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Research Paper 126 **March 2004**

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ISSN 1441-8010 www.qfrc.uts.edu.au

THE EVALUATION OF POINT BARRIER OPTIONS IN A PATH INTEGRAL FRAMEWORK

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Abstract. The pricing of point barrier or discretely monitored barrier options is a difficult problem. In general, there is no known closed form solution for pricing such options. In this paper we develop a path integral approach to the evaluation of barrier options. This leads to a backward recursion functional equation linking the pricing functions at successive barrier points. We solve this functional equation by expanding the pricing functions in Fourier-Hermite series. The backward recursion functional equation then becomes the backward recurrence relation for the coefficients in the Fourier-Hermite expansion of the pricing functions. We thus obtain a very efficient and accurate method for generating the pricing function at any barrier point.

We perform a number of numerical experiments with the method in order to gain some understanding of the nature of convergence. We present results for various volatility values and different numbers of basis functions in the Fourier-Hermite expansion. Comparisons will be given between pricing of point barriers in the path integral framework and by use of finite difference methods.

Date: March 10, 2005.

1. Introduction

Barrier options are derivative securities with values contingent on the relationship between the value of the underlying asset and one or more barrier levels. In this paper we consider the pricing of barrier options that are monitored at particular points over the life of the contract, also known as point barrier options. These types of exotic options have become a prominent feature of modern financial markets over the last decade with many variations heavily traded in the foreign exchange, equity and fixed income markets. The emergence of such securities has also provided challenging research problems in the area of efficient pricing and hedging of such securities. Most pricing models (Merton (1973), Rubinstein & Reiner (1991), Heynen & Kat (1996)) consider barrier options whose barrier level is continuously monitored at every instant in time over the life of the option, allowing the derivation of closed form solutions. However most traded barrier options are monitored discretely, rather than on a continuous basis. The application of the solutions derived assuming continuous monitoring to the pricing of discretely monitored barrier options results in substantial pricing errors (Chance (1994), Kat (1995) and Levy & Mantion (1997)). Hence the need for a method to accurately and efficiently evaluate discretely monitored barrier options. Traditional lattice and Monte Carlo methods have difficulties in incorporating discrete monitoring principally because of the misalignment of the monitoring points.

Several papers have appeared in the literature proposing various methods to handle discrete monitoring. Variations to the traditional binomial and trinomial methods were proposed by Figlewski & Gao (1997), Tian (1996) and Boyle & Tian (1997). These include Broadie, Glasserman & Kou (1997*a*) which proposes a method based on the price of a continuous barrier options with a continuity correction for discrete monitoring. Broadie, Glasserman & Kou (1997*b*) incorporate the correction term in developing a lattice method for determining accurate prices of discrete and continuous path-dependent options. As far as the pricing discretely monitored barrier options is concerned Wei (1998) proposes an interpolation method where as Sullivan (2000) proposes a method that reduces the discrete time multidimensional integration required to a sequential numerical integration.

In this paper we examine a potentially powerful alternative to existing pricing methods involving the expansion of the value of the discretely monitored barrier option in a Fourier-Hermite series in terms of the price of the underlying asset. We derive recurrence relations involving orthogonal polynomials under a given measure. The proposed method works well for various point barrier structures including, single and double barriers, constant and time varying barrier levels and for a number of payoff structures such as vanillas, digital and powers. The method can be made arbitrarily accurate by taking a sufficient number of terms in the expansion.

The techniques used extend the work of Chiarella, El-Hassan & Kucera (1999) where Fourier-Hermite series expansions were applied to the valuation of European and American options. The novel aspect of our contribution is the expansion of the derivative security price at each time step in a Fourier-Hermite series expansion so that it is obtained as a continuous and differentiable function of the price of the underlying asset. The actual implementation of our method then becomes a question of determining the coefficients of the Fourier-Hermite series expansion at each time step. Using the orthogonalisation condition, it turns out that these can be generated recursively by working backwards from one time step to the next, by use of the recurrence relations which generate the Hermite polynomials. The implementation of these recurrence relations is in fact very efficient.Here we apply the method to pricing discretely monitored barrier options. Since we obtain a continuous and differentiable representation of the price, the hedge ratios delta and gamma can be obtained to a high level of accuracy and also very cheaply in terms of calculation time.

In recent years it has become appreciated that the path integral technique of statistical physics can be applied to derivative security valuation. We refer in particular to Linetsky (1997) who provides an overview of the path integral concept and its application to financial problems. The wide application of path integral techniques to financial modelling and in particular to the pricing and hedging of options, including path-dependent and exotic options, was first studied by Dash (1988). Dash's contribution to the area was largely in the formulation of many derivative security pricing problems in the path integral framework, including standard equity options, exotic options, path dependent options and the pricing of bond options under a number of popular term structure models. The framework provides an intuitive description of the value of derivative securities using relatively simple mathematics.

However, the application and implementation of solution techniques of path integrals to finance problems has been limited, with the cited authors focusing on a general framework and pointing to the potential for the application of these techniques to financial pricing problems. Path integrals can be evaluated in a number of ways including analytic approximations by means of moment expansions in a perturbation series, deterministic discretisation schemes of the path integral and Monte-Carlo simulation (Makici (1995)) methods. The method chosen will in general depend on the problem at hand and the stability of the solution technique used. We also refer to the contribution of Eydeland (1994) who provides a computational algorithm based on Toeplitz matrix structure and fast Fourier transforms for evaluating financial securities in a path integral framework. This technique was successfully applied by Chiarella & El-Hassan (1997) to evaluate European and American bond options in an HJM framework.

The organisation of the paper is as follows: In section 2, we define the general barrier structure and outline the backward recursion procedure. In section 3, we show how the path integral formulation for the value of a discretely monitored barrier defined in section 2 can be represented as an expansion of the value function in a series of orthogonal polynomials and reduced to a backward recursion procedure. In section 4, we report some numerical results and in section 5 we make some concluding remarks.

2. The General Barrier Structure

We denote the underlying asset price by S and assume that under the risk-free measure it follows a geometric Brownian motion given by

$$dS = (r - q)Sdt + \sigma SdW,$$

where σ is the volatility, r is the risk-free rate of interest, q is the continuous dividend yield on the underlying asset and $W(t)$ is a standard Wiener process. By allowing for a continuous dividend yield q our framework may be applied to barrier options on indices, or to options on foreign exchange by setting $q = r_f$ where r_f is the risk free rate of interest in the foreign economy.

The implementation of the path integral method, to be described below, requires expansion of pricing functions in terms of Hermite polynomials. These are defined on an infinite interval, for this reason we need to transform the asset price to a variable defined on an infinite interval. This is most conveniently done by introducing the change of variable

$$\xi = \frac{1}{\sigma}\ln(S). \tag{2.1}$$

A straight forward application of Ito's lemma reveals that ξ satisfies the stochastic differential equation

$$d\xi = \frac{1}{\sigma}\left((r-q) - \frac{1}{2}\sigma^2\right)dt + dW(t). \tag{2.2}$$

We recall that (2.2) implies that the transition probability density for ξ between two times $t', t(t' < t)$, denoted $\pi\left(\xi_t, t|\xi_{t'}, t'\right)$ is normally distributed and is in fact given by

$$\pi\left(\xi_t, t|\xi_{t'}, t'\right) = \frac{1}{\sqrt{2\pi(t-t')}}\exp\left[\frac{-\left[\xi_t - \sqrt{2(t-t')}\mu(\xi_{t'}, t-t')\right]^2}{2(t-t')}\right], \tag{2.3}$$

where it is convenient to define

$$\mu(\xi, t) = \frac{1}{\sqrt{2t}}\left(\xi + \frac{1}{\sigma}\left(r - q - \frac{1}{2}\sigma^2\right)t\right). \tag{2.4}$$

We divide the time interval from initial time to option maturity into K subintervals $(t^{k-1}, t^k), (k = 0, 1, \cdots, K)$. The spacings between the barrier observation points, t^k, need not be constant. We set $\Delta t^k = t^k - t^{k-1}$ for $k = K, \cdots, 1$ with the implied notation that $t^0 = 0$ and $t^K = T$ so that $T = \sum_{k=1}^{K}\Delta t^k$.

We allow for barrier levels that can be time dependent. Thus at each time t^k there will be an upper barrier level, b_u^k, and a lower barrier level b_l^k, for $k = K-1, \cdots 1$ Here we note that we do not include barriers at expiry since they are part of the pay-off definition. With this notation we may handle the case of no lower barrier by setting $b_l^k = 0$, and the case of no upper barrier by letting $b_u^k \to \infty$. Figure 1 illustrates in the S, t plane a typical discretisation with a variety of possible barriers at the discretisation points.

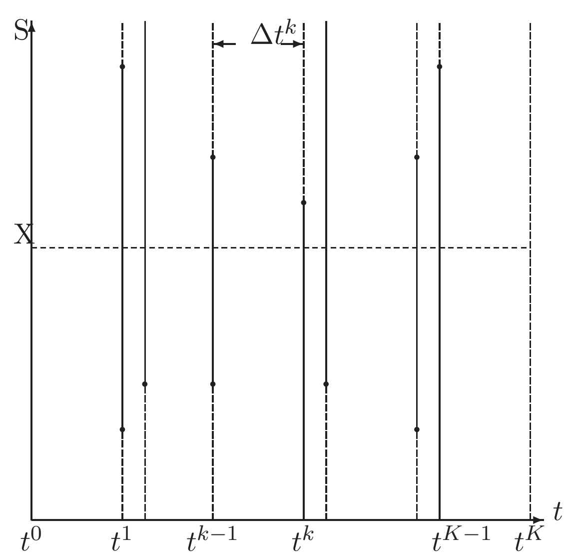

FIGURE 1 : *The Discretisation Scheme*

Between the barrier observation points, t^k, the option is assumed to be path-independent. Furthermore, the market parameters, r, q, and σ are assumed to stay constant between any two observation points. However, they can change from time interval to time interval. Thus for $t\epsilon[t_{k-1}, t_k]$ we set

$$r(t) = r^k, q(t) = q^k, \qquad \text{and} \qquad \sigma(t) = \sigma^k. \tag{2.5}$$

Because we have chosen to let the market parameters be time dependent (albeit constant across each time interval) when propagating the price function from $t = t^k$ back to $t = t^{k-1}$ the integration variable ξ^k is written on each sub-interval $[t_{k-1}, t_k]$ as

$$\xi^k = \frac{ln(S)}{\sigma^k}, \qquad \text{for} \qquad k = K, \cdots, 1 \tag{2.6}$$

where σ^k is the spot volatility as given on that particular interval.

3. Barrier as a Functional Recurrence Equation

Under the risk neutral measure the price of a derivative security at any point in time is the discounted expected payoff at the next point at which it may be exercised. The transition probability density function (2.3) is the one that is required to calculate the expected payoff. We use $F^k(\xi^k)$ to denote the value function of the barrier option at monitoring point t^k as a function of ξ^k, the volatility scaled log price. Then the discounted expected value relation between the value functions at two successive monitoring points t^{k-1}, t^k is give by

$$F^{k-1}\left(\xi^{k-1}\right) = e^{-r^k \Delta t^k} \int_{\ln(b_l^k)/\sigma^k}^{\ln(b_n^k)/\sigma^k} \pi(\xi^k, t^k | \xi^{k-1}, t^{k-1}) F^k(\xi^k) d\xi^k. \tag{3.1}$$

Substituting equation (2.3) and making a change of integration variable we obtain

$$F^{k-1}(\xi^{k-1}) = \frac{e^{-r^k \Delta t^k}}{\sqrt{\pi}} \int_{z_l^k}^{z_u^k} e^{-(\xi^k - \mu(\xi^{k-1}, \Delta t^k))^2} F^k(\sqrt{2\Delta t^k} \xi^k) d\xi^k, \tag{3.2}$$

where, due to the the scaling factor $\sqrt{2\Delta t^k}$ in F^k, the limits of integration z_l^k and z_u^k are given by

$$z_l^k = \frac{\ln(b_l^k)}{\sigma^k \sqrt{2\Delta t^k}} \qquad \text{and} \qquad z_u^k = \frac{\ln(b_u^k)}{\sigma^k \sqrt{2\Delta t^k}}, \tag{3.3}$$

for $k = K - 1 \cdots, 1$.

Because of our geometric Brownian motion assumption for the underlying asset price the option pricing function is homogenous in S/X. Hence for a call option we need only consider a pay-off function of the form $\max(S - 1, 0)$ and for a put option a pay-off function $\max(1 - S, 0)$.

Under the log transformation the pay-off for a call option function becomes $\max[0, e^{\sigma^k \xi^k} - 1]$. Similarly the pay-off function for a put option is $\max[0, 1 - e^{\sigma^k \xi^k}]$. We use $F^K(\xi^K)$ to denote the payoff function at the final time t^K.

For ease of clarity and notation, when performing the path integrations across all the time intervals, we will denote ξ^k as x, ξ^{k-1} as ξ, z_l^k as z_l, z_u^k as z_u, r^k as r, q^k as q, σ^k as σ, Δt^k as Δt.

Thus the notation implicitly carries the time dependance of the problem. With the above notation, equation (3.2) can be written as

$$F^{k-1}(\xi) = e^{-r\Delta t}\frac{1}{\sqrt{\pi}}\int_{z_l}^{z_u} e^{-(x-\mu(\xi,\Delta t))^2} F^k(\sqrt{2\Delta t}\, x)dx, \qquad k = K, \cdots, 1. \quad (3.4)$$

Equation (3.4) is the functional recurrence equation that we need to solve. The task at hand involves successive iteration of this equation from the given pay-off function, $F^K(\xi^K)$ through the barrier points back to $t = 0$. On completing the iterations, it is a simple matter to invert the log transformation and return to the price variable S and evaluate F^0 at the required spot value. Figure 3 illustrates the region of integration and the concept of the backward propagation of the price function.

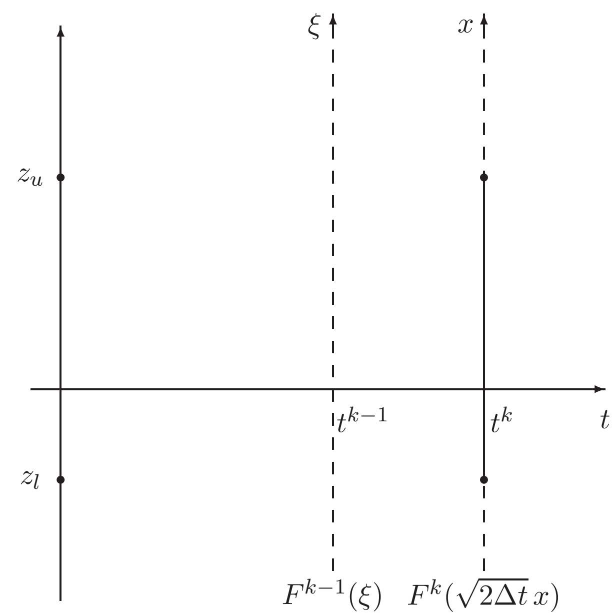

FIGURE 3 : *Propogating the Price Function Back from t^k to t^{k-1}*

4. The Obvious Fourier-Hermite Series Expansion

The basic problem in implementing (3.4) is to obtain a convenient way to evaluate the integral on the right-hand side and to then build up successively the value functions $F^k(\xi)\,(k = K-1, \cdots, 1)$. In this paper we solve this problem by expanding both $F^k(x)$ and $F^{k-1}(\xi)$ in Fourier-Hermite series. Thus we set

$$F^k(x) \approx \sum_{n=0}^{N} \alpha_n^k H_n(x),$$

and

$$F^{k-1}(\xi) \approx \sum_{m=0}^{N} \alpha_m^{k-1} H_m(\xi), \quad (4.1)$$

for N sufficiently large to ensure convergence of the series. By substituting the Fourier-Hermite expansions (4.1) into the backward recurrence (3.4) and making use of the orthogonality property of Hermite polynomials, it is possible to obtain a backward recurrence relation for the coefficients in the expansions (4.1). In this way, we are able to construct the value functions $F^k(\xi)$. We state this key result in proposition I.

Before stating this proposition it is useful to introduce some special notation for quantities which occur frequently in the calculations below.

Notation

First introduce the functions

$$L_m(x) = \frac{H_m(x)}{2^m m! v^m}, \tag{4.2}$$

which are easily shown to satisfy the recurrence equation

$$L_m(x) = \frac{x L_{m-1}(x)}{mv} - \frac{L_{m-2}(x)}{2(m-1)v^2}, \tag{4.3}$$

with $L_0(x) = 1$ and $L_1(x) = x/v$.

Next we set (recall $n(x) = e^{-x^2/2}/\sqrt{2\pi}$)

$$R_{m,n}(x) = \sqrt{2} L_m(x) H_n(vx + \beta) n(\sqrt{2}x), \tag{4.4}$$

from which we define

$$Q_{m,n}(x, y) = R_{m,n}(x) - R_{m,n}(y). \tag{4.5}$$

Finally we define

$$P(x, y) = N(\sqrt{2}x) - N(\sqrt{2}y), \tag{4.6}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du,$$

is the cumulative normal density function.

Proposition I

The known coefficients α_n^k and the to be calculated coefficients α_m^{k-1} at the time t^{k-1} are connected by the recurrence relation

$$\alpha_m^{k-1} = e^{-r\Delta t} \sum_{n=0}^{N} a_{m,n}^k \alpha_n^k, \qquad \text{for} \qquad k = K, \cdots, 1 \tag{4.7}$$

or in matrix notation:

$$\vec{\alpha}^k = e^{-r\Delta t} A^k \vec{\alpha}^k, \qquad \text{for} \qquad k = K, \cdots, 1 \tag{4.8}$$

where

$$A^k = [a_{m,n}^k] \qquad \text{for} \qquad m = 0, 1, \cdots, N \qquad \text{and} \qquad n = 0, 1, \cdots, N \tag{4.9}$$

The coefficients $a_{m,n}^k$ are generated by the equations outlined in the proof below.

Proof. Substituting the two series expansions (4.1) into the functional equation (3.4), using the orthogonality properties of the Hermite polynomials we find after some algebraic manipulations that

$$\alpha_m^{k-1} = e^{-r\Delta t} \sum_{n=0}^{N} \frac{n}{2^m m!} \frac{\alpha_n^k}{\sqrt{\pi}} \int_{z_u}^{z_l} H_n(\sqrt{2\Delta t}\, x) I_m(x) dx, \tag{4.10}$$

where

$$I_m(x) = \frac{\sqrt{2\Delta t}}{v^{m+1}} H_m\left(\frac{\sqrt{2\Delta t}\,x - \beta}{v}\right) exp\left[-\left(\frac{\sqrt{2\Delta t}x - \beta}{v}\right)^2\right], \tag{4.11}$$

with

$$v = \sqrt{1 + 2\Delta t} \qquad \text{and} \qquad \beta = \frac{1}{\sigma}\left(r_d - r_f - \frac{1}{2}\sigma^2\right)\Delta t. \tag{4.12}$$

Now, introducing the transformation

$$z = \frac{\sqrt{2\Delta t}x - \beta}{v}, \tag{4.13}$$

equation (4.10) can be written as

$$\alpha_m^{k-1} = e^{-r\Delta t}\sum_{n=0}^{N} a_{m,n}^k \alpha_n^k, \tag{4.14}$$

where the elements $a_{m,n}^k$ are given by

$$a_{m,n}^k = \frac{1}{2^m m! v^m}\frac{1}{\sqrt{\pi}}\int_{x_l}^{x_u} e^{-x^2} H_m(x) H_n(vx + \beta)dx. \tag{4.15}$$

The new limits of integration x_l and x_u are given by

$$x_l = \frac{\sqrt{2\Delta t}z_l - \beta}{v} \qquad \text{and} \qquad x_u = \frac{\sqrt{2\Delta t}z_u - \beta}{v}. \tag{4.16}$$

We note that at final time t^K for a call, the integration limits become

$$x_l = 0 \qquad \text{and} \qquad x_u \to \infty, \tag{4.17}$$

whilst for a put, when k=K,

$$x_l \to -\infty \qquad \text{and} \qquad x_u = 0. \tag{4.18}$$

At final time t^K, it is convenient to define w such that $w = 1(-1)$ applies to a Call(Put). Thus taking into consideration the foregoing limits the first four elements are given by

$$\begin{aligned}
a_{0,0}^K &= \frac{w}{\sqrt{\pi}}\int_0^{w\infty} e^{-x^2} dx = \frac{1}{2} \\
a_{0,1}^K &= \frac{w}{\sqrt{\pi}}\int_0^{w\infty} e^{-x^2} 2(vx + \beta)dx = \frac{w}{2\sqrt{\pi}} + \beta \\
a_{1,0}^K &= \frac{w}{\sqrt{\pi}}\int_0^{w\infty} e^{-x^2} 2x dx = \frac{w}{2v\sqrt{\pi}} \\
a_{1,1}^K &= \frac{w}{\sqrt{\pi}}\int_0^{w\infty} e^{-x^2} .2x.2(vx + \beta)dx = \frac{1}{2} + \frac{wb}{v\sqrt{\pi}}.
\end{aligned} \tag{4.19}$$

For $m = 2, 3, \cdots, N$ with $n = 0$

$$a_{m,0}^K = \frac{w}{2^m m! v^m \sqrt{\pi}}\int_0^{w\infty} e^{-x^2} H_m(x)dx = \frac{w}{2mv\sqrt{\pi}} L_{m-1}(0). \tag{4.20}$$

For $n = 2, 3, \cdots, N$ with $m = 0$

$$a_{0,n}^K = \frac{w}{\sqrt{\pi}}\int_0^{w\infty} e^{-x^2} H_n(vx + \beta)dx, \tag{4.21}$$

which upon use of the recurrence equation for Hermite polynomials yields the recurrence equation

$$a^K_{0,n} = \frac{wv}{\sqrt{\pi}} H_{n-1}(\beta) + 2\beta a^K_{0,n-1} + 2(v^2-1)(n-1)a_{0,n-2}. \tag{4.22}$$

For $m = 2, 3, \cdots, N$ and $n = 2, 3, \cdots, N$ application of the recurrence equation for Hermite polynomials yields

$$a^K_{m,n} = \frac{w\sqrt{2}}{2mv\sqrt{\Pi}} L_{m-1}(0) H_n(b) + \frac{n}{m} a^K_{n-1,m-1}. \tag{4.23}$$

Next we consider generation of the coefficients $a^k_{m,n}$ at any general time step k.

Then the first four elements are given by

$$\begin{aligned}
a^k_{0,0} &= \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} dx = P(x_u, x_l), \qquad \text{(use of 2.6 - see appendix 1)} \\
a^k_{0,1} &= \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} 2(vx+\beta) dx = vQ_{0,0}(x_l, x_u) + 2\beta P(x_u, x_l), \qquad \text{(use of 2.5)} \\
a^k_{1,0} &= \frac{1}{2v\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} 2x dx = \frac{1}{2v} Q_{0,0}(x_l, x_u), \\
a^k_{1,1} &= \frac{1}{2v\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} .2x.2(vx+\beta) dx = \frac{1}{2v} Q_{0,1}(x_l, x_u) + P(x_u, x_l).
\end{aligned} \tag{4.24}$$

To alleviate the notation we shall henceforth set

$$P = P(x_u, x_l) \qquad \text{and} \qquad Q_{m,n} = Q_{m,n}(x_l, x_u).$$

For $m = 2, 3, \cdots, N$ with $n = 0$

$$a^k_{m,0} = \frac{1}{2mv} Q_{m-1,0}. \tag{4.25}$$

For $n = 2, 3, \cdots, N$ with $m = 0$

$$a^k_{0,n} = vQ_{0,n-1} + 2\beta a^k_{0,n-1} + 2(v^2-1)(n-1)a^k_{0,n-2}. \tag{4.26}$$

For $m = 2, 3, \cdots, N$ and $n = 2, 3, \cdots, N$

$$a^k_{m,n} = \frac{1}{2mv} Q_{m-1,n} + \frac{n}{m} a^k_{m-1,n-1}. \tag{4.27}$$

Alternatively, if we introduce a new function:

$$Z_{m,n} = \frac{1}{2(m+1)v} Q_{m,n}(x_l, x_u), \tag{4.28}$$

then the first four elements are given by

$$\begin{aligned}
a^k_{0,0} = P \qquad & a^k_{0,1} = 2v^2 Z_{0,0} + 2\beta P, \\
a^k_{1,0} = Z_{0,0} \quad & a^k_{1,1} = Z_{0,1} + P.
\end{aligned} \tag{4.29}$$

For $m = 2, 3, \cdots, N$ and $n = 0$

$$a^k_{m,0} = Z_{m-1,0}. \tag{4.30}$$

For $n = 2, 3, \cdots, N$ and $m = 0$

$$a^k_{0,n} = 2v^2 Z_{0,n-1} + 2\beta a^k_{0,n-1} + 2(v^2-1)(n-1)a^k_{0,n-2}. \tag{4.31}$$

For $m = 2, 3, \cdots, N$ and $n = 2, 3, \cdots, N$

$$a^k_{m,n} = Z_{m-1,n} + \frac{n}{m} a^k_{m-1,n-1}. \tag{4.32}$$

All of the elements of the matrix A of the proposition are now defined. □

Proposition I has specified the coefficients of the matrix A which determine the backward transition of the $\overrightarrow{\alpha}^k$ coefficients from final time back to initial time. The quantity that we need to initialise the entire backward propogation process is $\overrightarrow{\alpha}^K$, the set of α coefficients at final time. These are determined by the payoff function at final time. In fact they are computed by expanding the payoff function itself in Fourier-Hermite series.

Recalling the payoff functions for calls and puts (see Figure 2) and the notation $w = 1(-1)$ to indicate call (put) the backward recursion from final payoff at t^k to the time step t^{K-1} may be written

$$F^{K-1}(\xi) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_l}^{z_u} e^{-(x-\mu(\xi,\Delta t))^2} w(e^{\sigma x} - 1) dx. \tag{4.33}$$

In the case of a call $z_u = \infty$ and $z_l = 0$. In the case of a put $z_u = 0$ and $z_l = -\infty$. we note however that the notation employed for the limits of integration in (4.33) also allows us to cater for the situation when there are point barriers at final time. In this case z_u and z_l would be determined by the barrier points.

Thus in order to calculate the coefficients α^K_n we first need to expand $e^{\sigma x}$ in a Fourier-Hermite series.

First we note the result that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) e^{\sigma x} dx = \sigma^n e^{\frac{\sigma^2}{4}}. \tag{4.34}$$

Then forming the Fourier-Hermite series

$$e^{\sigma x} = \sum_{n=0}^{\infty} \beta_n H_n(x), \tag{4.35}$$

we apply the orthogonality condition to obtain

$$\beta_n = \frac{1}{2^n n!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) e^{\sigma x} dx, \tag{4.36}$$

which by use of (4.33) reduces to

$$\beta_n = \frac{\sigma^n}{2^n n!} e^{\frac{\sigma^2}{4}}. \tag{4.37}$$

Therefore, for a call,

$$\alpha^K_n = \beta_n \qquad \text{for} \qquad n = 1, 2, \cdots, N \tag{4.38}$$

and $\alpha^K_0 = e^{\sigma^2} - 1$.

While for a Put,

$$\alpha^K_n = -\beta_n \qquad \text{for} \qquad n = 1, 2, \cdots, N \tag{4.39}$$

and $\alpha^K_0 = 1 - e^{\sigma^2}$.

5. Result

In this section we present some preliminary results obtained from implementing the path integral framework for pricing discretely monitored barrier (point barrier) options. The method was tested for parameter values similar to those used by Broadie et al. (1997*a*) and Wei (1998). The method was also tested on barrier puts and calls with digital type payoff functions. To ascertain the accuracy of the method, a Crank-Nicholson was used to generate *true* prices for the options investigated. Hence, the Crank-Nicholson scheme was used with very fine discretisations. The state variable was taken out to five times the value of the strike (scaled to unity) with 2000 steps *per unit*. The time variable was discretised to 100 steps *per day*.

	Path Integral m=40	Path Integral m=100	Crank-Nicholson
$\sigma = 0.60$	9.4895	9.4904	9.4905
$\sigma = 0.40$	7.0388	7.0393	7.0394
$\sigma = 0.20$	4.4336	4.4342	4.4344

TABLE 1. **Discretely Monitored Down-and-Out Call Option**

Parameters: *S = K = 100, T = 0.2 year, r = 0.1,*

q = 0, barrier = 95, monitoring frequency = 4.

Table 1 gives results of the path integral method for a set of parameter values which differ in their volatility value. The results are presented using 40 and 100 basis functions. It is evident that even with 40 basis functions the method is relatively accurate when compared with the true values, with the results approaching the true values when using 100 basis functions. Note that the parameter values used in this test include high volatility as well as barrier level close to the current asset price which typically present problems in other pricing methods for discretely monitored barrier options.

Table 2, table 3 and table 4 show results of the method for at-the-money, out-of-the-money and in-the-money down-and-out options under two monitoring frequencies - monthly and weekly. Results are presented for various barrier levels including values approaching the current asset values. Also presented are percentage-pricing errors relative to the true prices obtained using the Crank-Nicholson method. These give the percentage error of the option prices calculated using the path integral method relative to the true value. Following Wei (1998), we also give the percentage errors of prices calculated using the Broadie-Glasserman-Kou (BGK) continuity correction Broadie et al. (1997*a*). Comparing the percentage errors of the prices derived under the path integral method with those derived under the BGK methods, show that relative errors are of comparable magnitude when the barrier level lies away from the current asset price. However, as the barrier level approaches the current asset price, the relative errors producing using the path integral method are many much smaller than those using the BGK method.

Barrier Level	Path Integral Solution	True Solution	Percentage Pricing Error: Path Integral	Percentage Pricing Error: BGK Correction
Monthly Monitoring				
85	8.1859	8.1861	-0.003%	-0.011%
90	7.8406	7.8403	0.004%	-0.008%
95	6.7450	6.7463	-0.019%	0.036%
99.5	4.9323	4.9338	-0.031%	-10.926%
99.9	4.7460	4.7474	-0.030%	-16.023%
Weekly Monitoring				
85	8.1248	8.1250	-0.003%	0.006%
90	7.5761	7.5763	-0.003%	0.080%
95	5.8936	5.8946	-0.016%	-0.870%
99.5	3.0078	3.0093	-0.050%	-13.570%
99.9	2.7599	2.7354	0.895%	-16.527%

TABLE 2. **At-the-money Down and Out Call Options**

Parameters: *S = K = 100, T = 0.5 year, r = 0.05,*

q = 0, σ = 0.25, 100 basis function

Barrier Level	Path Integral Solution	True Solution	Percentage Pricing Error: Path Integral	Percentage Pricing Error: BGK Correction
Monthly Monitoring				
85	10.8029	10.8052	-0.022%	-0.010%
90	9.8831	9.8865	-0.035%	-0.007%
95	7.4328	7.4381	-0.071%	0.043%
99.5	3.6611	3.6623	-0.034%	-11.061%
99.9	3.3172	3.3181	-0.027%	-16.160%
Weekly Monitoring				
85	10.9187	10.921	-0.021%	0.009%
90	10.3118	10.3139	-0.021%	0.088%
95	8.6362	8.6381	-0.022%	-0.970%
99.5	6.1197	6.1213	-0.027%	-14.001%
99.9	5.8716	5.8732	-0.027%	-16.960%

TABLE 3. **In-the-money Down and Out Call Option**

Parameters: *S =100, K = 105, T = 0.5 year, r = 0.05,*

q = 0, σ = 0.25, 100 basis functions.

Also, the size of the relative error under the path integral method is relatively stable across barrier levels.

Although not reported in this version of the paper, the computation time required for the path integral method is one of its strengths. For a 1-year-down-and-out barrier option with weekly monitoring, with 100 basis functions, the algorithm in its current form takes less than 0.5 seconds to run for the level of accuracy given in this paper. This is extremely efficient when compared to a trinomial model or a Crank-Nicholson method. The computation times for the path integral method increase with the number of basis functions and with the monitoring frequency. A more detailed and comprehensive

Barrier Level	Path Integral Solution	True Solution	Percentage Pricing Error: Path Integral	Percentage Pricing Error: BGK Correction
Monthly Monitoring				
85	5.9226	5.9237	-0.018%	-0.014%
90	5.6068	5.6081	-0.022%	-0.005%
95	4.4945	4.4979	-0.075%	0.058%
99.5	2.3851	2.3847	0.018%	-10.882%
99.9	2.1751	2.1751	0.001%	-15.962%
Weekly Monitoring				
85	5.9545	5.9548	-0.005%	0.000%
90	5.7630	5.7642	-0.021%	0.080%
95	5.0803	4.0814	-0.022%	-0.825%
99.5	3.8346	3.8356	-0.026%	-13.343%
99.9	3.7011	3.7021	-0.026%	-16.258%

TABLE 4. **Out-of-the-money Down and Out Call Options**

Parameters: *S =100, K = 95, T = 0.5 year, r = 0.05,*

q = 0, σ *= 0.25, 100 basis functions.*

analysis of the computation times and the complexity of the algorithm relative to other available methods.

6. Conclusion

In this paper we have presented a pricing method for the valuation of point barrier options which are discretely monitored barrier options in a path integral framework. We show how the backward recursion algorithm of such derivative securities in this framework may be efficiently evaluated by expanding the price in a Fourier-Hermite series as a function of the underlying asset price. The method has the advantage of giving the price as a continuous function of the underlying asset price, hence the hedge ratios can be calculated to a high degree of accuracy with minimal additional computational effort. The method can handle various barrier structures with constant and time varying barrier levels for a variety of option payoffs. The method can be made arbitrarily accurate by increasing the number of basis functions in the expansions. Preliminary numerical results show that the method presented is relatively accurate and efficient.

A natural and simple extension of the work presented in this article would be to apply this method to discretely monitored lookback and Parisian options. A more detailed comparison of computation times and robustness of this method against other methods for pricing discretely monitored barrier options is required. The results of these extensions and comparisons will be reported in forthcoming articles.

Appendix A. The coefficient $a^k_{0,0}$

By definition,

$$a^k_{0,0} = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} dx, \tag{A.1}$$

which by change of variable becomes

$$\begin{aligned} a_{0,0}^k &= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}x_l}^{\sqrt{2}x_u} e^{-z^2/2}dz, \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{2}x_u} e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{2}x_l} e^{-z^2/2}dz. \end{aligned} \quad (A.2)$$

This can be expressed as

$$a_{0,0}^K = N(\sqrt{2}x_u) - N(\sqrt{2}x_l), \quad (A.3)$$

or using the notation as given by equation (4.6) we have that

$$a_{0,0}^k = P(x_l, x_u). \quad (A.4)$$

It is worth noting some ***some special values*** for this coefficient

At the pay-off, $k = K$ we have in the case of a <u>call</u> that $x_l = 0$ and $x_u \to \infty$ so that

$$a_{0,0}^k = P(0, \infty) = \frac{1}{2}. \quad (A.5)$$

In the case of a <u>put</u>,we have that $x_l \to -\infty$ and $x_u = 0$ so that

$$a_{0,0}^k = P(-\infty, 0) = \frac{1}{2}. \quad (A.6)$$

Consider *the general time step,* $k = (K-1, \cdots, 1)$, there are two special cases:

If we have <u>only an upper barrier</u>, then $x_l \to -\infty$ and

$$a_{0,0}^k = P(-\infty, x_u) = N(\sqrt{2}x_u). \quad (A.7)$$

If we have <u>only a lower barrier</u>, then $x_u \to \infty$ and

$$a_{0,0}^k = P(x_l, \infty) = 1 - N(\sqrt{2}x_l). \quad (A.8)$$

Appendix B. The coefficient $a_{1,0}^k$

By definition,

$$a_{1,0}^k = \frac{1}{2v}\frac{1}{\sqrt{\pi}}\int_{x_l}^{x_u} 2xe^{-x^2}dx \quad (B.1)$$

which easily evaluates to

$$a_{1,0}^k = \frac{1}{2v}\sqrt{2}\left[\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}x_l)^2/2} - \frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}x_u)^2/2}\right], \quad (B.2)$$

which upon use of the notation $n(x) = e^{-x^2/2}/\sqrt{2\pi}$, becomes

$$a_{1,0}^k = \frac{1}{2v}\left[n(\sqrt{2}x_l) - n(\sqrt{n}x_u)\right]. \quad (B.3)$$

Using the notation of equation (4.5) we can also write

$$a^k_{1,0} = \frac{1}{2v} Q_{0,0}(x_l, x_u). \tag{B.4}$$

We note ***some special values***.

At the pay-off, $k = K$ we have in the case of a call that $x_l = 0$ and $x_u \to \infty$, then

$$a^K_{1,0} = Q_{0,0}(0, \infty) = \frac{1}{2v\sqrt{\pi}} \tag{B.5}$$

In the case of a put, $x_l \to -\infty$ and $x_u = 0$. then,

$$a^k_{1,0} = Q_{0,0}(-\infty, 0) = \frac{-1}{2v\sqrt{\pi}}. \tag{B.6}$$

Here we see that the difference between the two cases is a simple sign change, that is

$$a^k_{1,0}(Call) = -a^K_{1,0}(Put). \tag{B.7}$$

At the general time step k, there are two special cases of interest:-
If we have only an upper barrier so that $x_l \to -\infty$ then

$$\begin{aligned} a^k_{1,0} &= \frac{1}{2v} Q_{0,0}(-\infty, x_u), \\ &= \frac{1}{2v}\left[R_{0,0}(-\infty) - R_{0,0}(x_u)\right], \\ &= \frac{-\sqrt{2}}{2v} n(\sqrt{2}x_u). \end{aligned} \tag{B.8}$$

If we have only a lower barrier so that $x_u \to \infty$ then

$$\begin{aligned} a^k_{1,0} &= \frac{1}{2v} Q_{0,0}(x_l, \infty), \\ &= \frac{1}{2v}\left[R_{0,0}(x_l) - R_{0,0}(\infty)\right], \\ &= \frac{\sqrt{2}}{2v} n(\sqrt{2}x_l). \end{aligned} \tag{B.9}$$

Appendix C. The coefficient $a^k_{0,1}$

By definition

$$a^k_{0,1} = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} 2(vx + \beta) dx, \tag{C.1}$$

which can be written

$$a^k_{0,1} = vI_1 + 2\beta I_2, \tag{C.2}$$

where

$$I_1 = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} 2xe^{-x^2} dx, \qquad I_2 = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} dx. \tag{C.3}$$

Using results from Appendices A and B, we find that

$$\begin{aligned} I_1 &= \sqrt{2}\left[n(\sqrt{2}x_l) - n(\sqrt{2}x_u)\right], \\ I_2 &= \left[N(\sqrt{2}x_u) - N(\sqrt{2}x_l)\right]. \end{aligned} \tag{C.4}$$

Hence on using equation (4.5) and (4.6)

$$a_{0,1}^k = vQ_{0,0}(x_l, x_u) + 2\beta P(x_l, x_u). \tag{C.5}$$

We consider some ***special values.*** *At the pay-off,* $k = K$. In the case of a call we have $x_l = 0$ and $x_u \to \infty$ so that

$$a_{0,1}^k = vQ_{0,0}(0,\infty) + 2bP(0,\infty) = \frac{1}{2v\sqrt{\pi}} + \beta. \tag{C.6}$$

In the case of a put we have $x_l \to -\infty$ and $x_u = 0$ so that

$$a_{0,1}^K = vQ_{0,0}(-\infty,0) + 2\beta P(-\infty,0) = \frac{-1}{2v\sqrt{\pi}} + \beta. \tag{C.7}$$

At the general time step k, there are two special cases of interest.

The case of only an upper barrier ($x_l \to -\infty$) so that

$$\begin{aligned} a_{0,1}^k &= vQ_{0,0}(-\infty, x_u) + 2\beta P(-\infty, x_u), \\ &= v\left[R_{0,0}(-\infty) - R_{0,0}(x_u)\right] + 2\beta P(-\infty, x_u), \\ &= -v\sqrt{2}n(\sqrt{2}x_u) + 2bN(\sqrt{2}x_u). \end{aligned} \tag{C.8}$$

In the case of only a lower barrier ($x_u \to \infty$) we have

$$\begin{aligned} a_{0,1}^k &= vQ_{0,0}(x_l, \infty) + 2\beta P(x_l, \infty), \\ &= v\left[R_{0,0}(x_l) - R_{0,0}(\infty)\right] + 2\beta P(x_l, \infty), \\ &= v\sqrt{2}n(\sqrt{2}x_l) + 2\beta\left[1 - N(\sqrt{2}x_l)\right]. \end{aligned} \tag{C.9}$$

Appendix D. The coefficient $a_{1,1}^k$

By definition

$$a_{1,1}^k = \frac{1}{2v\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} 2x.2(vx+\beta)dx, \tag{D.1}$$

which when integrating by parts and using (4.5) and (4.6) can be written as

$$a_{1,1}^k = \frac{1}{2v} Q_{0,1}(x_l, x_u) + P(x_l, x_u). \tag{D.2}$$

At the pay-off, $k = K$, we have ***two special cases.*** The call for which $x_l = 0$ and $x_u \to \infty$ so that

$$\begin{aligned} a_{1,1}^K &= \frac{1}{2v} Q_{0,1}(0,\infty) + P(0,\infty), \\ &= \frac{\beta}{2v\sqrt{\pi}} + \frac{1}{2}, \end{aligned} \tag{D.3}$$

and the put for which $x_l \to -\infty$ and $x_u = 0$ so that

$$\begin{aligned} a_{1,1}^K &= \frac{1}{2v} Q_{0,1}(-\infty,0) + P(-\infty,0), \\ &= \frac{-\beta}{v\sqrt{\pi}} + \frac{1}{2}. \end{aligned} \tag{D.4}$$

At the general time step k, there are ***two special cases*** of interest. Only upper barrier ($x_l \to -\infty$) so that

$$\begin{aligned} a_{1,1}^k &= \frac{1}{2v} Q_{0,1}(-\infty, x_u) + P(-\infty, x_u), \\ &= \frac{1}{2v}\left[R_{0,1}(-\infty) - R_{0,1}(x_u)\right] + P(-\infty, x_u), \\ &= \frac{-\sqrt{2}}{2v} H_1(vx_u + b) + N(\sqrt{2}x_u). \end{aligned} \tag{D.5}$$

The case of only lower barrier ($x_u \to \infty$) so that

$$\begin{aligned} a_{1,1}^k &= \frac{1}{2v} Q_{0,1}(x_l, \infty) + P(x_l, \infty), \\ &= \frac{1}{2v}\left[R_{0,1}(x_l) - R_{0,1}(\infty)\right] + P(x_l, \infty), \\ &= \frac{\sqrt{2}}{2v} H_1(vx_l + \beta) + \left[1 - N(\sqrt{2}x_l)\right]. \end{aligned} \tag{D.6}$$

Appendix E. The coefficient $a_{m,o}^k$

By definition

$$a_{m,o}^k = \frac{1}{2^m m! v^m} \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} H_m(x) dx. \tag{E.1}$$

Using the three-term reccurence relation,

$$H_m(x) = 2xH_{m-1}(x) - 2(m-1)H_{m-2}(x) \tag{E.2}$$

equation (E.1) can be written as

$$a_{m,0}^k = W - \frac{1}{2v^2} a_{m-2,o}^k, \tag{E.3}$$

where we set

$$W = \frac{1}{2^m m! v^m} \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} 2xe^{-x^2} H_{m-1}(x) dx. \tag{E.4}$$

Integrating by parts we find that

$$W = \frac{\sqrt{2}}{2^m m! v^m} \left[H_{m-1}(x_l) n(\sqrt{2}x_l) - H_{m-1}(x_u) n(\sqrt{2}x_u) \right] + \frac{1}{2mv^2} a^k_{m,o}, \tag{E.5}$$

which when substituted back into (E.3) gives

$$a^k_{m,o} = \frac{\sqrt{2}}{2^m m! v^m} \left[H_{m-1}(x_l) n(\sqrt{2}x_l) - H_{m-1}(x_u) n(\sqrt{2}x_u) \right]. \tag{E.6}$$

Use of (4.5) finally allows us to write

$$a^k_{m,0} = \frac{1}{2mv} Q_{m-1,o}(x_l, x_u). \tag{E.7}$$

At the pay-off, $k = K$, we have ***two special cases.*** The call for which $x_l = 0$ and $x_u \to \infty$ so that

$$\begin{aligned} a^K_{m,o} &= \frac{1}{2mv} Q_{m-1,o}(0,\infty), \\ &= \frac{1}{2mv} \left[R_{m-1,0}(0) - R_{m-1,0}(0) \right], \\ &= \frac{1}{2^m m! v^m} \frac{H_{m-1}(0)}{\sqrt{\pi}}, \qquad \text{(use of (4.4))}, \\ &= \frac{1}{2mv\sqrt{\pi}} L_{m-1}(0), \qquad \text{(use of (4.2))}. \end{aligned} \tag{E.8}$$

The put for which $x_l \to -\infty$ and $x_u = 0$ so that

$$\begin{aligned} a^K_{m,o} &= \frac{1}{2mv} Q_{m-1,o}(-\infty, 0), \\ &= \frac{1}{2mv} \left[R_{m-1,0}(-\infty) - R_{m-1,0}(0) \right], \\ &= \frac{-1}{2^m m! v^m} \frac{H_{m-1}(0)}{\sqrt{\pi}}, \qquad \text{(use of (4.4))}, \\ &= \frac{-1}{2mv\sqrt{\pi}} L_{m-1}(0), \qquad \text{(use of (4.2))}. \end{aligned} \tag{E.9}$$

At the *general time step k,* there are ***two special cases*** of interest. Only upper barrier ($x_l \to -\infty$) when,

$$\begin{aligned} a^k_{m,o} &= \frac{1}{2mv} Q_{m-1,o}(-\infty, x_u), \\ &= \frac{1}{2mv} \left[R_{m-1,o}(-\infty) - R_{m-1,o}(x_u) \right], \\ &= \frac{-\sqrt{2}}{2^m m! v^m} H_{m-1}(x_u) n(\sqrt{2}x_u), \qquad \text{(use of (4.4))} \\ &= \frac{-\sqrt{2}}{2mv} L_{m-1}(x_u) n(\sqrt{2}x_u), \qquad \text{(use of (4.2))}. \end{aligned} \tag{E.10}$$

Only <u>lower barrier</u> ($x_u \to \infty$) when

$$
\begin{aligned}
a^k_{m,o} &= \frac{1}{2mv} Q_{m-1,o}(x_l,\infty), \\
&= \frac{1}{2mv}\left[R_{m-1,o}(x_l) - R_{m-1,o}(\infty)\right], \\
&= \frac{\sqrt{2}}{2^m m! v^m} H_{m-1}(x_l) n(\sqrt{2}x_l), \qquad \text{(use of (4.4))}
\end{aligned}
$$

so finally

$$
= \frac{\sqrt{2}}{2mv} L_{m-1}(x_l) n(\sqrt{2}x_l), \qquad \text{(use of (4.2)).} \tag{E.11}
$$

Appendix F. The coefficient $a^k_{0,n}$

By definition

$$
a^k_{o,n} = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} e^{-x^2} H_n(vx+b)dx. \tag{F.1}
$$

Using the three-term reccurance relation,

$$
H_n(vx+b) = 2(vx+\beta)H_{n-1}(vx+b) - 2(n-1)H_{n-2}(vx+b), \tag{F.2}
$$

it follows that

$$
a^k_{0,n} = vW + 2\beta a^k_{o,n-1} - 2(n-1)a^k_{o,n-2}, \tag{F.3}
$$

where we set

$$
W = \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} 2xe^{-x^2} H_{n-1}(vx+\beta)dx. \tag{F.4}
$$

Integrating by parts, we find that

$$
W = \left[-H_{n-1}(vx+\beta)\frac{e^{-x^2}}{\sqrt{\pi}} \right]_{x_l}^{x_u} + 2v(n-1)a^k_{0,n-2}, \tag{F.5}
$$

which reduces (F.3) to

$$
\begin{aligned}
a^k_{o,n} &= v\sqrt{2}\left[H_{n-1}(vx_l+\beta)n(\sqrt{2}x_l) - H_{n-1}(vx_u+\beta)n(\sqrt{2}x_u)\right] \\
&+ 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}
\end{aligned} \tag{F.6}
$$

which by use of (4.5) can be written as

$$
a^k_{o,n} = vQ_{o,n-1}(x_l,x_u) + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}. \tag{F.7}
$$

At the pay-off, $k = K$, we have ***two special cases.*** The <u>call</u> for which $x_l = 0$ and $x_u \to \infty$, so that

$$
\begin{aligned}
a^K_{o,n} &= vQ_{o,n-1}(0,\infty) + 2\beta a^K_{o,n-1} + 2(v^2-1)(n-1)a^K_{o,n-2} \\
&= v\left[R_{o,n-1}(0) - R_{o,n-1}(\infty)\right] + 2\beta a^K_{o,n-1} + 2(v^2-1)(n-1)a^K_{o,n-2},
\end{aligned} \tag{F.8}
$$

which by use of (4.5) finally reduces to

$$a^K_{o,n} \;=\; \frac{v}{\sqrt{\pi}} H_{n-1}(\beta) + 2\beta a^K_{o,n-1} + 2(v^2-1)(n-1)a^K_{o,n-2}. \tag{F.9}$$

The put for which $x_l \to -\infty$ and $x_u = 0$ so that

$$\begin{aligned} a^k_{o,n} &= vQ_{o,n-1}(-\infty,0) + 2\beta a^K_{o,n-2} + 2(v^2-1)(n-1)a^K_{o,n-2}, \\ &= v\left[R_{o,n-1}(-\infty) - R_{o,n-1}(0)\right] + 2ba^K_{o,n-1} + 2(v^2-1)(n-1)a^K_{o,n-2}, \end{aligned} \tag{F.10}$$

which by use of (4.5)finally reduces to

$$a^K_{o,n} \;=\; \frac{-v}{\sqrt{\pi}} H_{n-1}(b) + 2\beta a^K_{o,n-1} + 2(v^2-1)(n-1)a^K_{o,n-2}. \tag{F.11}$$

At the general time step k, there are ***two special cases*** of interest. The case of only upper barrier ($x_l \to -\infty$), when

$$\begin{aligned} a^k_{o,n} &= vQ_{o,n-1}(-\infty,x_u) + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}, \\ &= v\left[R_{o,n-1}(-\infty) - R_{o,n-1}(x_u)\right] + 2ba^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}, \end{aligned} \tag{F.12}$$

which by use of (4.5) reduces to

$$a^k_{o,n} \;=\; -v\sqrt{2}H_{n-1}(vx_u+\beta)n(\sqrt{2}x_u) + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}. \tag{F.13}$$

The case of only lower barrier ($x_u \to \infty$), when

$$\begin{aligned} a^k_{o,n} &= vQ_{o,n-1}(x_l,\infty) + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}, \\ &= v\left[R_{o,n-1}(x_l) - R_{o,n-1}(\infty)\right] + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}, \end{aligned} \tag{F.14}$$

which by use of (4.5) reduces to

$$a^k_{o,n} \;=\; v\sqrt{2}H_{n-1}(vx_l+\beta)n(\sqrt{2}x_l) + 2\beta a^k_{o,n-1} + 2(v^2-1)(n-1)a^k_{o,n-2}. \tag{F.15}$$

Appendix G. The coefficient $a^k_{m,n}$

By definition

$$a^k_{m,n} = \frac{1}{2^m m! v^m}\frac{1}{\sqrt{\pi}}\int_{x_l}^{x_u} e^{-x^2}H_m(x)H_n(vx+b)dx. \tag{G.1}$$

Using the three-term reccurence relation,

$$H_m(x) = 2xH_{m-1}(x) - 2(m-1)H_{m-2}(x), \tag{G.2}$$

equation (G.1) can be written as

$$a^k_{m,n} = W - \frac{1}{2mv^2}a^k_{m-2,n}, \tag{G.3}$$

where we set

$$W = \frac{1}{2^m m! v^m} \frac{1}{\sqrt{\pi}} \int_{x_l}^{x_u} 2xe^{-x^2} H_{m-1}(x) H_n(vx+b)dx. \tag{G.4}$$

Integrating by parts we find that

$$\begin{aligned} W &= \frac{\sqrt{2}}{2^m m! v^m} \left[H_{m-1}(x_l) H_n(vx_l+\beta) n(\sqrt{2}x_l) - H_{m-1}(x_u) H_n(vx+\beta) n(\sqrt{2}x_u) \right] \\ &+ \frac{1}{2mv^2} a^k_{m-2,n} + \frac{n}{m} a^k_{m-1,n-1}, \end{aligned} \tag{G.5}$$

which by use of (4.6) can be written as

$$W = \frac{1}{2mv} Q_{m-1,n} + \frac{1}{2mv^2} a^k_{m-2,n} + \frac{n}{m} a^k_{m-1,n-1}. \tag{G.6}$$

Finally, substituting (G.6) back into (G.3) yields

$$a^k_{m,n} = \frac{1}{2mv} Q_{m-1,n} + \frac{n}{m} a^k_{m-1,n-1}. \tag{G.7}$$

At the pay-off, $k = K$, we have ***two special cases.*** The call for which $x_l = 0$ and $x_u \to \infty$, so that

$$\begin{aligned} a^K_{m,n} &= \frac{1}{2mv} Q_{m-1,n}(0,\infty) + \frac{n}{m} a^K_{m-1,n-1}, \\ &= \frac{1}{2mv} \left[R_{m-1,n}(0) - R_{m-1,n}(\infty) \right] \qquad \text{(use of (4.5))} \\ &+ \frac{n}{m} a^k_{m-1,n-1}, \end{aligned} \tag{G.8}$$

which by use of (4.4) reduces to

$$a^K_{m,n} = \frac{\sqrt{2}}{2mv\sqrt{\pi}} L_{m-1}(0) H_m(\beta) + \frac{n}{m} a^K_{m-1,n-1}. \tag{G.9}$$

The put for which $x_l \to -\infty$ and $x_u = 0$, so that

$$\begin{aligned} a^K_{m,n} &= \frac{1}{2mv} Q_{m-1,n}(-\infty,0) + \frac{n}{m} a^K_{m-1,n-1}, \\ &= \frac{1}{2mv} \left[R_{m-1,n}(-\infty) - R_{m-1,n}(0) \right] + \frac{n}{m} a^K_{m-1,n-1}, \end{aligned} \tag{G.10}$$

which by use of (4.4) reduces to

$$a^K_{m,n} = \frac{-\sqrt{2}}{2mv\sqrt{\pi}} L_{m-1}(0) H_m(\beta) + \frac{n}{m} a^K_{m-1,n-1}. \tag{G.11}$$

At the general time step k, there are ***two special cases.*** The case of only *upper barrier* ($x_l \to -\infty$), when

$$\begin{aligned} a^k_{m,n} &= \frac{1}{2mv} Q_{m-1,n}(-\infty, x_u) + \frac{n}{m} a^k_{m-1,n-1} \\ &= \frac{1}{2mv} \left[R_{m-1,n}(-\infty) - R_{m-1,n}(x_u) \right] + \frac{n}{m} a^k_{m-1,n-1} \end{aligned} \tag{G.12}$$

which by use of (4.4) finally reduces to

$$a^k_{m,n} = \frac{-\sqrt{2}}{2mv} L_m(x_u) H_n(vx_u + \beta) n(\sqrt{2}x_u) + \frac{n}{m} a^k_{m-1,n-1}. \tag{G.13}$$

The case of only lower barrier ($x_u \to \infty$), when

$$a^k_{m,n} = \frac{1}{2mv} Q_{m-1,n}(x_l, \infty) + \frac{n}{m} a^k_{m-1,n-1} \tag{G.14}$$

$$= \frac{1}{2mv} \left[R_{m-1,n}(x_l) - R_{m-1,n}(\infty) \right] + \frac{n}{m} a^k_{m-1,n-1}, \tag{G.15}$$

which by use of (4.4) finally reduces to

$$a^k_{m,n} = \frac{\sqrt{2}}{2mv} L_{m-1}(x) H_n(vx_l + b) + \frac{n}{m} a^k_{m-1,n-1}. \tag{G.16}$$

Appendix H. Useful Notation

$$P(x_l, x_u) = N(\sqrt{2}x_u) = N(\sqrt{2}x_l) \tag{H.1}$$

$$Q_{m,n}(x_l, x_u) = R_{m,n}(x_l) - R_{m,n} x_u \tag{H.2}$$

where

$$R_{m,n}(x) = \sqrt{2} L_m(x) H_n(vx + b) n(\sqrt{2}x) \tag{H.3}$$

with

$$L_m(x) = \frac{1}{2^m m! V^m} H_m(x) \tag{H.4}$$

and

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \qquad N(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{\frac{-\xi^2}{2}} d\xi \tag{H.5}$$

furthermore,

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x) \tag{H.6}$$

with

$$H_o(x) = 1 \qquad \text{and} \qquad H_1(x) = 2x \tag{H.7}$$

$$L_m(x) = \frac{x}{mV} L_{m-1}(x) - \frac{1}{2mV^2} L_{m-2}(x) \tag{H.8}$$

with

$$L_o(x) = 1 \qquad \text{and} \qquad L_1(x) = \frac{x}{V} \tag{H.9}$$

Let us define

$$L_m(x) = \frac{1}{2^m m! V^m} H_m(x) \tag{H.10}$$

Using the three term reccurance relation

$$H_m(x) = 2x H_{m-1}(x) - 2(m-1) H_{m-2}(x) \tag{H.11}$$

and nothing that

$$\frac{2}{2^m m! V^m} = \frac{1}{mV}\left[\frac{1}{2^{m-1}(m-1)!V^{m-1}}\right] \tag{H.12}$$

$$\frac{2(m-1)}{2^m m! V^m} = \frac{1}{2mV^2}\left[\frac{1}{2^{m-2}(m-2)!V^{m-2}}\right] \tag{H.13}$$

we instantly have that

$$L_m(x) = \frac{x}{mV}L_{m-1}(x) - \frac{1}{2mV^2}L_{m-2}(x) \tag{H.14}$$

with

$$L_o(x) = 1 \tag{H.15}$$

$$L_1(x) = \frac{x}{v} \tag{H.16}$$

thus the L_m's can be generated easily using (H.14), (H.15), (H.16).

Now, let us define

$$R_{m,n}(x) = \sqrt{2}L_m(x)H_n(vx+b)n(\sqrt{2}x) \tag{H.17}$$

Special values ($x = 0$)

$$R_{m,n}(0) = \frac{1}{\sqrt{\pi}}L_m(0)H_n(b) \tag{H.18}$$

and in particular,

$$R_{m,0}(0) = \frac{1}{\sqrt{\pi}}L_m(0) \tag{H.19}$$

$$R_{0,n}(0) = \frac{1}{\sqrt{\pi}}H_n(b) \tag{H.20}$$

Special values ($x \to \infty, x \to -\infty$)

$$\lim_{x\to-\infty} R_{m,n}(x) = 0 \tag{H.21}$$

$$\lim_{x\to\infty} R_{m,n}(x) = 0 \tag{H.22}$$

Furthermore, using the above notation,

$$Q_{m,n}(x_l, x_u) = R_{m,n}(x_l) - R_{m,n}(x_u) \tag{H.23}$$

In general,

$$P(x_l, x_u) = \left[N\left(\sqrt{2}x_u\right) - N\left(\sqrt{2}x_l\right)\right] \tag{H.24}$$

Special cases:

$$P(-\infty, \infty) = 1 \tag{H.25}$$

$$P(-\infty, 0) = \frac{1}{2} \tag{H.26}$$

$$P(0, \infty) = \frac{1}{2} \tag{H.27}$$

$$P(-\infty, x_u) = N(\sqrt{2}x_u) \tag{H.28}$$

$$P(x_l, \infty) = 1 - N(\sqrt{2}x_l) \tag{H.29}$$

$$P(x_l, 0) = \frac{1}{2} - N(\sqrt{2}x_l) \tag{H.30}$$

$$P(o, x_u) = N(\sqrt{2}x_u) - \frac{1}{2} \tag{H.31}$$

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