

Pricing American Interest Rate Options in a Heath-Jarrow-Morton Framework Using Method of Lines

by

Carl Chiarella and Nadima El-Hassan
School of Finance and Economics
University of Technology, Sydney
PO Box 123, Broadway NSW2007
AUSTRALIA

Fax: +61 2 9514 7722

email: carl.chiarella@uts.edu.au
nadima.el-hassan@uts.edu.au

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ABSTRACT We consider the pricing of American bond options in a Heath-Jarrow-Morton framework in which the forward rate volatility is a function of time to maturity and some underlying interest rate. It turns out that in this case the resulting pricing partial differential operators are two dimensional in the spatial variables. In this paper we investigate an efficient numerical method to solve these partial differential equations for American option prices and the corresponding free exercise surface. We consider in particular the method of lines which other investigators (eg Carr and Faguet (1994) and Van der Hoek and Meyer (1997)) have found to be efficient for American option pricing when there is one spatial variable. In extending this method for the two dimensional case, we solve the pricing equation by discretising the time variable and one of the state variables and use the spot rate of interest as a continuous variable. We compare our method with the lattice method of Li, Ritchken and Sankarasubramanian (1995).¹

1 Introduction

The Heath-Jarrow Morton approach to modeling the term structure of interest rates provides a complete and consistent theoretical framework for the evaluation of interest rate contingent claims. This approach mirrors for interest rate markets the Black-Scholes framework for the stock option. However, the main implementation difficulty of the Heath-Jarrow-Morton model is that for the most general specifications of the forward rate volatility function, the stochastic process for the instantaneous spot rate of interest is non-Markovian. That is, the evolution of the instantaneous spot rate depends on the all the paths taken by the term structure since the initial date. Hence, popular lattice models used for pricing interest rate options in this framework will be non-recombining and tend to grow exponentially with the number of steps in the lattice.

A great deal of research has therefore gone into determining which specification of the forward rate volatility functions allow the stochastic dynamics driving the term

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structure of interest rates to be reduced to Markovian form. In particular Cheyette (1992), Ritchken and Sankarasubramanian (1995), Carverhill(1994) and, Bhar and Chiarella(1997), show that this is possible when the forward rate volatility is a function of the spot rate of interest and time to maturity. Inui and Kijima (1997), and Chiarella and Kwon (2004) have shown that this reduction is also possible when the functional dependence on the spot rate is extended to functional dependence on a whole set of forward rates. In this way the dynamics of the entire yield curve can be reasonably well captured by the dynamic evolution of a finite number of rates (usually one would use the most liquidly traded rates). An important result in this literature is that with this quite general volatility specification it is possible to obtain a closed form expression for zero coupon bond prices in terms of the underlying rates.

The reduction to Markovian form comes at the cost of increasing the dimension of the state space by introducing integrated variance type quantities. A feature of the Markovian representation is that it is possible to obtain the infinitesimal generator for the transition probability density of the stochastic process driving the term structure of interest rates, from which it is then possible to obtain the partial differential equations determining the value of interest rate contingent claims.

In Chiarella and El-Hassan (1997), we have considered and solved by the method of lines partial differential equation for American options on zero coupon bonds in the case where the forward rate volatility is a deterministic of time to maturity. In this paper we extend that work by allowing the forward rate volatility to be a function of the instantaneous spot rate of interest and time to maturity. We know from Ritchken and Sanakarasubrnaian (1995) and Bhar and Chiarella (1997) that in this case the Markovian system driving the term structure of interest rates depends on the spot rate itself and an accumulated variance quantity. Li, Ritchken and Sankarasubramanian(1995) also consider the problem of pricing interest rate contingent claims in this framework. They develop a binomial-type lattice model which involves maintaining a vector of information on the accumulated variance at each node of the tree.

In this paper, we also consider the same problem but from the point of view of considering the partial differential equation implied by the infinitesimal generator of the two state variable stochastic process. Our approach allows us to obtain the partial differential equation which is the analogue of the Black-Scholes partial differential equation in that is also preference free. It also allows us to apply the full gamut of the numerical techniques for the solutions of partial differential equations to the evaluation of interest rate contingent claims. We believe this framework clarifies the pricing of interest rate contingent claims in the HJM framework and adds to the tool kit for pricing such claims. Here the numerical technique we employ is the method of lines which has been successively applied to the American put options on common stock by Goldenberg and Schmidt (1994), Meyer and Van der Hoek (1997) and Carr and Faguet (1996). The method of lines is found to be accurate and relatively efficient. An advantage of the method is that the delta of the option is a by-product of the numerical procedure. Furthermore, the solution generates a value surface for a variety of values of the underlying state variables for a given point in time. The method also facilitates the determination of the early exercise surface for the American option problem. We test the accuracy of the method by using it to solve the partial differential equation for bond prices for which there is a known analytic solution.

The plan of the paper is as follows. In section 2, we discuss the reduction to Markovian form, the bond pricing formula and derivation of the partial differential equation for the value of contingent claims. In section 3, we outline the method of lines as applied to the American put bond options problems. Section 4, discusses numerical results and sections 5 draws some conclusion and makes suggestions for further research.

2 Markovian HJM Models

The driving state variable of the HJM approach is $f(t, T)$, the forward rate at time t for instantaneous borrowing at time T . HJM show that under the equivalent martingale measure the forward rate dynamics can be expressed as

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T, \cdot) ds, \quad (1)$$

where

$$\alpha(t, T, \cdot) = \sigma^f(t, T, \cdot) \int_t^T \sigma^f(t, u, \cdot) du. \quad (2)$$

$\sigma^f(t, T, \cdot)$ is the forward rate volatility function whose third argument indicates possible dependence on a stochastic variable such as $r(t)$ or $f(t, T)$ itself. Finally, $\tilde{w}(t)$ is a Wiener process under the equivalent martingale measure.

From equation (1), we readily obtain the process for the instantaneous spot rate, $r(t) [= f(t, t)]$, as the stochastic integral equation

$$r(t) = f(0, t) + \int_0^t \sigma^f(u, t, \cdot) \int_u^t \sigma^f(u, y, \cdot) dy du + \int_0^t \sigma^f(u, t, \cdot) d\tilde{w}(s), \quad (3)$$

or the stochastic differential equation

$$dr(t) = \left[f_2(0, t) + \frac{\partial}{\partial t} \int_0^t \sigma^f(u, t, \cdot) \int_u^t \sigma^f(u, y, \cdot) dy du - \int_0^t \sigma_2^f(u, t, \cdot) d\tilde{w}(u) \right] + \sigma^f(t, t, \cdot) d\tilde{w}(t). \quad (4)$$

The price of a pure discount bond $P(t, T) \left[= \exp \left(- \int_t^T f(t, s) ds \right) \right]$, by Ito's lemma satisfies the stochastic differential equation

$$dP(t, T) = r(t)P(t, T)dt + \left[- \int_t^T \sigma^f(t, u, \cdot) du \right] P(t, T) d\tilde{w}(t) + \int_0^t \sigma^f(s, T, \cdot) d\tilde{w}(s). \quad (5)$$

The non-Markovian nature of the foregoing stochastic dynamical system stems from the third term in the drift coefficient of the stochastic differential equation (4) for $r(t)$. This term is an integral over the history of the noise process. A number of authors have shown how to reduce the non-Markovian system to a Markovian system of higher dimension by making various assumptions about the volatility function $\sigma^f(t, u, \cdot)$; see Cheyette (1992), Caverhill (1994), Ritchken and Sanakarasubramanian (1995), Bhar and Chiarella (1997, 1998).

Bhar and Chiarella (1997) in particular show that if

$$\sigma^f(t, T, \cdot) = p_n(T-t)e^{-\lambda(T-t)}G(r(t)), \quad \sigma > 0, \quad (6)$$

where $p_n(u)$ is a polynomial,

$$p_n(u) = a_0 + a_1u + \cdots + a_nu^n,$$

and G is some reasonably well-behaved function, then the system dynamics may be expressed in Markovian form. The cost of this reduction is the introduction of some supplementary state variables that summarize various statistical properties of the path history.

Here we shall focus on the special case of (6) having the form

$$\sigma^f(t, T, \cdot) = \sigma e^{-\lambda(T-t)}G(r(t)) \quad (\sigma > 0). \quad (7)$$

Typically we will take

$$G(r) = r^\gamma \quad (0 \leq \gamma), \quad (8)$$

so that the case $\gamma = \frac{1}{2}$ allows us to draw a link to the generalised Cox-Ingersoll-Ross model of Hull and White (1984) and the square root process of Duffie and Kan (1996).

Ritchken and Sankarasubramanian (1995) and Bhar and Chiarella (1997) show that for the forward rate volatility (7) the stochastic dynamics for $r(t)$ may be expressed in the Markovian form

$$dr = [D(t) + \sigma^2\phi - \lambda r] dt + \sigma G(r)d\tilde{w}, \quad (9)$$

$$d\phi = [G(r)^2 - 2\lambda\phi] dt, \quad \phi(0) = 0, \quad (10)$$

where we set

$$D(t) \equiv f_2(0, t) + \lambda f(0, t),$$

we note that the quantity $\phi(t)$ can also be expressed as

$$\phi(t) = \int_0^t G(r(s))e^{-2\lambda(T-s)} ds, \quad (11)$$

which is a function of the history of the r process up to time t .

Following the standard HJM argument interest rate contingent claims can be expressed as expectations with respect to the probability distribution generated by the

stochastic systems (9), (10) that we denote by \tilde{E} . For instance the price of a discount bond maturing at any time T_B is given by

$$P(r, \phi, t, T_B) = \tilde{E}_t \left[\exp \left\{ - \int_t^{T_B} r(y) dy \right\} \right]. \quad (12)$$

Similarly, the price of a European call option on this bond with exercise price K and which matures at time T ($< T_B$) is given by

$$C(r, \phi, t, T) = \tilde{E}_t \left[\exp \left\{ - \int_t^T r(y) dy \right\} (P(r, \phi, T, T_B) - K)^+ \right]. \quad (13)$$

We use $\pi(r_T, \phi_T, T | r_t, \phi_t, t)$ denote the transition probability density of the diffusion process (9), (10) between times t and T ($t < T$), this quantity satisfies the Kolmogorov backward equation

$$K\pi + \frac{\partial \pi}{\partial t} = 0, \quad (14)$$

where K , the infinitesimal generator of the diffusion process (9), (10) is defined by

$$K\pi \equiv \frac{1}{2} \sigma^2 G(r)^2 \frac{\partial^2 \pi}{\partial r^2} + [D(t) + \sigma^2 \phi - \lambda r] \frac{\partial \pi}{\partial r} + [G(r)^2 - 2\lambda \phi] \frac{\partial \pi}{\partial \phi}. \quad (15)$$

For a general interest rate dependent claim, $F(r, \phi, t, T)$ maturing at time T , application of the Feynman-Kac formula to (12) yields the partial differential equation that must be obeyed by the value of that claim (with appropriate boundary conditions).

For instance, if the claim is a zero coupon bond, the bond price must satisfy

$$KP + \frac{\partial P}{\partial t} - rP = 0, \quad (16)$$

which is to be solved on some time interval $t_0 \leq t \leq T_B$ subject to $P(r, \phi, T_B, T_B) = 1$. However we know from Ritchken and Sankarasubramanian (1995) that it is possible to obtain a closed form solution for the bond price in this case. In fact

$$P(r, \phi, t, T_B) = \frac{P(r_0, 0, 0, T_B)}{P(r_0, 0, 0, t)} \exp \left\{ - \frac{1}{2} \beta^2(t, T) \phi(t) + \beta(t, T) [f(0, t) - r(t)] \right\}, \quad (17)$$

where

$$B(t, T) = \frac{1}{\lambda} \left(1 - e^{-\lambda(T-t)} \right).$$

Similarly the partial differential equation for the price of the bond option $C(r, \phi, t, T)$ in (13) satisfies the partial differential equation

$$KC + \frac{\partial C}{\partial t} - rC = 0, \quad (18)$$

which is to be solved on the time interval $0 \leq t \leq T$ subject to

$$C(r, \phi, T, T) = [P(r, \phi, T, T_B) - K]^+. \quad (19)$$

3 The Solution Algorithm: Method of Lines with Invariant Imbedding

3.1 Formulation

In section 2 of this paper, we presented the two-state variable partial differential equation (18) for the price of bond options that is both preference free and matches the initial term structure of interest rates. The derivation of equation (18) was facilitated by the choice of the forward rate volatility of the form given in (7) which renders the dynamics of the spot rate of interest in the HJM framework Markovian. Hence, the price of any claim, where the underlying state variable is the spot rate of interest (and the accumulated variance ϕ), must satisfy equation (18) subject to appropriate boundary conditions.

Let $U(r, \phi, t, T)$ denote the price of an American option on a pure discount bond with exercise price K and expiry T . The maturity of the underlying discount bond is T_B , where $T \leq T_B$. Note that both the price of the discount bond and the value of the option on the bond are functions of the same state variables, namely the stochastic spot rate of interest, r , and the accumulated variance, ϕ , whose dynamics are given by (9) and (10) for the volatility function specified in (7). Hence, the price of the American option, $U(r, \phi, t, T)$ must satisfy the partial differential equation (18). In particular, the price of the American put option on a pure discount bond must satisfy the partial differential equation

$$KU + \frac{\partial U}{\partial t} - rU = 0, \quad (20)$$

in the continuation region C .

The continuation region is defined as

$$C = \{(r, \phi, s) | 0 < r < r^*, 0 < \phi < \phi^*, t \leq s \leq T\}. \quad (21)$$

In this region, the optimal strategy for the American put is to hold rather than exercise the option. Also in this region, the price of the pure discount bond is greater than the time dependent critical price of the bond, $P(r^*, \phi^*, t^*, T_B)$. This is the bond price at, $r(\cdot) = r^*(\cdot)$, $\phi = \phi^*$ and $t^* = \inf[s; s > t, r(s) \geq r^*(s)] \wedge T$ (see Chesney, Elliott and Gibson (1993)). Hence, the continuation region can be redefined in terms of the discount bond price as

$$C = \left\{ (r, \phi, s) | P(r, \phi, s, T_B) > P(r^*, \phi^*, s, T_B), \right. \\ \left. t \leq s \leq T \right\}. \quad (22)$$

The complement of the continuation region, the stopping region, S is defined as

$$S = \{(r, \phi, s) | r \geq r^*, t \leq s \leq T\}, \quad (23)$$

or in terms of the bond price

$$S = \left\{ (r, \phi, s) | P(r, \phi, s, T_B) \leq P(r^*, \phi^*, s, T_B), \right. \\ \left. t \leq s \leq T \right\}. \quad (24)$$

The optimal strategy in this region is to exercise the option with the price of the put option given by its intrinsic value, $[K - P(r, \phi, t)]$.

Equation (20) must be solved subject to

$$U(r, \phi, s, T) > \max[K - P(r, \phi, s, T_B), 0], \quad (25)$$

i.e. $(r, \phi, s) \in C$,

$$U(r, \phi, T, T) = \max[K - P(r, \phi, T, T_B)], \quad (26)$$

$$U(r^*, \phi^*, t^*, T) = K - P(r^*, \phi^*, t^*, T_B), \quad (27)$$

$$\text{grad}U(r^*, \phi^*, t^*, T) = -\text{grad}[K - P(r^*, \phi^*, t^*, T_B)]. \quad (28)$$

(This last condition translates to the smooth pasting condition $\frac{\partial U}{\partial P} \Big|_{P=P(r^*, \phi^*, t^*, T_B)} = -1$.)

Let us note here that since ϕ is not an observable quantity but an endogenous variable the question of when to exercise simply translates to the observation of the interest rate with respect to time. Thus the grad operator defaults to the one-dimensional derivative. Hence, the condition for the American option collapses to the one variable case where one only requires to smoothly paste to the surface with respect to r , independently of ϕ . Note this is not to say that one collapses to only a free line but the intersection of a surface with a plane that is characterised only by r and t . The intersection between the two surfaces gives rise to a ϕ dependent surface. One can view this as a three dimensional manifold being cut by a two dimensional surface. This is an important point to note. The intersection of the two surfaces yields an (r, ϕ, t) manifold despite the fact that one is pasting against an otherwise two dimensional (ie ϕ independent) surface.

The more subtle boundary condition for the American option is at $r = 0$. Unlike, in the case of an equity option, one cannot set the value of the option to zero at $r = 0$ since the value of the option in fact migrates upward as a function of time. This migration is in general not known a priori, therefore one has to adopt special solution techniques to handle the migrating boundary. We have found that the most consistent estimate of the value of the option at $r = 0$ could be obtained by setting the derivative of the option value function to zero at $r = 0$.

The value of options with an American style exercise feature is the solution of a free boundary value problem and requires the determination of the optimal early exercise boundary as well as the value of the option. Analytic formulas do not exist in general for American type options. Hence, the valuation of such options reduces to solving the system (20) subject to (25) – (28) by a range of fast and accurate numerical solution techniques such as the method of lines (Meyer, (1977, 1980, 1981), Goldenberg and Schmidt (1994), Meyer and Van Der Hoek (1997)), linear complementarity method or variational inequality techniques (Wilmott, Dewynne and Howison, (1993)). Another popular numerical technique used is that of binomial (or trinomial) trees.

Here, we apply the method of lines to evaluate American interest rate options in the two-dimensional state variable Markovian framework described in section 2. Li, Ritchken and Sankarasubramanian (1995) propose an algorithm for solving this problem in a binomial tree framework. Our objective here is to formulate and solve the problems in a partial differential equation framework.

The method of lines with invariant imbedding is a numerical technique used for solving partial differential equations and can be applied to free boundary problems by tracking the time dependent free boundary. In general, the technique involves discretisation of the time variable, thus replacing the time derivative with its discrete approximate analogue at each time step. This reduces the partial differential equation to a sequence of second order non-homogenous ordinary differential equations, which must be successively solved at each time step. By applying a Ricatti transformation, each second order non-homogenous boundary value problem can be transformed into a system of three first order ordinary differential equations, thus reducing second order boundary value problems to first order initial value problems with the obvious advantages.

In the case of our problem, where we have two state variables, r and ϕ , and the time variable, both the ϕ and time variables are discretised (while maintaining continuity case for the r variable) and their partials replaced with difference quotients. The multi-point free boundary problem for the resulting system of second order ordinary differential equations is then solved. At each time step of this algorithm a free boundary problem for an ordinary differential equation must be solved by conversion to an initial value problem through invariant imbedding (or sweeping method). The free boundary is found as the root of a function derived from the boundary conditions.

The advantages of using a numerical technique such as the method of lines for solving the American option problem include relative efficiency and accuracy, and the ability to handle coefficients of the partial differential equation which are functions of the state variables and time (Meyer, 1977). The method is well suited to free boundary problems as it is relatively simple to determine the free boundary or free surface as part of the solution algorithm. The method of lines can be applied to both one-dimensional and multidimensional free boundary problems. The formulation and subsequent of the American option in a partial differential equation framework is very useful as the resulting solutions are in the form of value surfaces. This gives the solution values for a large number of underlying state variables simultaneously.

3.2 The Solution Algorithm

The method of lines technique was applied to the problem of American put options on stocks by Goldenberg and Schmidt (1994), Meyer and Van Der Hoek (1997) and Carr and Faguet (1995). The complete algorithm and implementation details in Goldenberg and Schmidt (1994), Meyer and Van Der Hoek (1997) form the basis of the application of the method of lines in multidimensional form to this problem, as summarised below.

For illustrative purposes, we rewrite the parabolic operator K as

$$a(r) \frac{\partial^2 U}{\partial r^2} + b(r, \phi, t) \frac{\partial U}{\partial r} + c(r, \phi) \frac{\partial U}{\partial \phi}, \quad (29)$$

where the coefficients $a(r)$, $b(r, \phi, t)$ and $c(r, \phi)$ are given by

$$a(r) = \frac{1}{2} \sigma^2 G(r)^2, \quad (30)$$

$$b(r, \phi, t) = [f_2(0, t) + \lambda f(0, t) + \sigma^2 \phi - \lambda r], \quad (31)$$

$$c(r, \phi) = G(r) - 2\lambda \phi, \quad (32)$$

with

$$G(r) = r^\gamma,$$

and

$$\phi = \int_0^t G(r(s))^2 e^{-2\lambda(t-s)} ds. \quad (33)$$

Using the following discretisations on ϕ and t ,

$$t = m\Delta t, \text{ for } m = M, \dots, 0, \text{ with } M = \frac{T}{\Delta t}, \quad (34)$$

and

$$\phi = k\Delta\phi, \text{ for } k = K, \dots, 0 \text{ with } K = \frac{\Phi}{\Delta\phi}, \quad (35)$$

noting that we do not discretise r at this stage.

For the rest of this section the above discretisation scheme will be used with the following notation

$$U(r, k\Delta\phi, m\Delta t) = U_k^m(r), \quad (36)$$

where the $U(r)$ are twice differentiable functions. In explaining the implementation of the solution algorithm, $U(r, k\Delta\phi, m\Delta t) (= U_k^m(r))$ should be taken to indicate the value of any interest rate contingent claim.

$$b(r, k\Delta\phi, m\Delta t) = b_k^m(r), \quad (37)$$

$$c(r, k\Delta\phi) = c_k(r), \quad (38)$$

with $a(r)$ staying the same because there is no discretisation of r at this stage.

As discussed above, we approximate the partial derivatives with respect to ϕ and t with difference quotients as follows:

$$\frac{\partial P}{\partial t} \approx \frac{U_k^{m+1}(r) - U_k^m(r)}{\Delta t}, \quad (39)$$

$$\frac{\partial P}{\partial \phi} \approx \frac{U_{k+1}^m(r) - U_k^m(r)}{\Delta\phi}. \quad (40)$$

Note, at first glance it would appear that we have taken a forward difference in time, however this is not so. Since time runs from T down to 0, we have in fact taken a backward difference, the same as for ϕ .

$$\begin{aligned} \frac{U_k^{m+1}(r) - U_k^m(r)}{\Delta t} &+ a(r) \frac{d^2 U_k^m(r)}{dr^2} + b(r, \phi, t) \frac{dU_k^m(r)}{dr} \\ &+ c(r, \phi) \frac{U_{k+1}^m(r) - U_k^m(r)}{\Delta\phi} - rU_k^m(r) = 0. \end{aligned} \quad (41)$$

Now, substituting the notation given by equations (37) – (40) into (41) we have that

$$\begin{aligned} \frac{U_k^{m+1}(r) - U_k^m(r)}{\Delta t} &+ a(r) \frac{d^2 U_k^m(r)}{dr^2} + b_k^m(r) \frac{dU_k^m(r)}{dr} \\ &+ c_k(r) \left[\frac{U_{k+1}^m(r) - U_k^m(r)}{\Delta\phi} \right] - rU_k^m(r) = 0. \end{aligned} \quad (42)$$

Collecting coefficients of $U_k^m(r), U_k^{m+1}(r)$ and $U_{k-1}^m(r)$ together, we obtain the second-order ordinary differential equation:

$$\left\{ \begin{aligned} & a(r) \frac{d^2 U_k^m(r)}{dr^2} \\ & + b_k^m(r) \frac{dU_k^m(r)}{dr} - \left(r + \frac{1}{\Delta t} + \frac{c_k(r)}{\Delta \phi} \right) \end{aligned} \right\} U_k^m(r) + \left(\frac{1}{\Delta t} U_k^{m+1}(r) + \frac{1}{\Delta \phi} c_k(r) U_{k+1}^m \right) = 0. \quad (43)$$

The first part of equation (43) constitutes the current unknown variable $U_k^m(r)$. The second and third boxed terms represent known values from the previous time step $U_k^{m+1}(r)$ and the previous ϕ values $U_{k+1}^m(r)$.

We can write equation (15) as

$$\frac{d^2 U_k^m(r)}{dr^2} = A_k^m(r) \frac{dU_k^m(r)}{dr} + B_k^m(r) U_k^m(r) + C_k^m(r), \quad (44)$$

where the coefficients $A_k^m(r), B_k^m(r)$ and $C_k^m(r)$ are given by

$$A_k^m(r) = -\frac{b_k^m(r)}{a(r)}, \quad (45)$$

$$B_k^m(r) = \left(r + \frac{1}{\Delta t} + \frac{1}{\Delta \phi} c_k(r) \right) / a(r), \quad (46)$$

$$C_k^m(r) = -\left[\frac{1}{\Delta \phi} c_k(r) U_{k+1}^m + \frac{1}{\Delta t} U_k^{m+1}(r) \right] / a(r). \quad (47)$$

with boundary conditions

$$U_k^m(R_{\max}) = \beta_k^m \text{ (and } U_k^m(0) = \alpha_k^m \text{)},$$

and initial condition ($m = M$)

$$U_k^M(r) = \xi.$$

The second order ordinary differential equation in (44) can be reduced to a first order system by means of the transformations

$$V_k^m(r) = \frac{dU_k^m(r)}{dr}, \quad (48)$$

$$\frac{dV_k^m(r)}{dr} = A_k^m(r) V_k^m(r) + B_k^m(r) U_k^m(r) + C_k^m(r), \quad (49)$$

where U is assumed to be a Riccati transform of V such that

$$U_k^m(r) = R_k^m(r) V_k^m(r) + W_k^m(r). \quad (50)$$

The representation in (23) implies the following Riccati equation:

$$\frac{dR_k^m(r)}{dr} = 1 - A_k^m(r) R_k^m(r) - B_k^m(r) (R_k^m(r))^2, \quad (51)$$

$$\frac{dW_k^m(r)}{dr} = -R_k^m(r) [B_k^m(r) W_k^m(r) - C_k^m(r)], \quad (52)$$

where A_k^m , B_k and C_k^m are given in (18) and (19).

Equations (51) and (52) can be solved numerically as indicated in Appendix 2 to determine R and W at each point in time (see Goldenberg and Schmidt (1994), Meyer and Van der Hoek (1997)). The Ricatti transformation holds for all values of the state variables, r , including the free boundary. Hence, once R and W are known, the critical value of r at time step t_m is determined as the root of equation (50) using the boundary conditions (27) and (28) for the option. Having determined the values of R and W , V is found by substituting (50) into (49) and integrating numerically. The value of the contingent claim at this step is determined by substituting the R , W and V into the Ricatti transformation (50).

Under the method of lines, the hedge parameters are also determined as part of the solution. In particular, $V(r)$ is the delta of the option.

Figure 1 illustrates the determination of the critical value r^* at a particular point in the (ϕ, t) grid. Figure 2 indicates the sequence in which points in the (ϕ, t) are stepped through. Figure 3 illustrates the free surface in (r, ϕ, t) space.

4 Results

In this section we present some preliminary results obtained by applying the method of lines with invariant imbedding to the valuation of American put option on zero coupon bonds. The bond prices required in the boundary conditions (27) and (28) are calculated using equation (17).

Time Steps	Exercise 65	Price 70	Run Time (secs)
100	6.79	9.48	1.04
500	6.85	9.53	5.12
1000	6.90	9.54	10.91
2000	6.91	9.55	17.34

Table 1: *Option values calculated using method of lines with invariant imbedding.*

Three year American put option on a 10 year zero coupon bond. Bond face value = 100; The volatility structure is given by equations (7) and (8) with $\sigma = 0.02$, $\lambda = 0.1$, and $\gamma = 0.4$. The initial term structure of interest rates is assumed to be flat at 5%.

To gain some insight into the relative accuracy and computational efficiency of the method of lines technique as applied to the problem proposed in this paper, we perform some comparison of results with the Li, Ritchken and Sankarasubramanian (1995) method for solving American bond options in this framework. Their method consists of a binomial-type lattice model to evaluate American options in the two-dimensional state variable Markovian HJM framework. Their model provides a discretised approximation for the spot rate process $r(t)$. The lattice is constructed after a transformation is made to convert the spot rate process into a constant volatility process. A reconnecting lattice is then constructed for the transformed spot rate process, while maintaining a

vector at each node to represent the process for the accumulated variance. Derivative prices can be calculated on the lattice by means of backward recursion. The algorithm converges to the continuous time limit if the time and ϕ partitions are made arbitrarily fine.

Time Steps	Exercise 65	Price 70	Run Time (secs)
100	6.78	9.47	2.33
500	6.82	9.50	8.72
1000	6.89	9.52	18.96
2000	6.91	9.55	22.38

Table 2: Option values calculated using the lattice model.

Three year American put option on a 10 year zero coupon bond. Bond face value = 100; The volatility structure is given by equations (7) and (8) with $\sigma = 0.02$, $\lambda = 0.1$, and $\gamma = 0.4$. The initial term structure of interest rates is assumed to be flat at 5%. The maximum number of ϕ values allowed at each node is 8.

Comparison of Table 1 with Table 2, show that the method of lines with invariant imbedding is slightly faster for the same level of accuracy. However a more meaningful comparison would be to compare the complexity of the algorithms which will be part of future work on this model. Furthermore, it should be noted that the method of lines provides the option value surface, the free boundary surface and the delta of the option simultaneously.

5 Conclusion

We have set up the problem of pricing contingent claims under a specific assumption about the forward rate volatility function as the solution of a partial differential equation. We have shown that the method of lines is an efficient method to price American claims in this framework. Comparison with the quasi-analytical solution in the special square root process indicates that the method is very accurate. The method also allows us to generate as a by-product option deltas and early exercise surfaces. Computational time compares favorable with the lattice method of Li, Ritchken and Sanakarasubramanian and is somewhat faster for a given level of accuracy. The method we propose also has the advantage of more readily handling quite general initial term structures in comparison to lattice models.

Further research will focus on the case where the forward rate volatility function is a function of not only instantaneous spot rate but also a series of discrete forward rates. The infinitesimal generator for this case can be easily obtained from the framework of Chiarella and Kwon (2004). If for example, one were to take the case in which the forward rate volatility function depends on the instantaneous spot rate of interest and one forward rate (eg a long rate), then one would have a preference free Brennan-Schwartz type two-factor model. It turns out that in this case the infinitesimal generator

depends on three state variables, namely the two rates and the accumulated variance. The method of lines or indeed other method for the numerical solution of partial differential equations could be applied to the problem of pricing in such a framework. It is for such higher dimensional problems that the methods of lines may display its true advantages when compared to existing methods for handling American bond options.

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Appendix

Solving for R , V and W

To solve for $R_k^m(r)$, $V_k^m(r)$ and $W_k^m(r)$ we use a discrete mesh of the space variable r , which we denote by

$$r_i = i\Delta r \text{ for } i = 0, \dots, N, \quad (53)$$

Solving for $R_k^m(r)$ using

$$\frac{dR_k^m(r)}{dr} = 1 - A_k^m(r)R_k^m(r) - B_k(r)(R_k^m(r))^2, \quad (54)$$

To solve for $R_k^m(r)$ we use a mesh of the space variable r ,

$$r_i = i\Delta r, \text{ for } i = 0, \dots, N, \quad (55)$$

and employ the following notation

$$R_i = R_k^m(r_i), R_{i+1} = R_k^m(r_{i+1}), \quad (56)$$

$$A_i = A_k^m(r_i), A_{i+1} = A_k^m(r_{i+1}), \quad (57)$$

$$B_i = B_k(r_i), B_{i+1} = B_k(r_{i+1}), \quad (58)$$

where A_k^m , B_k and C_k^m are given in (18) and (19).

Using the notation defined by equations (55) – (58), equation (54) can be solved for $R(r)$ by integrating using the implicit trapezoidal rule.

$$\frac{R_{i+1} - R_i}{\Delta r} = \frac{1}{2} \left\{ \begin{array}{l} [1 - A_{i+1}R_{i+1} - B_{i+1}R_{i+1}^2] \\ + [1 - A_iR_i - B_iR_i^2] \end{array} \right\}. \quad (59)$$

Now equation (30) can be written as

$$\bar{A}R_{i+1}^2 + \bar{B}R_{i+1} + \bar{C} = 0, \quad (60)$$

where the coefficients \bar{A} , \bar{B} and \bar{C} are given by

$$\bar{A} = B_{i+1}, \quad (61)$$

$$\bar{B} = \frac{2 + \Delta r A_{i+1}}{\Delta r}, \quad (62)$$

$$\bar{C} = B_i R_i^2 + \left(\frac{-2 + \Delta r A_i}{\Delta r} \right) R_i - 2. \quad (63)$$

Equation (60) which defines the new variable for R is a quadratic equation that can be solved analytically. The solution to the quadratic equation (60):

$$R_{i+1} = \frac{-\bar{B} + \sqrt{\bar{B}^2 - 4\bar{A}\bar{C}}}{2\bar{A}}. \quad (64)$$

Since $R(0) = 0$, i.e. $R_0 = 0$ we see that for the first step $\bar{C} = -2$. This is an important observation since in our case both $A_0 = 0$ and $B_0 = 0$ are singular at $r = 0$, thus we require to solve the reduced quadratic equation,

$$B_1 R_1^2 + \left(\frac{2 + \Delta r A_1}{\Delta r} \right) R_1 - 2 = 0. \quad (65)$$

Equation (64) can be written in more stable form as

$$R_{i+1} = \frac{-2\bar{C}}{\bar{B} + \sqrt{\bar{B}^2 - 4\bar{A}\bar{C}}}. \quad (66)$$

The Equation for W

$$\frac{dW}{dr} = -R(r) [B(r)W(r) - C(r)]. \quad (67)$$

To solve for W , we following the notation already given and again use the implicit trapezoidal rule so that

$$\frac{W_{i+1} - W_i}{\Delta r} = -\frac{1}{2} \{ R_{i+1} [B_{i+1}W_{i+1} + C_{i+1}] + R_i [B_i W_i + C_i] \}. \quad (68)$$

With W_i known, and all the R_i 's also known, it is a simple matter to solve equation (68) for W_{i+1} .

$$W_{i+1} = \frac{\hat{A}W_i + \hat{B}\Delta r}{\hat{C}}. \quad (69)$$

where the coefficients \hat{A} , \hat{B} and \hat{C} are given by the following expressions

$$\hat{A} = 1 - \frac{\Delta r R_i A_{i+1}}{\Delta r}, \quad (70)$$

$$\hat{B} = \frac{1}{2} [R_i C_i - R_{i+1} C_{i+1}], \quad (71)$$

$$\hat{C} = 1 + \frac{\Delta r R_{i+1} B_{i+1}}{2}. \quad (72)$$

As discussed for the Riccati equation, the equation for W is singular at $r = 0$. Therefore, in order to start the algorithm, we need to solve W_1 as a special case. Here we use the fact that $R_0 = 0$ and having already solved for R_1 , the special case for W_1 is given by

$$W_1 = \frac{W_0 + \hat{B}\Delta r}{\hat{C}}, \quad (73)$$

where

$$\hat{B} = -\frac{1}{2}R_{i+1}C_{i+1}, \quad (74)$$

and

$$\hat{C} = 1 + \frac{\Delta r R_{i+1} B_{i+1}}{2}, \quad (75)$$

since $A = 1$ by the fact that $R_0 = 0$.

The V equation

$$\frac{dV}{dr} = [A(r) + B(r)R(r)]V(r) + [B(r)W(r) + C(r)]V(r_{\max}) = 0. \quad (76)$$

Repeating the procedure that was used for solving $R(r)$ and $W(r)$

$$\frac{V_{i+1} - V_i}{\Delta r} = \frac{1}{2} \left\{ \begin{array}{l} ([A_{i+1} + B_{i+1}R_{i+1}]V_{i+1} + [B_{i+1}W_{i+1} + C_{i+1}]) \\ + ([A_i + B_iR_i]V_i + [B_iW_i + C_i]) \end{array} \right\}. \quad (77)$$

With $R_i, R_{i+1}, W_i, W_{i+1}$ and V_{i+1} known, we now solve for V_i

$$V_i = \frac{\tilde{A}V_{i+1} - \tilde{B}}{\tilde{C}}, \quad i = N - 1, \dots, 0, \quad (78)$$

where \tilde{A}, \tilde{B} and \tilde{C} are given by

$$\tilde{A} = 1 - \frac{1}{2}(A_{i+1} + B_{i+1}R_{i+1})\Delta r, \quad (79)$$

$$\tilde{B} = \frac{\Delta r}{2} [(B_iW_i + C_i) + (B_{i+1}W_{i+1} + C_{i+1})], \quad (80)$$

$$\tilde{C} = 1 + \frac{1}{2}(A_i + B_iR_i)\Delta r. \quad (81)$$

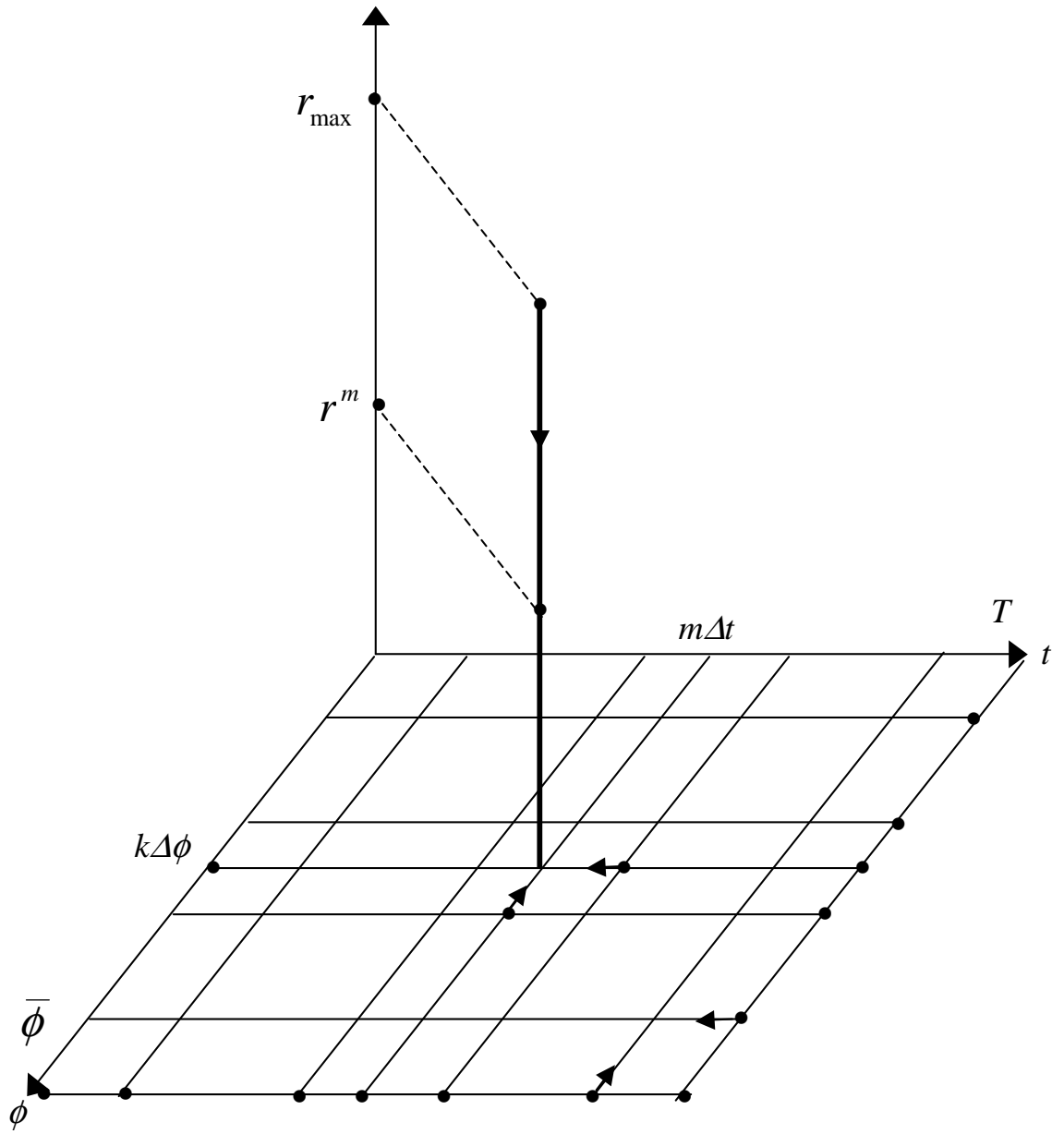


Figure 1: *Determining r^* at a particular point in the (ϕ, t) grid.*

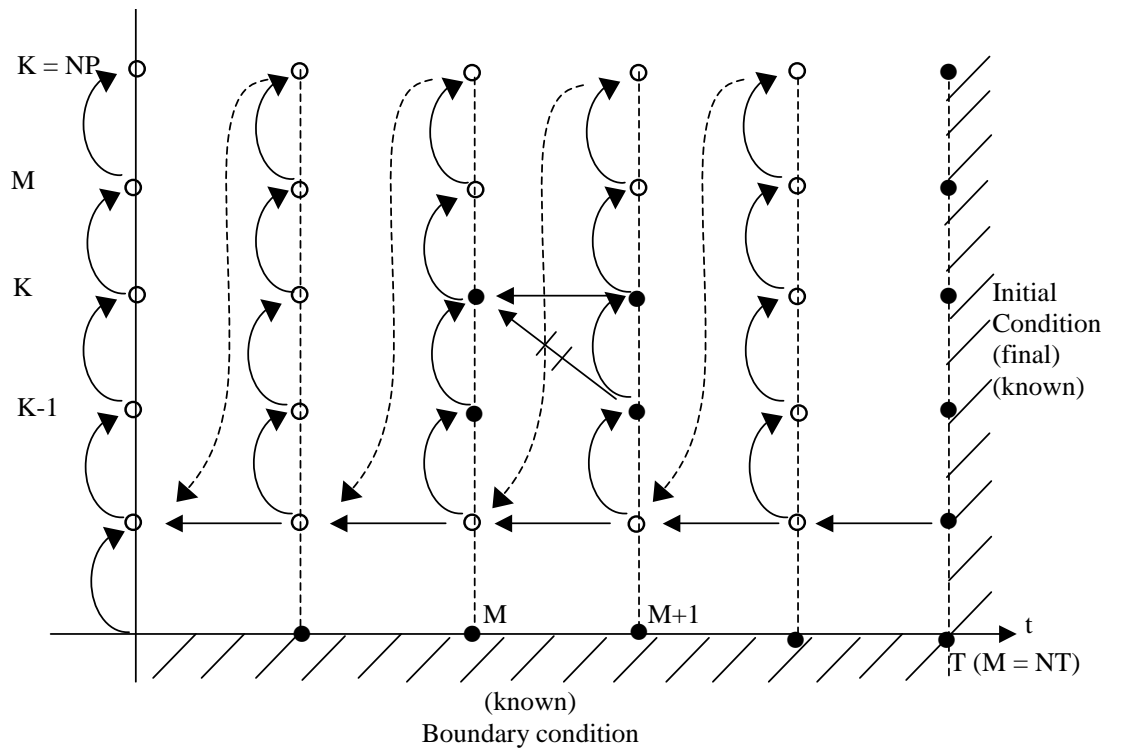


Figure 2: The way in which the algorithm steps through the (ϕ, t) grid.

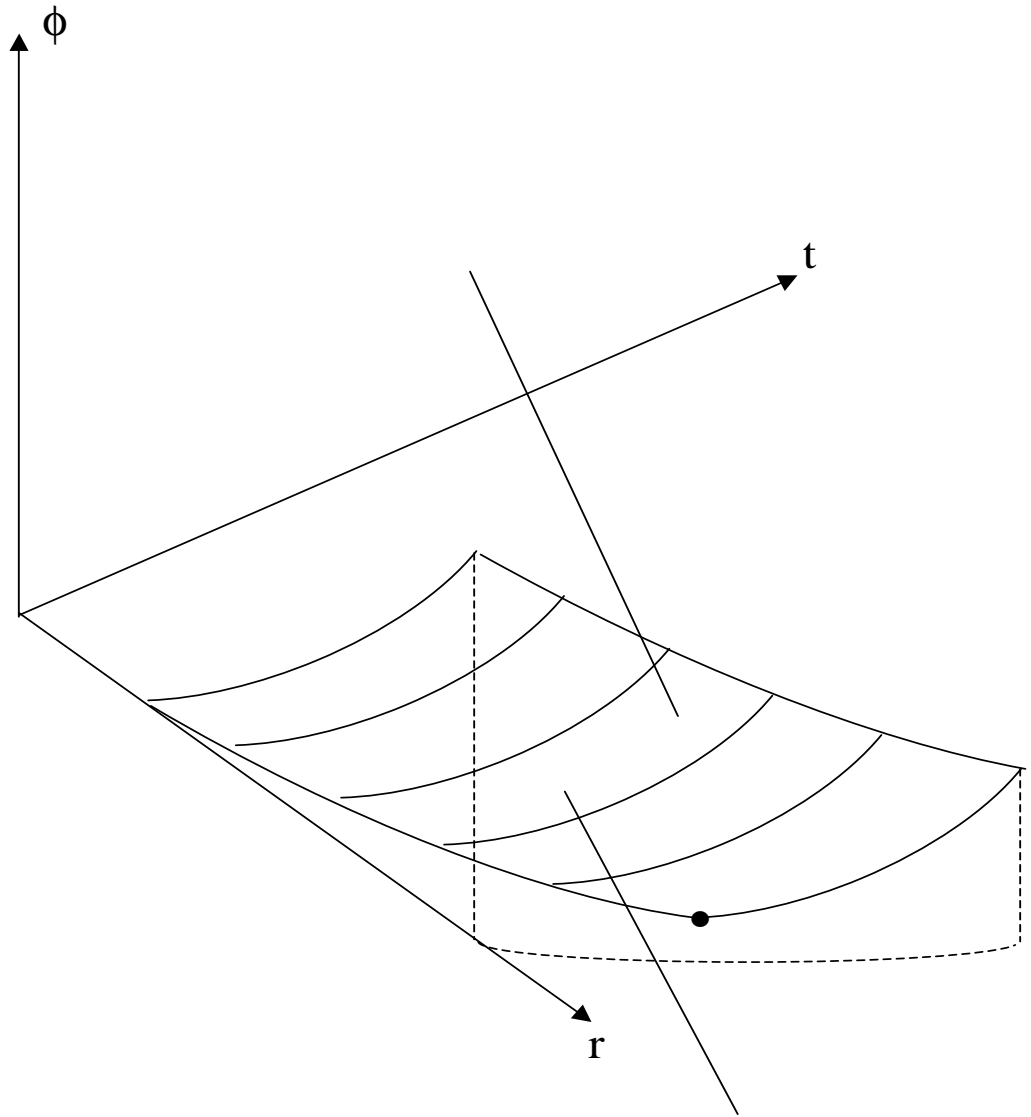


Figure 3: *Illustrating the free surface in (r, ϕ, t) space.*