine form of Duffie and Kan [DK96]. This observation provides an interesting link between the HJM framework and the Duffie-Kan framework in which the state variables consist of fixed tenor forward rates.

2.2. **Bond Price Formula.** For the volatility processes of the form (2.1), the following formula for the bond price $P(t,T)$ was obtained in [RS95] for the one-dimensional case, and subsequently generalised to the n-dimensional case in [IK98].

Theorem 2.1 ([RS95, p60], [IK98, p431])**.** *If $\sigma_i(t,T)$ are as given in (2.1), then the bond price is given by the formula*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -\frac{1}{2}\sum_{i=1}^{n} \gamma_i^2(t,T)\xi_i(t) - \sum_{i=1}^{n} \gamma_i(t,T)\zeta_i(t) \right\}, \tag{2.8}$$

where $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)\,dx} du$, for $1 \le i \le n$.

Proof. See Appendix A. □

3. State Variables $\xi_i(t)$ and $\zeta_i(t)$ as Functions of Forward Rates

The state variables $\xi_i(t)$ and $\zeta_i(t)$ introduced in (2.3) and (2.4), which appear in the bond price (2.8), are not directly observable in the market, and do not have immediate economic interpretation. Consequently, the connection between the bond price formula and the market observed variables is unclear. In this section, it is shown that the state variables $\xi_i(t)$ and $\zeta_i(t)$ are, in fact, expressible in terms of a finite set of fixed tenor forward rates.

For $1 \le i \le n$, define deterministic functions $\alpha_i(t,T)$ and $\beta_i(t,T)$ by

$$\alpha_i(t,T) = e^{-\int_t^T \kappa_i(x)\,dx}, \tag{3.1}$$

$$\beta_i(t,T) = \alpha_i(t,T)\gamma_i(t,T), \tag{3.2}$$

and note that for $s \le t \le T$, $\sigma_i(s,T)$ satisfies the identity

$$\sigma_i(s,T) = \sigma_i(s,t)\,\alpha_i(t,T). \tag{3.3}$$

Let $\triangle f(t,T) = f(t,T) - f(0,T)$. Then (1.1) can be rewritten

$$\begin{aligned} \triangle f(t,T) &= \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)\,\alpha_i(t,T) \left[\int_s^t \sigma_i(s,u)\,du + \int_t^T \sigma_i(s,t)\,\alpha_i(t,u)\,du \right] ds \\ &\quad + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)\,\alpha_i(t,T)\,dW_i(s) \\ &= \sum_{i=1}^{n} \alpha_i(t,T)\,\zeta_i(t) + \sum_{i=1}^{n} \beta_i(t,T)\,\xi_i(t). \end{aligned} \tag{3.4}$$

Setting $T = t + \varsigma$ in the above equation yields

$$\triangle f(t,t+\varsigma) = \sum_{i=1}^{n} \alpha_i(t,t+\varsigma)\,\zeta_i(t) + \sum_{i=1}^{n} \beta_i(t,t+\varsigma)\,\xi_i(t), \tag{3.5}$$

for the forward rate with fixed tenor ς. Now let $0 \le \tau_1 < \tau_2 < \cdots < \tau_{2n}$ be a fixed sequence of tenors, and consider the system of equations

$$\triangle f(t,\tau_1,\ldots,\tau_{2n}) = \left[\, \alpha(t,\tau_1,\ldots,\tau_{2n})\, \beta(t,\tau_1,\ldots,\tau_{2n}) \,\right] \begin{bmatrix} \zeta(t) \\ \xi(t) \end{bmatrix}, \tag{3.6}$$

where

$$\delta f(t,\tau_1,\dots,\tau_{2n}) = \begin{bmatrix} \triangle f(t,t+\tau_1) \\ \triangle f(t,t+\tau_2) \\ \vdots \\ \triangle f(t,t+\tau_{2n}) \end{bmatrix}, \quad \zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \vdots \\ \zeta_n(t) \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_n(t) \end{bmatrix},$$

$$\alpha(t,\tau_1,\dots,\tau_{2n}) = \begin{bmatrix} \alpha_1(t,t+\tau_1) & \alpha_2(t,t+\tau_1) & \cdots & \alpha_n(t,t+\tau_1) \\ \alpha_1(t,t+\tau_2) & \alpha_2(t,t+\tau_2) & \cdots & \alpha_n(t,t+\tau_2) \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1(t,t+\tau_{2n}) & \alpha_2(t,t+\tau_{2n}) & \cdots & \alpha_n(t,t+\tau_{2n}) \end{bmatrix},$$

$$\beta(t,\tau_1,\dots,\tau_{2n}) = \begin{bmatrix} \beta_1(t,t+\tau_1) & \beta_2(t,t+\tau_1) & \cdots & \beta_n(t,t+\tau_1) \\ \beta_1(t,t+\tau_2) & \beta_2(t,t+\tau_2) & \cdots & \beta_n(t,t+\tau_2) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_1(t,t+\tau_{2n}) & \beta_2(t,t+\tau_{2n}) & \cdots & \beta_n(t,t+\tau_{2n}) \end{bmatrix}.$$

If it is assumed that $\det[\,\alpha(t,\tau_1,\dots,\tau_{2n})\,\beta(t,\tau_1,\dots,\tau_{2n})\,] \neq 0$ for all t, then this system of equations can be inverted, and the state variables $\xi_i(t)$ and $\zeta_i(t)$ can be expressed as linear combinations of forward rates $f(t,t+\tau_j)$, $1 \leq j \leq 2n$. Furthermore, (3.5) can be used to write forward rates of *all* maturities in terms of the finite set $f(t,t+\tau_1)$, $f(t,t+\tau_2)$, $\dots$, $f(t,t+\tau_{2n})$. In particular, this shows that the entire forward rate curve is parametrised by a set of $2n$ fixed tenor forward rates.

3.1. **An Example.** Consider the 1-dimensional HJM model with volatility given by (2.1). Then the state variables for the resulting model are $\xi_1(t)$ and $\zeta_1(t)$, and the procedure outlined above yields the linear system

$$\begin{bmatrix} \triangle f(t,t+\tau_1) \\ \triangle f(t,t+\tau_2) \end{bmatrix} = \begin{bmatrix} \alpha_1(t,t+\tau_1) & \beta_1(t,t+\tau_1) \\ \alpha_1(t,t+\tau_2) & \beta_1(t,t+\tau_2) \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \xi_1(t) \end{bmatrix}$$

for $\xi(t)$ and $\zeta(t)$, where $0 \leq \tau_1 < \tau_2$. Simple inversion gives

$$\zeta_1(t) = \frac{\beta_1(t,t+\tau_2)\,\triangle f(t,t+\tau_1) - \beta_1(t,t+\tau_1)\,\triangle f(t,t+\tau_2)}{\alpha_1(t,t+\tau_1)\beta_1(t,t+\tau_2) - \beta_1(t,t+\tau_1)\alpha_1(t,t+\tau_2)},$$
$$\xi_1(t) = \frac{\alpha_1(t,t+\tau_1)\,\triangle f(t,t+\tau_2) - \alpha_1(t,t+\tau_2)\,\triangle f(t,t+\tau_1)}{\alpha_1(t,t+\tau_1)\beta_1(t,t+\tau_2) - \beta_1(t,t+\tau_1)\alpha_1(t,t+\tau_2)},$$

if $\alpha_1(t,t+\tau_1)\beta_1(t,t+\tau_2) - \beta_1(t,t+\tau_1)\alpha_1(t,t+\tau_2) \neq 0$. If it is further assumed that $\kappa_1(x) = \kappa$ is constant, then

$$\begin{aligned} \sigma_1(t,t+\varsigma) &= \sigma_1(t,t)e^{-\kappa\varsigma}, \\ \alpha_1(t,t+\varsigma) &= e^{-\kappa\varsigma}, \quad \text{and} \\ \beta_1(t,t+\varsigma) &= \frac{1}{\kappa}e^{-\kappa\varsigma}\left[1 - e^{-\kappa\varsigma}\right]. \end{aligned}$$

Without loss of generality, take $\tau_1 = 0$. Then $\alpha_1(t, t+\tau_1) = 1$, $\beta_1(t, t+\tau_1) = 0$, and $\gamma_1(t, t+\tau_1) = 0$. Letting $\tau_2 = \delta > 0$,

$$\begin{aligned}
\zeta_1(t) &= r(t) - r(0), \\
\xi_1(t) &= \kappa \frac{[f(t, t+\delta) - f(0, t+\delta)] - e^{-\kappa\delta}\,[r(t) - r(0)]}{e^{-\kappa\delta}\,[1 - e^{-\kappa\delta}]}
\end{aligned}$$

In view of (1.4), the bond price is given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-a(t,\delta,T)\,\triangle f(t,t+\delta) - b(t,\delta,T)\,\triangle r(t)\right],$$

where

$$\begin{aligned}
a(t,\delta,T) &= \frac{\kappa}{2e^{-\kappa\delta}\,[1 - e^{-\kappa(T-t)}]^2\,[1 - e^{-\kappa\delta}]}, \\
b(t,\delta,T) &= \frac{\kappa}{1 - e^{-\kappa(T-t)}} \left[\frac{\kappa_1^2}{2\left(1 - e^{-\kappa(T-t)}\right)\left(1 - e^{-\kappa\delta}\right)} + 1\right]
\end{aligned}$$

4. Multi-factor Generalisations of the Extended Vasicek Model

The extended Vasicek model of Hull-White is driven by one noise term, and the evolution of the spot rate is determined by the equation

$$dr(t) = \alpha(t)[\theta(t) - r(t)]\,dt + \rho(t)\,dz(t), \tag{4.1}$$

where $\alpha(t)$ and $\rho(t)$ are functions of t, and $z(t)$ is a standard Wiener process.

To obtain a multi-factor generalisation of the extended Vasicek model, assume that the volatilities given by (2.1) are deterministic so that $\varsigma_i(s, r(s)) = \varsigma_i(s)$. Then

$$\sigma_i(s,t) = \varsigma_i(s)\, e^{-\int_s^t \kappa_i(v)\,dv}, \tag{4.2}$$

and $\sigma_i(s,t)$ and $\xi_i(t) = \int_0^t \sigma_i^2(s,t)\,ds$ are deterministic functions. Further, (2.7) can be written

$$dr(t) = \left[\hat{\theta}(t) - \kappa_1(t) r(t)\right] dt + \sum_{i=1}^{n} \varsigma_i(t)\,dW_i(t), \tag{4.3}$$

where

$$\hat{\theta}(t) = \frac{\partial f(0,t)}{\partial t} + \kappa_1(t) f(0,t) + \sum_{i=1}^{n} \xi_i(t) + \sum_{i=2}^{n} \left[\kappa_1(t) - \kappa_i(t)\right] \zeta_i(t), \tag{4.4}$$

and from (2.6) the state variables $\zeta_i(t)$ satisfy

$$d\zeta_i(t) = [\xi_i(t) - \kappa_i(t)\zeta_i(t)]\,dt + \varsigma_i(t)\,dW_i(t), \tag{4.5}$$

for $2 \le i \le n$. In the special case $n = 1$, it is easily seen that the model reduces to the extended Vasicek model (4.1) with

$$\alpha(t) = \kappa_1(t), \quad \rho(t) = \varsigma_1(t), \quad \theta(t) = \frac{\hat{\theta}(t)}{\kappa_1(t)}. \tag{4.6}$$

The bond price formula for the multi-factor generalisation of the extended Vasicek models are given by (2.8), and since the models are Gaussian, European call option prices are given by [Rut96, Corollary 3.2].

Note that the derivation of the extended Vasicek model from the HJM framework provides a greater insight into the model parameters, in particular $\theta(t)$, and automatically provides a *risk neutral* formulation of the interest rate model.

5. MULTI-FACTOR GENERALISATION OF THE HULL-WHITE TWO FACTOR MODEL

In the Hull-White [HW94] two factor model, the spot rate process $r(t)$ is assumed to satisfy the stochastic differential equation

$$dr(t) = [\theta(t) + u(t) - a(t)\, r(t)]\, dt + \gamma_1(t)\, dz_1(t), \tag{5.1}$$

where the additional term $u(t)$ in the drift satisfies

$$du(t) = -b(t)\, u(t)\, dt + \gamma_2(t)\, dz_2(t), \tag{5.2}$$

and $z_1(t)$ and $z_2(t)$ are correlated Wiener processes with

$$\mathbb{E}[dz_1(t)\, dz_2(t)] = \rho\, dt. \tag{5.3}$$

The above model is a generalisation of the extended Vasicek model of §4 in which an additional stochastic factor has been introduced to accommodate a wider range of yield curves. Note that (5.1) and (5.2) can be rewritten in terms of *independent* Wiener processes $W_1(t)$ and $W_2(t)$ as follows

$$dr(t) = [\theta(t) + u(t) - a(t)r(t)]dt + \gamma_1(t)\sqrt{1-\rho^2}\, dW_1(t) + \gamma_1(t)\rho\, dW_2(t), \tag{5.4}$$
$$du(t) = -b(t)u(t) + \gamma_2(t)\, dW_2(t). \tag{5.5}$$

From (4.3), (4.4), and (4.5), it can be seen that the HJM framework considered in §4 *almost* reduces to the Hull-White two-factor model with $\zeta_i(t)$ playing the role of $u(t)$, except that the SDE (4.5) for $\zeta_i(t)$ does not have the required form (5.5). As will be seen, a minor change in the choice of state variables $\zeta_i(t)$ will result in the HJM n-factor generalisation of the Hull-White two-factor model.

In order to obtain the Hull-White two-factor model, take the volatility processes of the form

$$\sigma_i(s,t) = \hat{\sigma}_i(s) e^{-\int_s^t \kappa_i(v)\, dv}, \tag{5.6}$$

for $1 \le i \le n$, as in §4. For $1 \le i \le n$, define

$$\eta_i(t) = \int_0^t \sigma_i(s,t)\, dW_i(s). \tag{5.7}$$

Then as in (4.3),

$$dr(t) = \left[\hat{\theta}(t) + \sum_{i=2}^{n} [\kappa_1(t) - \kappa_i(t)]\, \eta_i(t) - \kappa_1(t)r(t)\right] dt + \sum_{i=1}^{n} \hat{\sigma}_i(t)\, dW_i(t), \tag{5.8}$$

where

$$\hat{\theta}(t) = \frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{n} \int_0^t \hat{\sigma}_i^2(s)\, e^{-2\int_s^t \kappa_i(v)\, dv} ds, \tag{5.9}$$

and the state variables $\eta_i(t)$, satisfy the SDE

$$d\eta_i(t) = -\kappa_i(t)\, \eta_i(t)\, dt + \hat{\sigma}_i(t)\, dW_i(t), \quad 2 \le i \le n. \tag{5.10}$$

In the special case $n = 2$, the model reduces to the Hull-White two-factor model with

$$\theta(t) = \hat{\theta}(t), \quad u(t) = \zeta_2(t), \quad a(t) = \kappa_1(t), \quad b(t) = \kappa_2(t), \tag{5.11}$$

$$\gamma_1(t)\sqrt{1-\rho^2} = \hat{\sigma}_1(t), \quad \gamma_1(t)\rho = \hat{\sigma}_2(t), \quad \gamma_2(t) = \hat{\sigma}_2(t). \tag{5.12}$$

Note that in the derivation of the Hull-White two-factor model from the HJM framework, the coefficient of $u(t)$ in (5.4) can be 1 *only* if $a(t)$ and $b(t)$ satisfy

$$a(t) - b(t) = 1. \tag{5.13}$$

Alternatively, if $a(t)$ and $b(t)$ are allowed to be arbitrary in the HJM framework, then the coefficient of $u(t)$ in (5.4) must be equal to $a(t) - b(t)$ rather than 1. Although this is perhaps a minor point, it illustrates the importance of deriving the models via the more consistent HJM framework. One not only obtains consistency restrictions for model parameters, but also valuable insight into the role played by the initial term structure in the determination of $\theta(t)$.

Note, finally, that the bond price in the multi-factor generalisation of the Hull-White two-factor model is once again given by (A.7), and the European call prices are given by [Rut96, Corollary 3.2].

6. Multi-factor Generalisations of the CIR Type Model

In the extended Cox-Ingersoll-Ross model of Hull-White, the evolution of the spot rate is determined by the equation

$$dr(t) = \alpha(t)[\theta(t) - r(t)]\,dt + \rho(t)\sqrt{r(t)}\,dz(t), \tag{6.1}$$

where $\alpha(t)$ and $\rho(t)$ are functions of t, and $z(t)$ is a standard Wiener process.

To obtain a multi-factor generalisation of the extended CIR type model, assume that the volatilities given by (2.1) are of the form $\varsigma_i(s, r(s)) = \varsigma_i(s)\sqrt{r(s)}$. Then

$$\sigma_i(s,t) = \varsigma_i(s)\sqrt{r(s)}\,e^{-\int_s^t \kappa_i(v)\,dv}. \tag{6.2}$$

Note that since $\sigma_i(s,t)$ are not deterministic, $\xi_i(t) = \int_0^t \sigma_i^2(s,t)\,ds$ are also not deterministic. Now, equation (2.7) can be written in the form

$$dr(t) = \left[\hat{\theta}(t) - \kappa_1(t)r(t)\right]dt + \sum_{i=1}^{n} \varsigma_i(t)\sqrt{r(t)}\,dW_i(t), \tag{6.3}$$

where

$$\hat{\theta}(t) = \frac{\partial f(0,t)}{\partial t} + \kappa_1(t)f(0,t) + \sum_{i=1}^{n} \xi_i(t) + \sum_{i=2}^{n} [\kappa_1(t) - \kappa_i(t)]\,\zeta_i(t), \tag{6.4}$$

and from (2.5) and (2.6) the state variables $\xi_i(t)$ and $\zeta_i(t)$ satisfy

$$d\xi_i(t) = \left[\varsigma_i^2(t)r(t) - 2\kappa_i(t)\xi_i(t)\right]dt, \tag{6.5}$$

$$d\zeta_i(t) = [\xi_i(t) - \kappa_i(t)\zeta_i(t)]\,dt + \varsigma_i(t)\sqrt{r(t)}\,dW_i(t), \tag{6.6}$$

for $1 \leq i \leq n$. In the special case $n = 1$, the model reduces to the extended CIR type model with

$$\alpha(t) = \kappa_1(t), \quad \rho(t) = \hat{\sigma}_1(t), \quad \theta(t) = \frac{\hat{\theta}(t)}{\kappa_1(t)}. \tag{6.7}$$

Note, however, that the state variable $\xi_1(t)$ does not appear in the original CIR formulation.

The bond price formula for the multi-factor generalisation of the extended CIR type models is given by (2.8), and since the models are Gaussian, European call prices are given by [Rut96, Corollary 3.2].

As previously, deriving these models from the HJM framework provides a greater insight into the model parameters, in particular $\theta(t)$, and automatically provides a *risk neutral* formulation of the interest rate model.

7. A General Framework

Unlike the models considered so far, in which the spot rate is assumed to follow a Gaussian process, the Black-Karasinski model assumes a log-normal process for the spot rate. In order to obtain the Black-Karasinski type models from the HJM framework, a more general framework must be developed in which the evolution of a function of the forward rate is modeled rather than the forward rate itself.

Let U and V be subsets of $\mathbb{R}$, and let $G\colon U \to V$ be an invertible function with inverse $H\colon V \to U$. We will write $G(x)$ and $H(x)$ for the two functions and write $G'(x)$, $H'(x)$ and $G''(x)$, $H''(x)$ for their first and second derivatives. Then, instead of beginning with the stochastic integral equation (1.1) for the forward rate $f(t, T, \omega)$, begin with a stochastic integral equation for $G[f(t, T, \omega)]$

$$G[f(t,T,\omega)] = G[f(0,T)] + \int_0^t \alpha(s,T,\omega)\,ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,T,\omega)\,dW_i(s). \tag{7.1}$$

By putting $T = t$, the corresponding stochastic integral equation for $G[r(t, \omega)]$ can be obtained as

$$G[r(t,\omega)] = G[f(0,t)] + \int_0^t \alpha(s,t,\omega)\,ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t,\omega)\,dW_i(s). \tag{7.2}$$

Since $f(t, T) = H[G[f(t, T)]]$,

$$\begin{aligned} f(t,T,\omega) = f(0,T) &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,T,\omega) H'\left[G[f(s,T)]\right]\,dW_i(s) \\ &+ \int_0^t \left[\alpha(s,T) H'\left[G[f(s,T)]\right] + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(s,T,\omega) H''\left[G[(s,T)]\right] \right] ds \end{aligned} \tag{7.3}$$

by Itô's lemma. Now if

$$\breve{\alpha}(s,T,\omega) = \alpha(s,T)H'\left[G[f(s,T)]\right] + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(s,T,\omega)H''\left[G[f(s,T)]\right], \tag{7.4}$$

$$\breve{\sigma}(s,T,\omega) = \sigma_i(s,T,\omega)H'\left[G[f(s,T)]\right], \tag{7.5}$$

then (7.3) can be written as

$$f(t,T,\omega) = f(0,T) + \int_0^t \breve{\alpha}(s,T,\omega)\,ds + \sum_{i=1}^{n}\int_0^t \breve{\sigma}_i(s,T,\omega)\,dW_i(s), \tag{7.6}$$

which is the standard HJM formulation, and the arguments of [HJM92] imply

$$\breve{\alpha}(t,T) = -\sum_{i=1}^{n}\breve{\sigma}(t,T)\left[\phi_i(t) - \int_t^T \breve{\sigma}_i(t,u)\,du\right], \tag{7.7}$$

where $\phi_i(t)$, $1 \leq i \leq n$, are the market prices of risk. The Wiener processes $\widetilde{W_i}(t)$ under the equivalent martingale measure $\widetilde{\mathbb{P}}$ are given by $\widetilde{W_i}(t) = W_i(t) - \int_0^t \phi_i(s)\,ds$, so that

$$d\widetilde{W_i}(t) = dW_i(t) - \phi_i(t)\,dt. \tag{7.8}$$

Substituting (7.7) into (7.4)

$$\begin{aligned}
&\alpha(s,T)H'\left[G[f(s,T)]\right] + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(s,T)H''\left[G[f(s,T)]\right] \\
&\qquad = -\sum_{i=1}^{n}\sigma_i(s,T)H'\left[G[f(s,T)]\right]\left[\phi_i(s) - \int_s^T \sigma_i(s,u)H'\left[G[f(s,u)]\right]\,du\right],
\end{aligned}$$

Rearranging the above equation gives for $\alpha(s,T)$ the expression

$$\begin{aligned}
\alpha(s,T) = &-\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(s,T)G'[f(s,T)]H''\left[G[f(s,T)]\right] \\
&-\sum_{i=1}^{n}\sigma_i(s,T)\left[\phi_i(s) - \int_s^T \sigma_i(s,u)H'\left[G[f(s,u)]\right]\,du\right],
\end{aligned} \tag{7.9}$$

since

$$H'\left[G[f(s,T)]\right]G'[f(s,T)] = 1. \tag{7.10}$$

Substituting (7.9) into (7.3) gives the stochastic integral equation

$$\begin{aligned}
G[f(t,T)] = G[f(0,T)] &- \frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,T)G'[f(s,T)]H''\left[G[f(s,T)]\right]\,ds \\
&+ \sum_{i=1}^{n}\int_0^t \sigma_i(s,T)\int_s^T \sigma_i(s,u)H'\left[G[f(s,u)]\right]\,du\,ds \\
&+ \sum_{i=1}^{n}\int_0^t \sigma_i(s,T)\,d\widetilde{W_i}(s),
\end{aligned} \tag{7.11}$$

for $G[f(t,T)]$ under the risk-neutral measure. Putting $T=t$, the equation for $G[r(t)]$ under the risk-neutral measure is

$$\begin{aligned} G[r(t)] = G[f(0,t)] &- \frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t)G'[f(s,t)]H''\left[G[f(s,t)]\right]\,ds \\ &+\sum_{i=1}^{n}\int_0^t \sigma_i(s,t)\int_s^t \sigma_i(s,u)H'\left[G[f(s,u)]\right]\,du\,ds \\ &+\sum_{i=1}^{n}\int_0^t \sigma_i(s,t)\,d\widetilde{W}_i(s). \end{aligned} \tag{7.12}$$

The corresponding SDE for $G[f(t,T)]$ is

$$\begin{aligned} dG[f(t,T)] = &\left\{-\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,T)G'[f(t,T)]H''\left[G[f(t,T)]\right]\right. \\ &\left.+\sum_{i=1}^{n}\sigma_i(t,T)\int_t^T \sigma_i(t,u)H'\left[G[f(t,u)]\right]\,du\right\}dt \\ &+\sum_{i=1}^{n}\sigma_i(t,T)\,d\widetilde{W}_i(t), \end{aligned} \tag{7.13}$$

and the corresponding SDE for $G[r(t)]$ is

$$\begin{aligned} dG[r(t)] = &\left\{\frac{\partial G[f(0,t)]}{\partial t} - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,t)G'[f(t,t)]H''\left[G[f(t,t)]\right]\right. \\ &-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial t}\left[\sigma_i^2(s,t)G'[f(s,t)]H''\left[G[f(s,t)]\right]\right]ds \\ &+\sum_{i=1}^{n}\left[\int_0^t \sigma_i^2(s,t)H'\left[G[f(s,t)]\right]ds\right] \\ &+\sum_{i=1}^{n}\left[\int_0^t \frac{\partial\sigma_i(s,t)}{\partial t}\int_s^t \sigma_i(s,u)H'\left[G[f(s,u)]\right]\,du\,ds\right] \\ &\left.+\sum_{i=1}^{n}\int_0^t \frac{\partial\sigma_i(s,t)}{\partial t}\,d\widetilde{W}_i(s)\right\}dt + \sum_{i=1}^{n}\sigma_i(t,t)\,d\widetilde{W}_i(t). \end{aligned} \tag{7.14}$$

The SDE (7.14) corresponds to models in which the dynamics of a function of the spot rate is specified rather than the dynamics of the spot rate itself.

8. MULTI-FACTOR GENERALISATION OF THE BLACK-KARASINSKI MODEL

In this section the Black-Karasinski [BK91] type models are obtained as special cases of the HJM framework using the techniques introduced in §7. In the Black-Karasinski model, the log of the spot rate, $\ln r(t)$, is assumed to satisfy the SDE

$$d\ln r(t) = \alpha(t)\left[\theta(t) - \ln r(t)\right]\,dt + \rho(t)\,dz(t), \tag{8.1}$$

where $\alpha(t)$, $\theta(t)$, and $\rho(t)$ are functions of t, and $z(t)$ is a standard Wiener process.

Now, to obtain a Black-Karasinski type model, take $G(x) = \ln x$ in the framework developed in §7 so that $H(x) = e^x$. Then

$$G'[f(t,T)] = \frac{1}{f(t,T)}, \tag{8.2}$$

$$H'\left[G[f(t,T)]\right] = H''\left[G[f(t,T)]\right] = f(t,T). \tag{8.3}$$

Substitution into (7.12) gives

$$\begin{aligned} \ln r(t) = \ln f(0,t) &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,t) \int_s^t \sigma_i(s,u) f(s,u)\,du\,ds \\ &- \frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t)\,ds + \sum_{i=1}^{n}\int_0^t \sigma_i(s,t)\,d\widetilde{W}_i(s), \end{aligned} \tag{8.4}$$

and substitution into (7.14) gives

$$\begin{aligned} d\ln r(t) = \Bigg\{ &\frac{1}{f(0,t)}\frac{\partial f(0,t)}{\partial t} - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,t) - \sum_{i=1}^{n}\int_0^t \sigma_i(s,t)\frac{\partial \sigma_i(s,t)}{\partial t}\,ds\,dt \\ &+ \sum_{i=1}^{n}\int_0^t \left[\sigma_i^2(s,t)f(s,t) + \frac{\partial \sigma_i(s,t)}{\partial t}\int_s^t \sigma_i(s,u)f(s,u)\,du\right] ds \\ &+ \sum_{i=1}^{n}\int_0^t \frac{\partial \sigma_i(s,t)}{\partial t}\,d\widetilde{W}_i(s)\Bigg\}\,dt + \sum_{i=1}^{n}\sigma_i(t,t)\,d\widetilde{W}_i(t). \end{aligned} \tag{8.5}$$

To obtain the Black-Karasinski model, consider the volatility processes

$$\sigma_i(s,t) = \varsigma_i(s) e^{-\int_s^t \kappa_i(v)\,dv}. \tag{8.6}$$

Then $\partial\sigma_i(s,t)/\partial t = -\kappa_i(t)\sigma_i(s,t)$, and (8.5) becomes

$$\begin{aligned} d\ln r(t) = \Bigg\{ &\frac{1}{f(0,t)}\frac{\partial f(0,t)}{\partial t} - \frac{1}{2}\sum_{i=1}^{n}\varsigma_i^2(t) + \sum_{i=1}^{n}\kappa_i(t)\int_0^t \sigma_i^2(s,t)\,ds \\ &+ \sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t) f(s,t)\,ds \\ &- \sum_{i=1}^{n}\kappa_i(t)\int_0^t \sigma_i(s,t)\int_s^t \sigma_i(s,u)f(s,u)\,du\,ds \\ &- \sum_{i=1}^{n}\kappa_i(t)\int_0^t \sigma_i(s,t)\,d\widetilde{W}_s^i\Bigg\}\,dt + \sum_{i=1}^{n}\varsigma_i(t)\,d\widetilde{W}_i(t). \end{aligned} \tag{8.7}$$

Now introduce variables $\zeta_i(t)$ by

$$\ln\zeta_i(t) = \int_0^t \sigma_i(s,t)\int_s^t \sigma_i(s,u)f(s,u)\,du\,ds + \int_0^t \sigma_i(s,t)\,d\widetilde{W}_i(s), \tag{8.8}$$

for $1 \le i \le n$. Then $\ln\zeta_i(t)$ satisfy the SDE

$$d\ln\zeta_i(t) = \left[\int_0^t \sigma_i^2(s,t)f(s,t)\,ds - \kappa_i(t)\ln\zeta_i(t)\right]dt + \varsigma_i(t)\,d\widetilde{W}_i(t), \tag{8.9}$$

and (8.4) and (8.7) can be written

$$\ln r(t) = \ln f(0,t) - \frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t)\,ds + \sum_{i=1}^{n}\zeta_i(t), \tag{8.10}$$

$$\begin{aligned} d\ln r(t) = \Bigg\{ & \frac{1}{f(0,t)}\frac{\partial f(0,t)}{\partial t} - \frac{1}{2}\sum_{i=1}^{n}\varsigma_i^2(t) + \sum_{i=1}^{n}\kappa_i(t)\int_0^t \sigma_i^2(s,t)\,ds \\ & + \sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t) f(s,t)\,ds \\ & - \kappa_1(t)\left[\ln r(t) - \ln f(0,t) + \frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t)\,ds\right] \\ & + \sum_{i=2}^{n}\left[\kappa_1(t) - \kappa_i(t)\right]\zeta_i(t)\Bigg\}\,dt + \sum_{i=1}^{n}\varsigma_i(t)\,d\widetilde{W}_i(t). \end{aligned} \tag{8.11}$$

Finally, (8.11) can be rewritten as

$$\begin{aligned} d\ln r(t) = & \left[\hat{\theta}(t) - \kappa_1(t)\ln r(t) + \sum_{i=2}^{n}[\kappa_1(t) - \kappa_i(t)]\ln\zeta_i(t)\right] dt \\ & + \sum_{i=1}^{n}\varsigma_i(t)\,d\widetilde{W}_i(t), \end{aligned} \tag{8.12}$$

where

$$\begin{aligned} \hat{\theta}(t) = & \frac{1}{f(0,t)}\frac{\partial f(0,t)}{\partial t} + \kappa_1(t)\ln f(0,t) - \frac{1}{2}\sum_{i=1}^{n}\varsigma_i^2(t) \\ & + \sum_{i=1}^{n}\left[\kappa_i(t) - \frac{1}{2}\kappa_1(t)\right]\int_0^t \sigma_i^2(s,t)\,ds + \sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t) f(s,t)\,ds, \end{aligned} \tag{8.13}$$

and $\ln\zeta_i(t)$ satisfies the SDE (8.9).

In the special case $n = 1$, the model reduces to a Black-Karasinski type model with

$$\alpha(t) = \kappa_1(t), \quad \rho(t) = \varsigma_1(t), \quad \theta(t) = \frac{\hat{\theta}(t)}{\kappa_1(t)}. \tag{8.14}$$

9. CONCLUSION

In this paper, it was shown that suitable specialisations of the forward rate volatility processes produce interest models that resemble many of the traditional interest rate models, such as the extended Vasicek, the two factor Hull-White model, the Cox-Ingersoll-Ross, and the Black-Karasinki model as special cases of the HJM framework. Future research will focus on the forms of volatility processes required to produce the traditional models.

It was also shown that the derivation of interest models from the HJM framework not only provides a better insight into the model parameters, but also results in models that are automatically calibrated to the initial term structure.

Appendix A. Derivation of the Bond Price

Lemma A.1. *Let $\sigma_i(t,T)$ be given by (2.1). Then the following identities hold:*

$$\sigma_i(s,t+u) = \sigma_i(s,t)\, e^{-\int_t^{t+u}\kappa_i(v)\,dv}, \tag{A.1}$$

$$\sigma_i(s,t) = \sigma_i(s,s)\, e^{-\int_s^{t}\kappa_i(v)\,dv}. \tag{A.2}$$

Proof. For the first identity, note that

$$\begin{aligned}\sigma_i(s,t+u) &= \varsigma_i\left(r(s)\right)\, e^{\int_s^{t+u}\kappa_i(x)\,dx} = \left(\varsigma_i\left(r(s)\right)\, e^{\int_s^{t}\kappa_i(x)\,dx}\right) e^{\int_t^{t+u}\kappa_i(x)\,dx} \\ &= \sigma_i(s,t)\, e^{\int_t^{t+u}\kappa_i(x)\,dx}.\end{aligned}$$

The second identity follows from setting $t=s$ and $t+u=t$. □

The following lemma is contained in [IK98, p37].

Lemma A.2. *Let $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)\,dx}du$. Then the following identities hold.*

$$\int_t^T e^{-\int_t^u \kappa_i(x)\,dx}\int_t^u e^{-\int_t^v \kappa_i(x)\,dx}dv\,du = \frac{1}{2}\gamma_i^2(t,T), \tag{A.3}$$

$$\int_t^T \sigma_i^*(s,u)\,du = \gamma_i(t,T)\,\sigma_i^*(s,t) + \frac{1}{2}\gamma^2(t,T)\,\sigma_i^2(s,t), \tag{A.4}$$

$$\int_t^T \sigma_i(s,u)\,du = \gamma_i(t,T)\,\sigma_i(s,t), \tag{A.5}$$

where $\sigma_i^*(s,u) = \sigma_i(s,u)\int_s^u \sigma_i(s,v)\,dv$.

Proof. Denote by $\chi_i(t,T)$ the term on the left hand side of (A.3). Then

$$\begin{aligned}\chi_i(t,T) &= \int_t^T e^{-\int_t^u \kappa_i(x)\,dx}\int_t^u e^{-\int_t^v \kappa_i(x)\,dx}dv\,du \\ &= \int_t^T \frac{d}{du}\left[\int_t^u e^{-\int_t^v \kappa_i(x)\,dx}dv\right]\left[\int_t^u e^{-\int_t^v \kappa_i(x)\,dx}dv\right]du \\ &= \int_t^T \gamma_i(t,u)\frac{d}{du}\gamma_i(t,u)\,du = \int_t^T d\left[\frac{1}{2}\gamma_i^2(t,u)\right] \\ &= \frac{1}{2}\left[\gamma_i(t,T) - \gamma_i(t,t)\right] \\ &= \frac{1}{2}\gamma_i^2(t,T), \quad \text{since } \gamma_i(t,t) = 0.\end{aligned}$$

Now consider (A.4). Using (A.1),

$$\begin{aligned}\int_t^T \sigma_i^*(s,u)\,du &= \int_t^T \sigma_i(s,u)\int_s^u \sigma_i(s,v)\,dv\,du\\ &= \sigma_i(s,t)\int_t^T e^{-\int_t^u \kappa_i(x)\,dx}\left[\int_s^t \sigma_i(s,v)\,dv + \int_t^u \sigma_i(s,v)\,dv\right] du\\ &= \sigma_i(s,t)\int_t^T e^{-\int_t^u \kappa_i(x)\,dx} du \int_s^t \sigma_i(s,v)\,dv + \sigma_i^2(s,t)\,\chi_i(t,T)\\ &= \gamma_i(t,T)\sigma_i(s,t)\int_s^t \sigma_i(s,v)\,dv + \frac{1}{2}\sigma_i^2(s,t)\gamma_i^2(t,T),\end{aligned}$$

which is (A.4). Similar arguments establish (A.5). □

Recall from (2.3) and (2.4) the definition of the state variables $\xi_i(t)$ and $\zeta_i(t)$. Note that in view of (1.1) and (1.4)

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[-\sum_{i=1}^n \int_t^T \zeta_i(u)\,du\right]. \tag{A.6}$$

The following bond price formula is contained in [RS95, p60] for the one factor case, and [IK98, p431] for the multi-factor case.

Theorem A.3. *If $\sigma_i(t,T)$ is as given in (2.1), then the bond price is given by the formula*

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left\{-\Phi(t,T) - \Psi(t,T) - \gamma_1(t,T)\left[r(t) - f(0,t)\right]\right\}, \tag{A.7}$$

where $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)\,dx} du$, for $1 \le i \le n$,

$$\begin{aligned}\Phi(t,T) &= \frac{1}{2}\sum_{i=1}^n \xi_i(t)\,\gamma_i^2(t,T), \quad \textit{and}\\ \Psi(t,T) &= \sum_{i=2}^n \zeta_i(t)\left[\gamma_i(t,T) - \gamma_1(t,T)\right].\end{aligned}$$

Proof. In view of (A.6), $\int_t^T \xi_i(u)\,du$ must be computed:

$$\begin{aligned}\int_t^T \xi_i(u)\,du &= \int_t^T \left[\int_0^t \sigma_i^*(s,u)\,ds + \int_0^t \sigma_i(s,u)\,d\widetilde{W}_i(s)\right] du\\ &= \int_0^t \int_t^T \sigma_i^*(s,u)\,du\,ds + \int_0^t \int_t^T \sigma_i(s,u)\,du\,d\widetilde{W}_i(s)\\ &= \int_0^t \left[\gamma_i(t,T)\,\sigma_i^*(s,t) + \frac{1}{2}\gamma^2(t,T)\,\sigma_i^2(s,t)\right] ds\\ &\qquad + \int_0^t \gamma_i(t,T)\,\sigma_i(s,t)\,d\widetilde{W}_i(s) \quad \text{by (A.4) and (A.5)}\\ &= \gamma_i(t,T)\,\zeta_i(t) + \frac{1}{2}\gamma_i^2(t,T)\,\xi_i(t),\end{aligned}$$

where Fubini Theorem was used in the third equality. It follows that

$$\begin{aligned}\sum_{i=1}^{n}\int_t^T \zeta_i(u)\,du &= \sum_{i=1}^{n}\left[\gamma_i(t,T)\,\zeta_i(t)+\frac{1}{2}\gamma_i^2(t,T)\,\xi_i(t)\right]\\ &= \Phi(t,T)+\sum_{i=1}^{n}\left[(\gamma_i(t,T)-\gamma_1(t,T))+\gamma_1(t,T)\right]\,\zeta_i(t)\\ &= \Phi(t,T)+\Psi(t,T)+\gamma_1(t,T)\sum_{i=1}^{n}\zeta_i(t)\\ &= \Phi(t,T)+\Psi(t,T)+\gamma_1(t,T)\,\left[r(t)-f(0,t)\right].\end{aligned}$$

This completes the proof. □

Note that the above formula applies to a larger class of volatility processes than those given by (2.1). The formula is, in fact, valid for all volatility processes that satisfy (A.1), including those that involve a finite number of fixed tenor forward rates, as shown in [CK98].

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