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CAPM and Option Pricing with Elliptical Distributions

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CAPM and Option Pricing with Elliptical Distributions

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Abstract

In this paper, we offer an alternative proof of the Capital Asset Pricing Model when the returns follow a multivariate elliptical distribution. Empirical studies continue to demonstrate the inappropriateness of the normality assumption in modelling asset returns. The class of elliptical distributions, which includes the more familiar Normal distribution, provides flexibility in modelling the thickness of tails associated with the possibility that asset returns take extreme values with non-negligible probabilities. Within this framework, we prove a new version of Stein's lemma for elliptical distribution and use this result to derive the CAPM when returns are elliptical. We also derive a closed form solution of call option prices when the underlying is elliptically distributed. We use the probability distortion function approach based on the dual utility theory of choice under uncertainty.

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1 Introduction and Motivation

This paper considers the general class of symmetric distributions in extending familiar results of Capital Asset Pricing Model (CAPM) and the theory of asset pricing. This class, called the class of Elliptical distributions, includes the familiar Normal distribution and shares many of its familiar properties. However, this class provides greater flexibility in modelling tails or extremes that are becoming commonly important in financial economics. Besides this flexibility, it preserves several properties of the Normal distribution which allows one to derive attractive explicit solution forms. As an illustration, the classical CAPM result

$$\mathsf{E}\left(R_k\right) = R_F + \beta\left[\mathsf{E}\left(R_M\right) - R_F\right] \tag{1}$$

which gives the expected return on an asset k as a linear function of the risk-free rate R_F and the expected return on the market, can be derived by assuming asset returns are multivariate normally distributed. See Sharpe (1964), Lintner (1965), and Mossin (1966). It has been demonstrated in Owen and Rabinovitch (1983) and again, in Ingersoll (1987) that relaxing this normality assumption into the wider class of elliptical distributions preserves the result in (1). This paper re-examines the CAPM result under this general class of elliptical distributions by offering a rigorous proof using a version of the Stein's Lemma for elliptical distributions. The Stein's Lemma for Normal distributions states that for a bivariate normal random variable (X, Y) we have

$$Cov\left(X, h\left(Y\right)\right) = \mathsf{E}\left[h'\left(Y\right)\right] \cdot Cov\left(X, Y\right) \tag{2}$$

for any differentiable h satisfying certain regularity conditions; see Stein (1973, 1981). In this paper, we extend this lemma into the case of bivariate elliptical random variables allowing us to prove the CAPM for elliptical distributions.

This paper also considers option pricing when the underlying is elliptically distributed. We use probability distortion function approach based on the dual theory of choice under uncertainty (Yaari 1987). We derive closed form solution of call option which collapsed to Black-Scholes option price in the special case when the elliptical distribution is Normal.

This paper is organized as follows. In Section 2, we introduce elliptical distributions, as in Fang, Kotz, and Ng (1990). We develop and repeat some results that will be used in later sections. Most results proved elsewhere are simply stated, but some basic useful results are also proved. In Section 3, we state and prove the Stein's lemma for elliptical distributions. Section 4 provides a re-derivation of the CAPM assuming multivariate elliptical distribution of returns. Section 5 discusses option pricing when the underlying is elliptically distributed . Section 7 provides and SDE representation of the dynamics of process which is Elliptically distributed. We conclude in Section 8.

2 Elliptical Distributions: Definition and Properties

The class of elliptical distributions consists mainly of the class of symmetric distributions and is widely becoming popular in actuarial science, insurance, and finance. It contains many distributions that are generally more leptukortic than the Normal distribution allowing us to model tails which are frequently observed in financial data; see Embrechts, et al. (2001) and Shmidt (2002).

In the financial literature, Bingham and Kiesel (2002) propose a semi-parametric model for stock-price and asset-return distributions based on elliptical distributions because as the authors observed, Gaussian or normal models provide mathematical tractability but are inconsistent with empirical data. In the following, we recall some basic definition and properties of elliptical distributions.

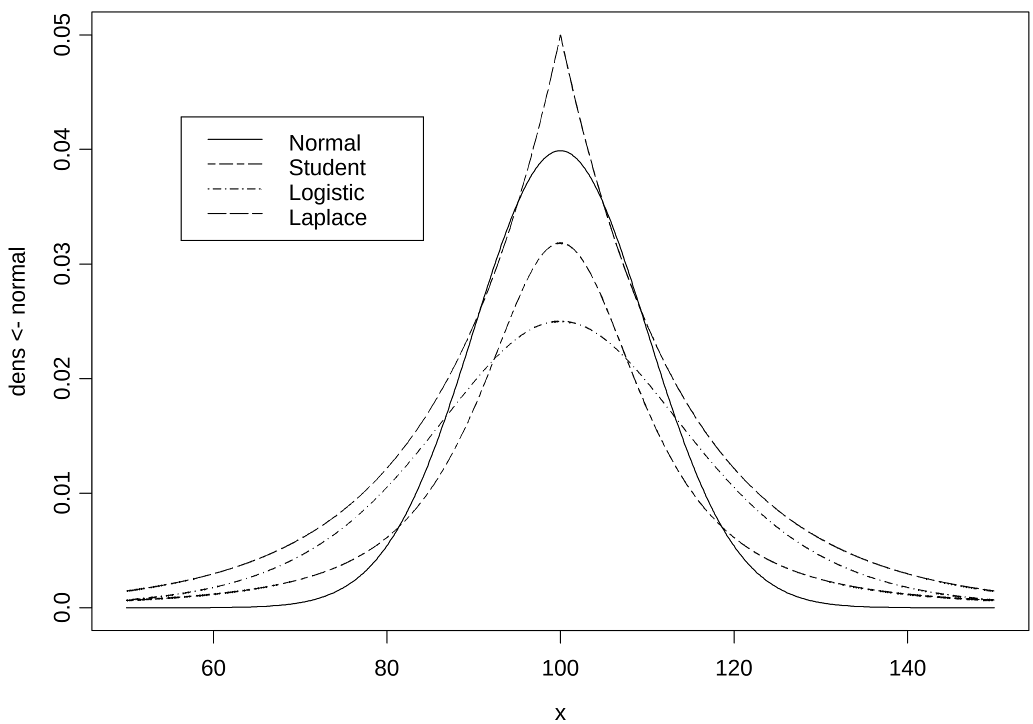


Fig. 1. *Some Well-Known Elliptical Distribution Densities*

2.1 Elliptical Density and Characteristic Function

It is well-known that a random vector $\mathbf{X} = (X, ..., X_n)^T$ is said to have a n-dimensional normal distribution if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{AZ},$$

where $\mathbf{Z} = (Z_1, ..., Z_m)^T$ is a random vector consisting of m mutually independent standard normal random variables, $\mathbf{A}$ is a $n \times m$ matrix, $\boldsymbol{\mu}$ is a $n \times 1$ vector and $\stackrel{d}{=}$ stands for "equality in distribution". Equivalently, one can say that $\mathbf{X}$ is normal if its characteristic function can be expressed as

$$\mathsf{E}\left[\exp\left(i\mathbf{t}^T\mathbf{X}\right)\right] = \exp\left(i\mathbf{t}^T\boldsymbol{\mu}\right)\ \exp\left(-\tfrac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right), \tag{3}$$

for some fixed vector $\boldsymbol{\mu}(n \times 1)$ and some fixed matrix $\boldsymbol{\Sigma}(n \times n)$, and where $\mathbf{t}^T = (t_1, t_2, \ldots, t_n)$. For random vectors belonging to the class of multivariate normal distributions with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, we use the notation $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. It is well-known that the vector $\boldsymbol{\mu}$ is the mean vector and that the matrix $\boldsymbol{\Sigma}$ is the variance-covariance matrix. Note that the relation between $\boldsymbol{\Sigma}$ and $\mathbf{A}$ is given by $\boldsymbol{\Sigma} = \mathbf{AA}^T$.

The class of multivariate elliptical distributions is a natural extension of the class of multivariate normal distributions.

Definition 1. *The random vector $\mathbf{X} = (X_1, ..., X_n)^T$ is said to have an elliptical distribution with parameters the vector $\boldsymbol{\mu}(n \times 1)$ and the matrix $\boldsymbol{\Sigma}(n \times n)$ if its characteristic function can be expressed as*

$$\mathsf{E}\left[\exp\left(i\mathbf{t}^T\mathbf{X}\right)\right] = \exp\left(i\mathbf{t}^T\boldsymbol{\mu}\right) \cdot \psi\left(\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right), \tag{4}$$

for some scalar function ψ and where $\mathbf{t}^T = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \mathbf{AA}^T \tag{5}$$

for some matrix $\mathbf{A}(n \times m)$*.*

If $\mathbf{X}$ has elliptical distribution, we write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ and say that $\mathbf{X}$ is elliptical. The function ψ is called the *characteristic generator* of $\mathbf{X}$ and hence, the characteristic generator of the multivariate normal is given by $\psi(u) = \exp(-u/2)$.

It is well-known that the characteristic function of a random vector always exists and that there is a one-to-one correspondence between distribution and characteristic functions. However, not every function ψ can be used to construct a characteristic function of an elliptical distribution. Obviously, this function ψ must fulfill the requirement $\psi(0) = 1$. A necessary and sufficient condition for the function ψ to be a characteristic generator of an n-dimensional elliptical distribution is given as Theorem 2.2 in Fang, et al. (1990).

The random vector $\mathbf{X}$ does not, in general, possess a density $f_{\mathsf{X}}(\mathbf{x})$, but if it does, it will have the form

$$f_{\mathsf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left[(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right], \tag{6}$$

for some non-negative function $g_n(\cdot)$ called the *density generator* and for some constant c_n called the normalizing constant. This density generator is subscripted with an n to emphasize that it may depend on the dimension of the vector. We shall drop this n, and simply write g, in the univariate case. It was demonstrated in Landsman and Valdez (2002) that the normalizing constant c_n in (6) can be explicitly determined using

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x) dx \right]^{-1} \tag{7}$$

The condition

$$\int_0^\infty x^{n/2-1} g_n(x) dx < \infty \tag{8}$$

guarantees $g_n(x)$ to be density generator (see Fang, et al. 1987) and therefore the existence of the density of $\mathbf{X}$. Alternatively, we may introduce the elliptical distribution via the density generator and we then write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$.

2.2 Mean and Covariance Property

As pointed out by Embrechts et al (1999, 2001), the linear correlation measure provides a canonical scalar measure of dependencies for elliptical distributions. Observe that the condition (8) does not require the existence of the mean and covariance of vector $\mathbf{X}$. However, if the mean vector exists, it will be $\mathsf{E}(\mathbf{X}) = \boldsymbol{\mu}$, and if the covariance matrix exists, it will be

$$Cov(\mathbf{X}) = -\psi'(0)\,\boldsymbol{\Sigma}, \tag{9}$$

where ψ' denotes the first derivative of the characteristic function. See Fang, et al. (1987). The characteristic generator can be chosen such that $\psi'(0) = -1$ leaving us with the variance-covariance $Cov(\mathbf{X}) = \boldsymbol{\Sigma}$. We shall denote the elements of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively by

$$\boldsymbol{\mu} = (\mu_1, ..., \mu_n)^T$$

and

$$\boldsymbol{\Sigma} = (\sigma_{ij}) \text{ for } i, j = 1, 2, ..., n.$$

The diagonals of $\boldsymbol{\Sigma}$ are often written as $\sigma_{kk} = \sigma_k^2$. Observe that the matrix $\boldsymbol{\Sigma}$ coincides with the covariance matrix up to a constant. However, this is not quite true for the correlation, because if we take any pairs (X_i, X_j), we have its correlation expressed as

$$\rho(X_i, X_j) = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i) \cdot Var(X_j)}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}} \cdot \sqrt{\sigma_{jj}}}. \tag{10}$$

In the special case where $\boldsymbol{\mu} = (0, ..., 0)^T$, the zero vector, and $\boldsymbol{\Sigma} = \mathbf{I}_n$, the identity matrix, we have the standard elliptical, oftentimes called spherical, random vector, and in which case, we shall denote it by $\mathbf{Z}$.

2.3 Sums and Linear Combinations of Elliptical

The class of elliptical distributions possesses the linearity property which is quite useful for portfolio theory. Indeed, an investment portfolio is usually a linear combination of several assets. The linearity property can be briefly summarized as follows: If the returns on assets are assumed to have elliptical distributions, then the return on a portfolio of these assets will also have an elliptical distribution.

From (4), it follows that if $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and A is some $m \times n$ matrix of rank $m \leq n$ and b some m-dimensional column vector, then

$$A\mathbf{X} + b \sim E_m\left(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^T, g_m\right). \tag{11}$$

In other words, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator ψ or from the same sequence of density generators $g_1, ...g_n$, corresponding to ψ. Therefore, any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator. In particular, for $k = 1, 2, ..., n$, $X_k \sim E_1(\mu_k, \sigma_k^2, g_1)$ so that its density can be written as

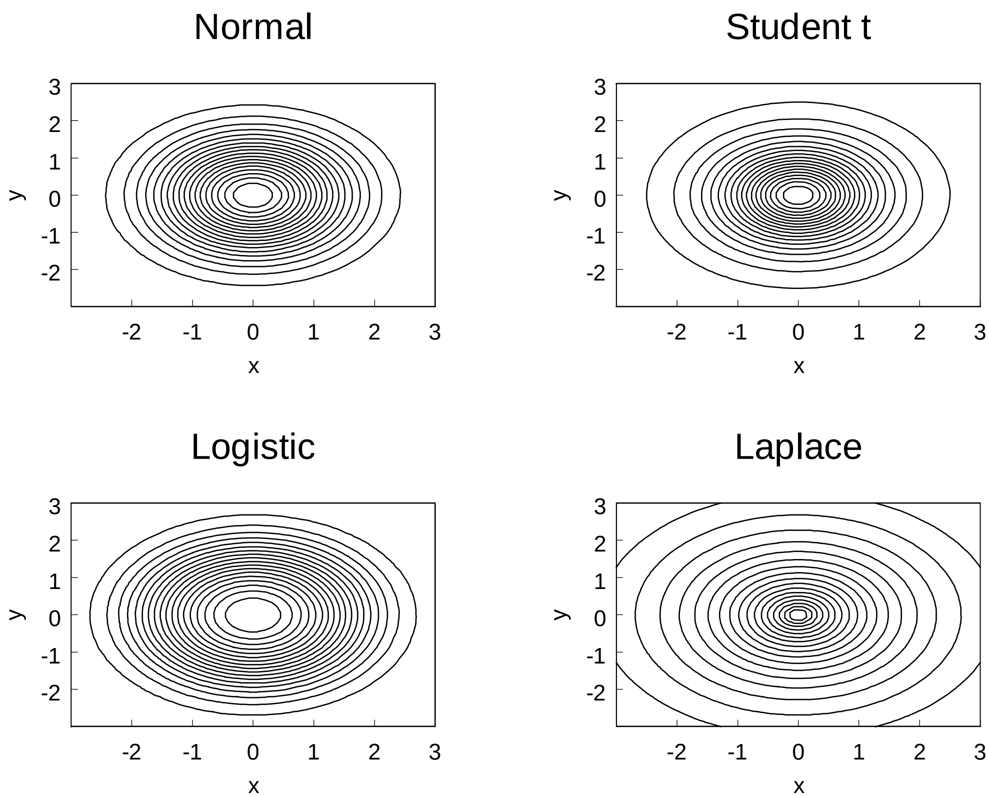
$$f_{X_k}(x) = \frac{c_1}{\sigma_k} g_1\left[\frac{1}{2}\left(\frac{x - \mu_k}{\sigma_k}\right)^2\right]. \tag{12}$$

If we define the sum $S = X_1 + X_2 + \cdots + X_n = \mathbf{e}^T\mathbf{X}$, where $\mathbf{e}$ is a column vector of ones with dimension n, then it immediately follows that

$$S \sim E_n\left(\mathbf{e}^T\boldsymbol{\mu}, \mathbf{e}^T\boldsymbol{\Sigma}\mathbf{e}, g_1\right). \tag{13}$$

2.4 Fat Tails Property of Elliptical Distributions

The following graph represents isoprobability contours of different distributions that belong to Elliptic class. Each ellipse represents the set of points which have the same probability under the distribution considered.



Family	Density $g_n(u)$ or characteristic $\psi(u)$ generators
Bessel	$g_n(u) = (u/b)^{a/2} K_a\left[(u/b)^{1/2}\right]$, $a > -n/2, b > 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind
Cauchy	$g_n(u) = (1+u)^{-(n+1)/2}$
Exponential Power	$g_n(u) = \exp\left[-r(u)^s\right]$, $r, s > 0$
Laplace	$g_n(u) = \exp(-\lvert u \rvert)$
Logistic	$g_n(u) = \dfrac{\exp(-u)}{\left[1+\exp(-u)\right]^2}$
Normal	$g_n(u) = \exp(-u/2)$; $\psi(u) = \exp(-u/2)$
Stable Laws	$\psi(u) = \exp\left[-r(u)^{s/2}\right]$, $0 < s \le 2, r > 0$
Student t	$g_n(u) = \left(1+\dfrac{u}{m}\right)^{-(n+m)/2}$, $m > 0$ an integer

Table 1

Some Well-Known Families of Elliptical Distributions with

their Characteristic Generator and/or Density Generators.

3 Stein's Lemma for Elliptical Distributions

Charles Stein (1973, 1981) used the property of the exponential function inherent in Normal distributions and integration by parts to prove the following result: If the random pair (X, Y) has a bivariate normal distribution and h is differentiable satisfying the condition that

$$\mathsf{E}\left[h'(X)|\right] < \infty,$$

then

$$Cov\left[h(X), Y\right] = Cov(X, Y) \cdot \mathsf{E}[h'(X)].$$

In this section, we extend Stein's lemma for elliptical distributions. Besides the advantage gained by proving a new result, this has also applications in proving the Capital Asset Pricing Model when the underlying returns are multivariate elliptical.

Lemma 3.1. *Let* $X \sim E_1\left(\mu_X, \boldsymbol{\sigma}_X^2, g\right)$ *and* h *be a differentiable function such that* $\mathsf{E}[|h'(X)|] < \infty$, *then*

$$\sigma_X^2 \mathsf{E}\left[h'(X)\right] = \frac{c}{c^*} \cdot \mathsf{E}\left[h(X^*)(X^* - \mu)\right] \tag{14}$$

where the random variable $X^* \sim E_1\left(\boldsymbol{\mu}, \sigma_X^2, -g'\right)$ *with* c^* *as the normalizing constant.*

Proof. We have

$$\mathsf{E}\left[h'(X)\right] = \int_{-\infty}^{\infty} h'(x) \frac{c}{\sigma_X} g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right] dx$$

where the normalizing constant c equals

$$c = \frac{\Gamma(1/2)}{\sqrt{2\pi}}\left[\int_0^{\infty} x^{-\frac{1}{2}} g(x) dx\right]^{-1}.$$

Applying integration by parts with

$$u = \frac{c}{\sigma_X} g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right] \qquad du = \frac{c}{\sigma_X} g'\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right]\frac{1}{\sigma_X^2}(x-\mu)dx$$

$$dv = h'(x)dx \qquad v = h(x)$$

we obtain

$$\mathsf{E}\left[h'(X)\right] = \frac{c}{\sigma_X} g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right] h(x)\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} h(x)\frac{c}{\sigma_X^3} g'\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right](x-\mu)dx$$

The first term of the above equality vanishes due to the conditions imposed on h and also the property of the density generator g. Thus we have:

$$\sigma_X^2 \mathsf{E}\left[h'(X)\right] = \int_{-\infty}^{\infty} h(x)(x-\mu)\frac{c}{\sigma_X}\left(-g'\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2\right]\right)dx$$

Define the random variable $X^* \sim E_1\left(\mu, {\sigma_X}^2, -g'\right)$ where the density generator of X^* is the negative derivative of the density generator of X. Recall that

$$c = \frac{\Gamma(1/2)}{\sqrt{2\pi}}\left[\int_0^{\infty} x^{-\frac{1}{2}} g(x)dx\right]^{-1} \text{ whereas } c^* = \frac{\Gamma(1/2)}{\sqrt{2\pi}}\left[\int_0^{\infty} x^{-\frac{1}{2}}\left(-g'(x)\right)dx\right]^{-1}$$

so that

$$\frac{c}{c^*} = \frac{\int_0^{\infty} x^{-\frac{1}{2}}\left(-g'(x)\right)dx}{\int_0^{\infty} x^{-\frac{1}{2}} g(x)dx}.$$

Therefore

$$\sigma_X^2 \mathsf{E}\left[h'(X)\right] = \frac{c}{c^*}\mathsf{E}\left[h(X^*)(X^*-\mu)\right]$$

where $X^* \sim E_1\left(\mu, {\sigma_X}^2, -g'\right)$.

Note that in the case of the Normal distribution, we have $g(x) = e^{-x}$ so that $g'(x) = -e^{-x}$. Hence $-g'(x) = -g(x)$ and therefore

$$\sigma_X^2 \mathsf{E}\left[h'(X)\right] = \mathsf{E}\left[h(X)(X-\mu)\right].$$

Note that in this case $X^* \overset{d}{=} X$. Furthermore, equation (14) in the lemma can be re-stated equivalently as

$$\sigma_X^2 \mathsf{E}\left[h'(\widetilde{X})\right] = \frac{\widetilde{c}}{c} \cdot \mathsf{E}\left[h(X)(X-\mu)\right] \tag{15}$$

where the random variable $\widetilde{X} \sim E_1\left(\mu, \sigma_X^2, -\int g\right)$ with $\widetilde{c}$ as its normalizing constant. This follows immediately by taking the density function of X to be the negative primitive, $-\int g$, of g.

Lemma 3.2 (Stein's Lemma for Elliptical). *Let the bivariate vector* $(X,Y) \sim E_2\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2\right)$ *with density generator denoted by* g_2 *and*

$$\boldsymbol{\mu} = \begin{pmatrix} \mathsf{E}(X) \\ \mathsf{E}(Y) \end{pmatrix} \quad \textit{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}.$$

Let h *be a differentiable function of* X*, then*

$$Cov\left[h(X), Y\right] = \frac{c}{\widetilde{c}} \cdot Cov\left(X,Y\right) \cdot \mathsf{E}\left[h'(\widetilde{X})\right] \tag{16}$$

where $\widetilde{X} \sim E_1\left(\mu, \sigma_X^2, -\int g\right)$.

Proof. Note that

$$\begin{aligned} Cov\left[h(X),Y\right] &= \mathsf{E}\left[\left(h(X)-\mathsf{E}\left[h(X)\right]\right)\left(Y-\mathsf{E}[Y]\right)\right] \\ &= \mathsf{E}\left[\left(h(X)-\mathsf{E}[h(X)]\right)\left(Y-\mathsf{E}[Y]\right]\right. \\ &= \mathsf{E}_X[\mathsf{E}_Y\left[\left(h(X)-\mathsf{E}[h(X)]\right)\left(Y-\mathsf{E}[Y]|X\right]\right. \\ &= \mathsf{E}_X\left[\left(h(X)-\mathsf{E}[h(X)]\right)\underbrace{\mathsf{E}_Y\left[Y-\mathsf{E}[Y]|X\right]}_{\text{Conditional variance}}\right]. \end{aligned}$$

It can be shown (see Dhaene and Valdez, 2003) that

$$\mathsf{E}_Y\left[Y-\mathsf{E}[Y]|X\right] = \frac{Cov(X,Y)}{\sigma_X^2}\left(X-\mathsf{E}[X]\right).$$

Thus,

$$\begin{aligned} Cov\left[h(X),Y\right] &= \frac{Cov(X,Y)}{\sigma_X^2}\mathsf{E}\left[\left(h(X)-\mathsf{E}[h(X)]\right)\left(X-\mathsf{E}[X]\right]\right. \\ &= \frac{Cov(X,Y)}{\sigma_X^2}\mathsf{E}\left[h(X)\left(X-\mathsf{E}[X]\right)\right]. \end{aligned}$$

Using the equation (15) resulting from the previous lemma, we can write

$$\mathsf{E}([h(X)\left(X-\mathsf{E}([X])\right)] = \frac{c}{\widetilde{c}}\cdot\sigma_X^2\cdot\mathsf{E}\left[h'(\widetilde{X})\right]$$

where $\widetilde{X}\sim E_1\left(\mu_X,\sigma_X^2,-\int g\right)$. Thus, we have

$$\begin{aligned} Cov\left[h(X),Y\right] &= \frac{Cov(X,Y)}{\sigma_X^2}\cdot\frac{c}{\widetilde{c}}\cdot\sigma_X^2\cdot\mathsf{E}\left[h'(\widetilde{X})\right] \\ &= \frac{c}{\widetilde{c}}\cdot Cov\left(X,Y\right)\cdot\mathsf{E}\left[h'(\widetilde{X})\right]. \end{aligned}$$

Note that in the case of the Normal distribution, we have $\widetilde{X} \overset{d}{=} X$ so that $\dfrac{c}{\widetilde{c}} = 1$ and therefore

$$Cov\left[h(X), Y\right] = Cov\left(X, Y\right) \cdot \mathsf{E}\left[h'(X)\right]$$

which gives the more familiar Stein's lemma. We shall $\widetilde{X}$ the integrated elliptical random variable associated with X.

4 C.A.P.M. with Elliptical Distributions

Much of the current theory of capital asset pricing is based on the assumption that asset prices (or returns) are multivariate normal random variables. Several empirical studies have indicated violation of this fundamental assumption. The class of elliptical distributions offers a more flexible framework for modelling asset prices or returns. Like with the Normal distribution, the dependence structure in an Elliptical distribution can be summarized in terms of the variance-covariance matrix but with also the characteristic generator. Because many of the properties of the Normal distribution extend to this larger class, existing results on asset pricing relying on the Normal distribution assumption may be preserved. This induces us to examine the validity of CAPM by relaxing the normality assumption and generalizing it to Elliptical distributions. Owen and Rabinovitch (1983) derive the Tobin's separation and the Ross's mutual fund separation theorems in the case when the underlying returns are Elliptical. Ingersoll (1987) derives the CAPM and portfolio allocation in this case.

In this section, we offer a more comprehensive proof of CAPM using the Stein's lemma for Elliptical distributions proved in the previous section.

4.1 Set-up

We adopt the "equilibrium pricing approach" used in both Panjer, et al. (1998) and Huang and Litzenberger (1988). Consider a one-period economy where ω denotes the state of nature at the end of the period. Assume there are I agents each with time-additive utility function

$$u_{i0}(c_{i0}) + u_{i1}(C_{i1}(\omega)), \quad \text{for } i = 1, 2, ..., I.$$

Expected utility is thus

$$u_{i0}(c_{i0}) + \sum_{\omega} p_i(\omega) u_{i1}(C_{i1}(\omega)).$$

Agents are expected utility maximizers. Assume there are Arrow-Debreu securities which pay 1 for each state ω and none for all other states. These Arrow-Debreu prices are denoted by Ψ_ω. Optimal consumption at equilibrium exists and are to be denoted by c_{i0}^* and $C_{i1}^*(\omega)$.

Now, consider a particular state, say ω_a, and suppose the agent buys additional α units at time 0 so that consumptions are $c_{i0}^* - \alpha\Psi_{\omega_a}$ at time 0 and $C_{i1}^*(\omega_a) + \alpha$ at time 1 in state ω_a. Expected utility becomes

$$u_{i0}(c_{i0}^* - \alpha\Psi_{\omega_a}) + \sum_{\omega \neq \omega_a} p_i(\omega) u_{i1}(C_{i1}^*(\omega)) + p_i(\omega_a) u_{i1}(C_{i1}^*(\omega_a) + \alpha)$$

and taking the first derivative with respect to α, we get

$$-\Psi_{\omega_a} u_{i0}'(c_{i0}^* - \alpha\Psi_{\omega_a}) + p_i(\omega_a) u_{i1}'(C_{i1}^*(\omega_a) + \alpha)$$

which must be equal to 0 (since already optimal) at $\alpha = 0$. It follows immediately that

$$\Psi_\omega = p_i(\omega) \frac{u_{i1}'(C_{i1}^*(\omega))}{u_{i0}'(c_{i0}^*)}$$

where we have dropped the subscript a without ambiguity. These are called the state prices.

Now using these state prices to price any other security, consider for example a security that pays 1 unit at time 1 in each state. This is precisely a unit discount bond that pays 1 unit at time 1, regardless of the state. We must then have

$$\sum_\omega \Psi_\omega = \sum_\omega p_i(\omega) \frac{u_{i1}'(C_{i1}^*(\omega))}{u_{i0}'(c_{i0}^*)} = \frac{1}{1+R_F}$$

where R_F is the risk-free interest rate. As yet another example, consider a security that pays $X(\omega)$ in state ω. Suppose $\pi(x)$ denotes the price for this security. Then, clearly it must be equal to

$$\pi(x) = \sum_\omega p_i(\omega) \frac{u_{i1}'(C_{i1}^*(\omega))}{u_{i0}'(c_{i0}^*)} X(\omega) = \mathsf{E}(ZX)$$

where $Z(\omega)$ is equal to $\dfrac{u_{i1}'(C_{i1}^*(\omega))}{u_{i0}'(c_{i0}^*)}$, sometimes called the price density or pricing kernel.

Note that the pricing formula above depends on the preferences and consumption allocation of a particular agent. To derive the pricing formula at equilibrium, we would have to maximize each agent's utility and then let market clear. Alternatively, if the subjective probabilities are the same across agents, we can simplify this procedure by maximizing a representative agent and then letting market clear by assuming this representative agent has all the aggregate consumption and aggregate endowment. The representative agent's utility function is thus $\upsilon_0(c_0^a) = \sum_{i=1}^I k_i u_{i0}(c_{i0})$ and $\upsilon_1(C_1^a) = \sum_{i=1}^I k_i u_{i1}(C_{i1})$ where c_0 and C_1 are the aggregate consumptions and $\sum_{i=1}^{I} k_i = 1$. Therefore, the state prices are

$$\Psi_\omega = p(\omega) \frac{\upsilon_1' (C_1^a(\omega))}{\upsilon_0' (c_0^a)}$$

and

$$Z = \frac{\upsilon_1' (C_1^a)}{\upsilon_0' (c_0^a)}. \tag{17}$$

4.2 Deriving the C.A.P.M.

Using the equilibrium approach, we derive the CAPM. Consider a security j that pays an amount of $X_j(\omega)$ at time 1 in state ω. Let π_j be the current price of the security. By arbitrage (two portfolios with equal payoffs have the same value), we have

$$\pi_j = \sum_\omega \Psi_\omega X_j(\omega) = \sum_\omega p(\omega) \frac{\upsilon_1' (C_1^a(\omega))}{\upsilon_0' (c_0^a)} X_j(\omega) = \mathsf{E}(Z X_j). \tag{18}$$

Denote by $R_j(\omega)$ the rate of return in state ω so that

$$R_j(\omega) = \frac{X_j(\omega) - \pi_j}{\pi_j}. \tag{19}$$

From equation (18), we have

$$\mathsf{E}\left(Z \frac{X_j}{\pi_j} \right) = 1$$

from equation (19), we get

$$\begin{aligned} \mathsf{E}\left(Z \frac{X_j}{\pi_j} \right) &= \mathsf{E}[Z(1+R_j)] = \mathsf{E}(Z) + \mathsf{E}(ZR_j) \\ &= \mathsf{E}(Z) + Cov(R_j, Z) + \mathsf{E}(Z)\,\mathsf{E}(R_j) \\ &= \mathsf{E}(Z)[1 + \mathsf{E}(R_j)] + Cov(R_j, Z). \end{aligned} \tag{20}$$

For the one-period bond, we have

$$\mathsf{E}(Z) = \sum_{\omega} \Psi_{\omega} = \frac{1}{1+R_F}. \tag{21}$$

where R_F denotes the one-period risk-free rate. Replacing (21) in (20), we obtain

$$1 = \frac{1}{1+R_F}\left[1+\mathsf{E}(R_j)\right] + Cov(R_j, Z).$$

Thus, we have

$$\mathsf{E}(R_j) - R_F = -(1+R_F)\, Cov(R_j, Z). \tag{22}$$

Because at equilibrium the total consumption will equal to the total wealth in the economy, the market rate of return can be expressed as

$$1 + R_m(\omega) = \frac{C_1^a(\omega)}{c_0^a}$$

so that this return also satisfies the same form of equation

$$\mathsf{E}(R_m) - R_F = -(1+R_F)\, Cov(R_m, Z). \tag{23}$$

Dividing the equation (22) by equation (23), we have

$$\frac{\mathsf{E}(R_j) - R_F}{\mathsf{E}(R_m) - R_F} = \frac{Cov(R_j, Z)}{Cov(R_m, Z)}$$

A re-arrangement leads us to the following CAPM formula:

$$\mathsf{E}(R_j) = R_F + \frac{Cov(R_j, Z)}{Cov(R_m, Z)} \cdot \left[\mathsf{E}(R_m) - R_F\right] = R_F + \beta_j \cdot \left[\mathsf{E}(R_m) - R_F\right]$$

where $\beta_j = \dfrac{Cov(R_j, Z)}{Cov(R_m, Z)}$. The problem with this equation is that the "beta" is unobservable. However, we can simplify this by imposing assumption of elliptical distributions on the returns.

Proposition 4.1 (CAPM with Multivariate Elliptical Return). *Assume a market with n securities and that all securities follow a multivariate elliptical distribution. The expected rate of return for security j can be expressed as*

$$\mathsf{E}(R_j) = R_F + \beta_j \cdot [\mathsf{E}(R_m) - R_F], \quad \textit{for } j = 1, 2, ..., n$$

where R_F is the risk-free rate, R_m is the market rate of return, and

$$\beta_j = \frac{Cov(R_j, R_m)}{Var(R_m)}.$$

Proof. From the property of elliptical, each R_j has an elliptical distribution. The rate of return in the market R_m is a linear combination of rates of return of all securities. Hence, it follows that R_m has also an elliptical distribution. Furthermore, each bivariate pair (R_j, R_m) will have a bivariate elliptical distribution. Using equation (17) to evaluate the covariances, we have

$$\begin{aligned} \frac{Cov(R_j, Z)}{Cov(R_m, Z)} &= \frac{Cov\left(R_j, \frac{\upsilon_1'(C_1^a)}{\upsilon_0'(c_0^a)}\right)}{Cov\left(R_m, \frac{\upsilon_1'(C_1^a)}{\upsilon_0'(c_0^a)}\right)} \\ &= \frac{Cov(R_j, \upsilon_1'(C_1^a))}{Cov(R_m, \upsilon_1'(C_1^a))} \\ &= \frac{Cov(R_j, \upsilon_1'(c_0^a(1+R_m)))}{Cov(R_m, \upsilon_1'(c_0^a(1+R_m)))}. \end{aligned}$$

Applying Stein's lemma for elliptical distribution, we simplify this to

$$\frac{Cov\left(R_j, Z\right)}{Cov\left(R_m, Z\right)} = \frac{(c/\widetilde{c}) \cdot Cov\left(R_j, R_m\right) \cdot \mathsf{E}\left[\upsilon_1''\left(c_0^a\left(1+\widetilde{R}_m\right)\right)\right]}{(c/\widetilde{c}) \cdot Cov\left(R_m, R_m\right) \cdot \mathsf{E}\left[\upsilon_1''\left(c_0^a\left(1+\widetilde{R}_m\right)\right)\right]} = \frac{Cov\left(R_j, R_m\right)}{Var\left(R_m\right)}$$

where $\widetilde{R}_m$ is the integrated elliptical random variable associated with R_m, and c and $\widetilde{c}$ are the normalizing constants corresponding to R_m and $\widetilde{R}_m$ respectively.

5 Option pricing using Probability Distortion Functions

The concept of probability distortion functions is widely used in insurance risk pricing. The idea is to modify the real world probability distribution of the contingent claim to adjust for risk. This concept is somehow related to change of measure, but the link is not evident in all cases. Probability distortion is used in Yaari (1987) in the theory of choice under uncertainty. The certainty equivalent[1] of a risk is computed as a mean of distorted cumulative distribution function of the underlying risk.

Wang (2000 [28]) proposed a class of probability distortion functions that aims to integrate financial and actuarial insurance pricing theories. The probability distortion function proposed is based on the standard cumulative normal distribution. In his paper Wang states that the new distortion function connects four different approaches:

1. the traditional actuarial standard deviation principle,
2. Yaari's (1987) economic theory of choice under uncertainty,
3. CAPM, and
4. option-pricing theory.

[1] The certainty equivalent of a risk is the amount which when received with certainty, is regarded as good as taking the risk itself

Let us recall some definitions of the probability distortion functions. Consider a random variable X with a decumulative distribution function $S_X(x) = P[X > x]$. Let $\Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ be the standard normal cumulative distribution function and define

$$g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$$

for p in $[0,1]$. This operator shifts the p^{th} quantile of X, assuming that X is normally distributed, by a positive or negative value α and re-evaluates the normal cumulative probability for the shifted quantile. Wang shows that $g_\alpha(p)$ is concave for positive α and convex for negative α. In fact it is easy to see that if $\alpha > 0$, then $g_\alpha(p) > p$, and if $\alpha < 0$, then $g_\alpha(p) < p$. Since g_α is continuous and $g_\alpha(p) \in [0,1]$, then it follows that

$$\begin{aligned} g_\alpha \text{ is convex if } \alpha &< 0 \\ g_\alpha \text{ is concave if } \alpha &> 0 \end{aligned}$$

Under this distortion function, an individual behaves pessimistically by shifting the quantiles to the left, thereby assigning higher probabilities to low outcomes, and behaves optimistically by shifting the quantiles to the right thereby assigning higher probabilities to high outcomes.

Wang (2000) defines the risk-adjusted premium for a risk X, by the Choquet integral representation

$$H[X;\alpha] = \int_{-\infty}^{0} \{g_\alpha[S_X(x)] - 1\} dx + \int_0^{\infty} g_\alpha[S_X(x)] dx$$

where X will be negative if it is an insurance loss and will take positive values for the pay-off from a limited-liability asset.

This new risk pricing measure has many advantages and seems to perform well if normality of the underlying risk is assumed. However it is not clear why it should work for non-normal case.

Hamada & Sherris (2003) applied Wang transform to price European call option written on a security with prices following a geometric Brownian motion and they derived Black and Scholes option price formula. This consistency with financial theory is not obtained when the underlying is not log-normal. The case of CEV process was considered to show this inconsistency. This is due to Wang's choice of the distortion function based on the cumulative normal distribution.

5.1 New Class of Probability Distortion Functions

If the underlying is not log-normal, then a fair price can be obtained by choosing the distortion function to be based on the cumulative distribution function of the underlying. Indeed, in Hamada & Sherris (2003) we have the following proposition:

Proposition 5.1. *Let Z be a random variable with a cumulative distribution function F and a probability density function symmetric around* 0. *And let the contingent claim $X(T)$ be a function of Z such that $X = h(Z)$ where h is a continuous, positive and increasing function, then the fair price of $X(T)$ at time* 0 *is given by:*

$$X(0) = e^{-rT} I\left[X(T); -\alpha_T\right]$$

where

$$I\left[X(T); -\alpha_T\right] = \int_0^{\infty} F[F^{-1}(\Pr[X(T) > s]) - \alpha_T] ds$$

and α_T is a parameter calibrated to the market price

The proposition above states that a fair price for the claim is given by its certainty equivalent. Where the certainty equivalent is defined as the mean of the distorted decumulative distribution function. This is consistent with insurance pricing theory, introduced by Yaari (1987).

The question that arises is wether this is an arbitrage-free price ?

If the underlying is Normally distributed, then $F = \Phi$ where Φ is the standard normal cdf. It is proven in Hamada and Sherris (2003) that indeed we obtain an arbitrage-free price using this probability distortion function, and more particularly, we obtain Black-Scholes prices for options.

Now, if the underlying is not Normal, then answer to the above question is not clear in all cases. This is due to the fact that non-normality of the underlying corresponds in most cases to incompleteness in the market. In this situation, no unique price exists, and utility based equilibrium pricing is used instead. It can be argued that the above pricing can be used since it is also founded on non-expected utility theory.

The above formula seems difficult to implement, however, for symmetric distributions, where elliptical are a special case, we have a simpler representation, given in the following proposition.

Proposition 5.2. *With the set-up in the previous proposition, we have:*

$$I\left[X(T); -\alpha_T\right] = \mathsf{E}[h(Z - \alpha_T)] \tag{24}$$

5.2 Pricing Option when the Underlying is Elliptically Distributed

Let the Elliptical variable $Z \sim E_1(0, 1, \psi)$ and $X_t = \mu t + \sigma\sqrt{t}Z \sim E_1(\mu t, \sigma^2 t, \psi)$ for each time $t \geq 0$

Put the price process : $S_t = S_0 e^{X_t}$. The security process S is adapted to the natural filtration of X.

At each time t,

$$S_t = S_0 e^{X_t} \sim LE_1(\ln S_0 + \mu t, \sigma^2 t, \psi)$$

From Proposition (5.1), the fair price at time 0 of a call option maturing at T

written on a security with price process S is:

$$C(0) = e^{-rT} I\left[(S_T - K)^+; -\alpha_T\right]$$

where

$$I[(S_T - K)^+; -\alpha_T] = \int_0^\infty F\left[F^{-1}(P[(S_T - K)^+ > s]) - \alpha_T\right] ds$$

Since the distribution of Z is symmetric around 0, and $(S_T - K)^+ = h(Z)$ where:

$$h(z) = \left(S_0 e^{\mu T + \sigma\sqrt{T} z} - K\right)^+$$

then, using Proposition (5.2), we have:

$$I[(S_T - K)^+; -\alpha_T] = \mathsf{E}[h(Z - \alpha_T)]$$

One can explicitly evaluate the above expectation and as a result obtain a fair price of the option :

Theorem 5.1. *The fair value of an option written on a security which has elliptical distribution with parameters defined above is given by :*

$$I(X, \alpha) = e^{\mu + \sigma\alpha} \psi(-\sigma^2) F_{U^*}\left(\frac{\mu + \sigma\alpha - \log K}{\sigma}\right) - K F_U\left(\frac{\mu + \sigma\alpha - \log K}{\sigma}\right)$$

where U is spherically distributed with characteristic generator ψ and α is calibrated to the market prices of the underlying.

Proof. We are going to calculate explicitly the expression above using Theorem 7 from Dahene & Valdez (2003):

If $Y \sim LE_1(\mu, \sigma^2, \psi)$ and $U \sim S_1(\psi)$ (spherical distribution) with density f_U and cdf F_U, then

$$\mathsf{E}\left[(Y-K)^+\right] = e^{\mu}\psi(-\sigma^2)F_{U^*}\left(\frac{\mu-\log K}{\sigma}\right) - KF_U\left(\frac{\mu-\log K}{\sigma}\right)$$

where U^* is a random variable with as density the Esscher transform with parameter

σ of U, i.e.

$$f_{U^*}(x) = \frac{f_U(x)e^{\sigma x}}{\psi(-\sigma^2)}$$

Now we evaluate the expression in 24 where $h(\widetilde{Z}+\alpha) = \left(e^{\mu+\sigma\alpha+\sigma\widetilde{Z}} - K\right)^+ = (Y-K)^+$ where $Y \sim LE_1(\mu+\sigma\alpha, \sigma^2, \psi)$.

Therefore, the insurance fair price at time 0, based on Generalised Wang transform is given by:

$$I(X,\alpha) = e^{\mu+\sigma\alpha}\psi(-\sigma^2)F_{U^*}\left(\frac{\mu+\sigma\alpha-\log K}{\sigma}\right) - KF_U\left(\frac{\mu+\sigma\alpha-\log K}{\sigma}\right)$$

This price looks like Black-Scholes option price. It is indeed Black-Scholes when the underlying is geometric Brownian motion.

6 Dynamics for Elliptical Distributions

The closed form solution derived above seems tractable. However, it assumes having a closed form of the cumulative distribution of the underlying. This might be difficult to find if one is given an SDE for the underlying. The idea is to start from a given function F that satisfies cdf properties (non-decreasing, null at zero and 1 at 1), then derive an SDE such that at each time, the distribution of the underlying admits F as a cdf.

The idea is to start from the standard Brownian motion and transform it in order to obtain another process which is elliptically distributed at each point of time. Let us consider the process:

$$X = F^{-1}\left(\Phi\left(\frac{B}{\sqrt{t}}\right)\right)$$

where F is a cumulative distribution function of an elliptical family, B is the standard Brownian motion and Φ is the standard normal cumulative distribution function. Fixing time t, Let B_t be the standard Brownian motion. We have:

$$\frac{B_t}{\sqrt{t}} \sim N(0,1)$$

Since Φ is the standard Normal cdf, then

$$\Phi\left(\frac{B_t}{\sqrt{t}}\right) \sim \mathcal{U}(0,1)$$

By a classic theorem in statistics it is obvious that

$$X_t = F^{-1}\left(\Phi\left(\frac{B_t}{\sqrt{t}}\right)\right)$$

is elliptically distributed with a cumulative distribution function F.

Put $\widetilde{B}_t = \frac{B_t}{\sqrt{t}}$, (standardized Brownian motion), we have:

$$d\widetilde{B}_t = -\frac{B_t}{2t\sqrt{t}}dt + \frac{1}{\sqrt{t}}dB_t$$

Let us define $G = F^{-1} \circ \Phi$, so that $X_t = G(\widetilde{B}_t)$ for each time t.

The dynamics of X_t is obtained by applying Ito's lemma :

$$dX_t = \frac{1}{2t}\left[G''(\widetilde{B}_t) - G'(\widetilde{B}_t)\widetilde{B}_t\right]dt + \frac{1}{\sqrt{t}}G'(\widetilde{B}_t)dB_t$$

This is a semi-martingale representation of the process X, which at each time t, has an Elliptical distribution $X_t \sim E_1(\mu, \sigma, f)$.

Now, for the above S.D.E to admit a strong solution $X_t = F_t^{-1}\left(\Phi\left(\frac{B_t}{\sqrt{t}}\right)\right)$, two conditions must be checked. This will impose restrictions on the choice of G, and therefore for the choice of F_t, the cdf of X_t.

Remark 1. *The case of geometric Brownian motion can be recovered for a suitable choice of F*

7 Concluding Remarks

In this paper we derived Stein's Lemma for a bivariate elliptical random variable and used it to re-derive the C.A.P.M. We also used the probability distortion functions approach to derive a closed form solution of a call option price when the underlying is elliptically distributed. This generalizes the work of Hamada & Sherris (2003) where consistency of Black-Scholes option pricing and probability distortion functions is proved in the case of normality. We finally derive an SDE of processes which are elliptically distributed at each time. This result is general and can be used for any other type of distributions. Further enhancement of the paper might consist of empirical test of the CAPM when the returns are elliptical. The Australian Stock Exchange data can be used for this purpose.

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