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Hedging Diffusion Processes by Local Risk-Minimisation with Applications to Index Tracking

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Abstract. The solution to the problem of hedging contingent claims by local risk-minimisation has been considered in detail in Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1991). However, given a stochastic process X_t and $\tau_1 \neq \tau_2$, the strategy that is locally risk-minimising for X_{τ_1} is in general not locally risk-minimising for X_{τ_2}. In the case of diffusion processes, this paper considers the problem of determining a strategy that is simultaneously locally risk-minimising for X_τ for all τ. That is, a strategy that is locally risk-minimising for the *entire process* X_t. The necessary and sufficient conditions under which this is possible are obtained, and applied to the problem of index tracking. In particular, a close connection between the local risk-minimising and the tracking error variance minimising strategies for index tracking is established, and leads to a simple criterion for the selection of optimal set of assets from which to form a tracker portfolio, as well as a value-at-risk type measure for the set of assets used.

Key words: Minimal martingale measure, local risk-minimisation, hedging, incomplete market, index tracking, portfolio selection

JEL Classification: D52, D81, G11

Mathematics Subject Classification (2000): 91B28, 91B70

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1 Introduction

Let $N \in \mathbb{N}_+$, $T \in \mathbb{R}_+$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a probability basis satisfying the usual conditions where the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ is generated by a standard N-dimensional $\mathbb{P}$-Wiener process w_t. Denote by S_t^0 the value of the money market account earning interest at the risk free rate, r_t, and let S_t^i be the price of risky assets for $1 \leq i \leq N$. Assume that the S_t^i satisfy the equations

$$S_t^0 = S_0^0 \exp\left(\int_0^t r_s\, ds\right), \tag{1}$$

$$S_t^i = S_0^i + \int_0^t S_s^i \mu_s^i\, ds + \sum_{j=1}^N \int_0^t S_s^i \sigma_s^{i,j}\, dw_s^j, \tag{2}$$

where $1 \leq i \leq N$, and r_t, μ_t^i and $\sigma_t^{i,j}$ are adapted to $(\mathscr{F}_t)_{0\leq t\leq T}$ and sufficiently regular to ensure the existence of a unique strong solution. In the sequel, N will be the number of assets in a benchmark stock index and S_t^i will be the price process for the i-th asset in the index. It will be assumed further that for any $A \subset \{1,,2\ldots,N\}$ the matrix process $\sigma_t^A \triangleq (\sigma_t^{i,j})_{i\in A, 1\leq j\leq N}$ has full rank for all $t \in [0,T]$ so that in particular $\sigma_t^A(\sigma_t^A)'$ is invertible for all $t \in [0,T]$. This condition ensures that the index does not contain any redundant assets for all $t \in [0,T]$. The set A will represent the set of indices for the assets that are used to track the benchmark stock index. Finally, suppose we are given a square integrable process S_t^I that satisfies an equation of the form

$$S_t^I = S_0^I + \int_0^t S_s^I \mu_s^I\, ds + \sum_{j=1}^N \int_0^t S_s^I \sigma_s^{I,j}\, dw_s^j, \tag{3}$$

where μ_t^I and $\sigma_t^{I,j}$ are adapted to $(\mathscr{F}_t)_{0\leq t\leq T}$ and sufficiently regular. In the sequel, S_t^I will be the value process for the benchmark stock index. Note that since the stock index is constructed from its constituent assets, S_t^I necessarily satisfies an equation of the form (3).

For any subset $A \subset \{1,2,\ldots,N\}$ let $\mathcal{S}(A)$ be the set of assets with price S_t^i, where $i \in A$. Now fix $A \subset \{1,2,\ldots,N\}$, $\tau \in [0,T]$ and consider a τ-measurable (square integrable) random variable S_τ^I, where the process S_t^I is given by (3). Then it is not possible in general to find a *self-financing* replicating strategy for S_τ^I using only the assets in $\mathcal{S}(A)$. An approach for addressing this problem due to Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1991) is to relax the self-financing condition and look for strategies that replicate S_τ^I with minimal local risk in some sense. For the intuition and the details of this approach, refer to the papers listed above.

Suppose now that with A fixed, a locally risk-minimising strategy exists for S_τ^I for all τ. If $\tau_1 \neq \tau_2 \in [0,T]$, then it is the case in general that the locally risk-minimising for $S_{\tau_1}^I$ does not coincide with the corresponding strategy for $S_{\tau_2}^I$ over the common interval $[0,\tau_1 \wedge \tau_2]$. This paper gives the necessary and sufficient conditions under which a locally risk-minimising strategy for S_T^I is locally risk-minimising for all S_τ^I with $\tau \in [0,T]$. That is, we give the necessary and sufficient conditions under which it is possible to hedge an entire process using a locally risk-minimising strategy.

The results are then applied to the problem of tracking a benchmark stock index to establish close links between the locally risk-minimising and the tracking error variance (TEV) minimising strategies for index tracking. In particular, it is shown that the TEV minimising strategy is locally risk-minimising if and only if it is an unbiased tracker for S^I_τ. Exploiting the links between the two approaches, we obtain a simple criterion for selecting the optimal set of assets with which to track the index, and a value-at-risk type measure for the set of assets used to track the index.

The structure of the remainder of the paper is as follows. The main results on hedging processes by local risk-minimisation is first presented in Section 2. The results are then applied to the problem of index tracking in Section 3, and the special case of deterministic coefficients is considered in Section 4 to obtain a value-at-risk type measure for tracker portfolios. Finally, the paper concludes with Section 5.

2 Hedging Diffusion Processes by Local Risk-Minimisation

Let $N \in \mathbb{N}_+$, $T \in \mathbb{R}_+$, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$, S^i_t and S^I_t be as defined above, and denote by Z^i_t and Z^I_t the discounted processes, $Z^i_t \triangleq (S^0_t)^{-1} S^i_t$ and $Z^I_t \triangleq (S^0_t)^{-1} S^I_t$ respectively, where $1 \le i \le N$. Note then that the Z^i_t satisfy the equation

$$Z^i_t = Z^i_0 + \int_0^t Z^i_s(\mu^i_s - r_s)\, ds + \sum_{j=1}^n \int_0^t Z^i_s \sigma^{i,j}_s\, dw^j_s, \tag{4}$$

for all $i \in \{1, 2, \ldots, N, I\}$. Now, let $A = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \ldots, N\}$, and define $Z^A_t \triangleq (Z^{i_1}_t, Z^{i_2}_t, \ldots, Z^{i_n}_t)'$, $\mu^A_t \triangleq (\mu^{i_1}_t, \mu^{i_2}_t, \ldots, \mu^{i_n}_t)'$, and let σ^A_t be as defined above. Then the equations for Z^A_t can be written more compactly in the form

$$Z^A_t = Z^A_0 + \int_0^t \mathrm{diag}(Z^A_s)(\mu^A_s - r_s \mathbf{1}_n)\, ds + \int_0^t \mathrm{diag}(Z^A_s)\sigma^A_s\, dw_s, \tag{5}$$

where $\mathbf{1}_n = (1, 1, \ldots, 1)' \in \mathbb{R}^n$.

A *trading strategy* is a pair $\xi_t = (\xi^0_t, \xi^A_t)$, where ξ^A_t is an $\mathbb{R}^n$-valued predictable square integrable process and ξ^0_t is an adapted $\mathbb{R}$-valued process. The *value* process, $V_t(\xi)$, associated to ξ_t is defined by

$$V_t(\xi) \triangleq \xi^0_t S^0_t + \xi^A_t \cdot S^A_t, \tag{6}$$

where $S^A_t = (S^{i_1}_t, S^{i_2}_t, \ldots, S^{i_n}_t)'$, and the corresponding discounted value process, $\bar{V}_t(\xi)$, is given by

$$\bar{V}_t(\xi) \triangleq (S^0_t)^{-1} V_t(\xi) = \xi^0_t + \xi^A_t \cdot Z^A_t. \tag{7}$$

Given $\tau \in [0, T]$ and a square integrable $\mathscr{F}_\tau$-measurable random variable H, the strategy ξ_t is said to be locally risk-minimising for H (with respect to A) if H admits the Föllmer-Schweizer decomposition

$$H = H_0 + \int_0^\tau \xi^A_s \cdot dZ^A_s + L^H_\tau, \tag{8}$$

where $H_0 \in \mathbb{R}$, and L_t^H is a square integrable $\mathbb{P}$-martingale such that $L_0^H = 0$ and $\mathbb{P}$-orthogonal to the martingale part,

$$M_t \triangleq \int_0^t \operatorname{diag}(Z_t^A)\sigma_t^A \, dw_t, \tag{9}$$

of Z_t^A. For the purposes of this paper, it will be the case that $H = Z_\tau^I$, the *discounted* index value at time τ. For notational convenience, define

$$\widehat{K}_t \triangleq \int_0^t (\mu_s^A - r_t \mathbf{1}_n)'[\sigma_s^A(\sigma_s^A)']^{-1}(\mu_s^A - r_t \mathbf{1}_n) \, ds. \tag{10}$$

We recall the following result from Pham, Rheinländer and Schweizer (1998).

Theorem 1 *If $\widehat{K}_t$ is continuous and bounded, then every square integrable $\mathscr{F}_\tau$-measurable random variable can be replicated using a locally risk-minimising strategy.*

Proof Refer to Pham, Rheinländer and Schweizer (1998) Corollary 5. □

To ensure the existence of locally risk-minimising strategies for all square integral random variables, it will be assumed henceforth that:

Assumption 1 *The process $\widehat{K}_t$ is continuous and bounded.*

The existence of locally risk-minimising strategies for square integrable random variables holds under less restrictive conditions, as shown for example in Schweizer (1991). However, for the purposes of this paper the above assumption is not unreasonable.

Recall that $\widehat{\mathbb{P}} \sim \mathbb{P}$ is a *minimal martingale* measure if any square integrable $\mathbb{P}$-martingale that is $\mathbb{P}$-orthogonal to M_t remains a martingale under $\widehat{\mathbb{P}}$. It is well known that the Girsanov density of $\widehat{\mathbb{P}}$ is given by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathscr{E}\left(-\int_0^T \widehat{\lambda}_s' \, dM_s\right), \tag{11}$$

where

$$\widehat{\lambda}_t \triangleq \operatorname{diag}(Z_t^A)^{-1}[\sigma_s^A(\sigma_s^A)']^{-1}(\mu_t^A - r_t \mathbf{1}_n). \tag{12}$$

A useful property of the minimal martingale measure is that if ξ_t a locally risk-minimising strategy for a square integrable $\mathscr{F}_\tau$-measurable random variable H, then

$$\bar{V}_t(\xi) = \widehat{\mathbb{E}}[H|\mathscr{F}_t]. \tag{13}$$

For the details, refer to Schweizer (1991) equation (3.8) and Theorem 3.2, and recall that $H = Z_\tau^I$ for the purposes of this paper.

Definition 1 *Let $(X_t)_{0 \le t \le T}$ be an $(\mathscr{F}_t)$-adapted square integrable process. Then a trading strategy ξ_t will be said to be locally risk-minimising for X_t if for all $\tau \in [0, T]$ the strategy $\xi_{t \wedge \tau}$ is the locally risk-minimising strategy for $X_{t \wedge \tau}$ over $[0, \tau]$.*

The process X_t should be identified with the *discounted* index process Z_t^I. The next simple proposition is the key result for the remainder of the paper.

Proposition 1 *Let X_t be as given above. Then X_t admits a locally risk-minimising strategy if and only if X_t is a $\widehat{\mathbb{P}}$-martingale.*

Proof Suppose X_t admits a locally risk-minimising strategy. Then there exists a strategy $(\xi_t)_{0\leq t\leq T}$ for which $\xi_{t\wedge\tau}$ is locally risk-minimising for $X_{t\wedge\tau}$, for all $\tau \in [0,T]$. For notational convenience, define $\xi_t^\tau = \xi_{t\wedge\tau}$. Then $\widehat{\mathbb{E}}[X_u|\mathscr{F}_t] = \bar{V}_t(\xi^u) = \bar{V}_t(\xi^t) = X_t$ for all $u \geq t$, and so X_t is a $\widehat{\mathbb{P}}$-martingale. Conversely, suppose X_t is a $\widehat{\mathbb{P}}$-martingale. Then since X_T is square integrable by assumption, X_T admits a locally risk-minimising strategy ξ_t. That is, X_T admits the Föllmer-Schweizer decomposition

$$X_T = \widehat{\mathbb{E}}[X_T|\mathscr{F}_0] + \int_0^T \xi_s \cdot dZ_s^A + L_T$$

of the form (8). Since each term in this equation is a $\widehat{P}$-martingale, we obtain by conditioning on $\mathscr{F}_\tau$ the Föllmer-Schweizer decomposition

$$X_\tau = \widehat{\mathbb{E}}[X_\tau|\mathscr{F}_0] + \int_0^\tau \xi_s \cdot dZ_s^A + L_\tau,$$

of X_τ. It follows that ξ_t^τ is a locally risk-minimising strategy for $X_{t\wedge\tau}$. □

Note that this result is valid under more general situations than the present case where the underlying asset price process satisfies (5). Since a locally risk-minimising strategy exists for the process Z_t^I if and only if Z_t^I is a $\widehat{\mathbb{P}}$-martingale by the above theorem, we now turn to obtaining explicit characterisation of the conditions under which Z_t^I is a $\widehat{\mathbb{P}}$-martingale.

Lemma 1 *Let Z_t^I and Z_t^A be as given in (4) and (5). Then Z_t^I is a $\widehat{\mathbb{P}}$-martingale if and only if*

$$\alpha_t \triangleq \mu_t^I - r_t - (\sigma_t^A \sigma_t^I)'[\sigma_t^A(\sigma_t^A)']^{-1}(\mu_t^A - r_t \mathbf{1}_n) \equiv 0, \tag{14}$$

where $\sigma_t^I = (\sigma_t^{I,1}, \sigma_t^{I,2}, \ldots, \sigma_t^{I,N})'$.

Proof Define $\widehat{w}_t = w_t + \int_0^t (\sigma_t^A)' \mathrm{diag}(Z_t^A)\widehat{\lambda}_t \, dt$. Then by (11) and Girsanov theorem, $\widehat{w}_t$ is a standard $\widehat{\mathbb{P}}$-Wiener process, and $dZ_t^I = Z_t^I \alpha_t \, dt + Z_t^I \sigma_t^I \cdot d\widehat{w}_t$. The result now follows since the drift term of Z_t^I vanishes if and only if $\alpha_t \equiv 0$. □

It will emerge later that the process α_t is the instantaneous tracking bias for the locally risk-minimising strategy using assets in $\mathcal{S}(A)$.

Corollary 1 *If Z_t^I and Z_t^A are as given above, then Z_t^I admits a locally risk-minimising strategy if and only if $\alpha_t \equiv 0$.*

The following characterisation of locally risk-minimising strategies is well-known.

Proposition 2 *Let H be a $\mathscr{F}_\tau$-measurable square integrable random variable. Then the locally risk-minimising strategy ξ_t^H for H is given by*

$$\xi_t^H = d\langle M\rangle_t^{-1} d\langle M, \bar{V}(\xi^H)\rangle_t, \tag{15}$$

where $\bar{V}_t(\xi^H) = \widehat{\mathbb{E}}[H|\mathscr{F}_t]$ for $t \in [0,\tau]$.

Proof This follows from considering the Föllmer-Schweizer decomposition (8) for H under $\widehat{\mathbb{P}}$ and projecting $\bar{V}_t(\xi^H)$ on the $\widehat{\mathbb{P}}$-martingale $\int_0^t \xi_s^H \cdot dZ_s^A$. For the details, refer to Föllmer and Schweizer (1991) Theorem 3.14. □

This enables us to determine explicitly the locally risk-minimising strategy for Z_t^I.

Corollary 2 *Suppose* $\alpha_t \equiv 0$. *Then the locally risk-minimising strategy for* Z_t^I *is*

$$\xi_t^A \triangleq Z_t^I \mathrm{diag}(Z_t^A)^{-1} [\sigma_t^A (\sigma_t^A)']^{-1} \sigma_t^A \sigma_t^I. \tag{16}$$

Proof Simple computations give $d\langle M \rangle_t = \mathrm{diag}(Z_t^A)\sigma_t^A(\sigma_t^A)'\mathrm{diag}(Z_t^A)dt$. Next, since $\alpha_t \equiv 0$, we have Z_t^I is a $\widehat{\mathbb{P}}$-martingale by Corollary 1 and so $\bar{V}_t(\xi^A) = Z_t^I$ and

$$d\langle \bar{V}(\xi^A), M \rangle_t = d\langle Z^I, M \rangle_t = Z_t^I \mathrm{diag}(Z_t^A)\sigma_t^A \sigma_t^I \, dt.$$

The result now follows from the previous proposition. □

3 Applications to Index Tracking

Managed funds, and in particular index related funds, have experienced rapid growth over the past decade, with the value of index related funds in the US exceeding $\$1.5$ trillion at the end of year 2000 as reported in Frino and Gallagher (2001). As the name suggests, one of the primary objectives of *index funds* is to replicate the return on a target benchmark index such as the S&P500. One obvious way to achieve this would be to hold *all* the assets in the index and in the same proportion as they appear in the index. However, this is impractical due to transaction costs, losses due to bid-ask spreads and the limited liquidity of smaller stocks. Hence, the funds are forced to choose a subset of assets with which to form their *tracker* portfolios.

Following Roll (1992), the predominant market practice is to select a set of liquid assets, and then periodically adjust the weights in order to minimise the *tracking error variance* or TEV. A brief description of this approach now follows. It is assumed in the subsequent discussion that there are no restrictions on the short sale of assets.

Noting that returns and covariances in practice are estimated from discrete market data, consider the Euler-Maruyama discretisation of the asset price sdes (2) and (3)

$$S_{t+\Delta t}^i \approx S_t^i + S_t^i \mu_t^i \Delta t + S_t^i \sum_{j=1}^n \sigma_t^{i,j} \Delta w_t^j, \tag{17}$$

from which the approximate return, ϱ_t^i, for the asset i over the interval $[t, t+\Delta t]$ can be obtained as

$$\varrho_t^i \triangleq \frac{S_{t+\Delta t}^i - S_t^i}{S_t^i} \approx \mu_t^i \Delta t + \sum_{j=1}^n \sigma_t^{i,j} \Delta w_t^j \tag{18}$$

for $i \in \{1, 2, \dots, N, I\}$. Now, let $A = \{i_1, i_2, \dots, i_n\}$ be defined as above, and let

$$\varsigma_t^{i,j} \triangleq \sigma_t^i \cdot \sigma_t^j \approx \frac{1}{\Delta t} \mathrm{cov}[\varrho_t^i, \varrho_t^j | \mathscr{F}_t]. \tag{19}$$

Furthermore, let $\Phi_t = (\varsigma_t^{i,j})_{i,j \in A \cup \{I\}}$ and define the $n \times n$ and $n \times 1$ submatrices $\Phi_t^{1,1}$ and $\Phi_t^{1,2}$ of Φ_t by

$$\Phi_t = \begin{bmatrix} \Phi_t^{1,1} & \Phi_t^{1,2} \\ (\Phi_t^{1,2})' & \varsigma_t^{I,I} \end{bmatrix}. \tag{20}$$

Given the weights $\pi_t = (\pi_t^{i_1}, \dots, \pi_t^{i_n})'$ for a portfolio constructed from $\mathcal{S}(A)$, define the tracking error, ε_t, over the interval $[t, t + \Delta t]$ by

$$\varepsilon_t = (\varrho_t^I - r_t \Delta t) - \pi_t \cdot (\varrho_t^A - r_t \Delta t \mathbf{1}), \tag{21}$$

where $\varrho_t^A = (\varrho_t^{i_1}, \varrho_t^{i_2}, \dots, \varrho_t^{i_n})'$. Accounting for the money market account, this is the difference in the index return and the corresponding return on the portfolio with weights π_t in $\mathcal{S}(A)$ and $1 - \pi_t \cdot \mathbf{1}_n$ in the money market account.

Given the set A, the standard market practice for selecting the weights π_t for the tracker portfolio is to choose, for each time t, the weights π_t^i such that the (instantaneous) *tracking error variance* (TEV)

$$\sigma_{\varepsilon_t}^2 = \frac{1}{\Delta t} \operatorname{var} \left[(\varrho_t^I - r_t \Delta t) - \pi_t \cdot (\varrho_t^A - r_t \Delta t \mathbf{1}) \right] \tag{22}$$

is minimised. Note that the division by Δt in (22) is for convenience only and does not affect the subsequent results in any way.

Proposition 3 *The weights for the TEV-minimising strategy are given by the vector*

$$\pi_t = (\Phi_t^{1,1})^{-1} \Phi_t^{1,2}.$$

Proof This follows from differentiating (22) with respect to $\pi_t^{i_j}$ and solving the resulting set of linear equations for π_t. □

Having obtained the weights, it is a simple exercise to compute the instantaneous mean and instantaneous variance of the tracking error. In analogy with (22) define the instantaneous mean of the tracking error by

$$\mu_{\varepsilon_t} \triangleq \frac{1}{\Delta t} \mathbb{E}[\varepsilon_t | \mathscr{F}_t]. \tag{23}$$

Proposition 4 *The instantaneous mean and variance of the tracking error for a TEV minimising strategy are*

$$\mu_{\varepsilon_t} = \mu_t^I - r_t - (\Phi_t^{1,2})'(\Phi_t^{1,1})^{-1}(\mu_t^A - r_t \mathbf{1}) = \alpha_t, \tag{24}$$

$$\sigma_{\varepsilon_t}^2 = \varsigma_t^{I,I} - (\Phi_t^{1,2})'(\Phi_t^{1,1})^{-1} \Phi_t^{1,2}, \tag{25}$$

where α_t is as defined in (14).

Now, suppose the index fund maintains the value of the tracker portfolio at the index value S_t^I for all $t \in [0, T]$ by injecting funds if necessary. Then the number of units of each asset is given by

$$\tilde{\xi}_t^A = S_t^I \operatorname{diag}(S_t^A)^{-1} (\Phi_t^{1,1})^{-1} \Phi_t^{1,2} = Z_t^I \operatorname{diag}(Z_t^A)^{-1} (\Phi_t^{1,1})^{-1} \Phi_t^{1,2}. \tag{26}$$

Note that $\tilde{\xi}_t$ is precisely the locally risk-minimising strategy, ξ_t^A, for S_t^I given in (16) if $\alpha_t \equiv 0$. Recalling that a tracker portfolio for an index is said to be *unbiased* if $\mu_{\varepsilon_t} \equiv 0$, this gives the following link between unbiased and locally risk-minimising strategies.

Corollary 3 *TEV-minimising strategy is locally risk-minimising for S_t^I if and only if it is unbiased.*

Proof Firstly, if $\mu_{\varepsilon_t} = \alpha_t \not\equiv 0$ then Z_t^I does not admit a locally risk-minimising strategy by Corollary 1 and so in particular TEV-minimising strategy is not locally minimising for S_t^I. Conversely, if $\mu_{\varepsilon_t} \equiv 0$, then by Corollary 2 the locally risk-minimising strategy for S_t^I coincides with TEV-minimising strategy. □

Hence, the standard market practice of forming the tracker portfolio by minimising the TEV is not optimal in general from the local risk-minimisation point of view. Noting that $\mu_{\varepsilon_t} = \alpha_t$ depends only on the set A, the above results give the following simple criterion for selecting the optimal set of assets from which to form a tracking strategy.

Criterion (Selection of Assets for Index Tracking I) *To ensure local risk-minimisation select a set of assets for which the TEV-minimising strategy is unbiased.*

4 Special Case of Deterministic Coefficients

In this section, we consider the special case where r_t, μ_t^i and $\sigma_t^{i,j}$ are deterministic. For any strategy ξ_t, define the associated discounted *cost* process, $\bar{C}_t(\xi)$, by

$$\bar{C}_t(\xi) = Z_t^I - \bar{V}_0(t) - \int_0^t \xi_s \cdot dZ_s^A, \tag{27}$$

and for a given $\tau \in [t, T]$, consider the remaining cost process, $R_t^\tau(\xi)$, given by

$$R_t^\tau(\xi) = Z_\tau^I - Z_t^I - \int_t^\tau \xi_s \cdot dZ_s^A = \bar{C}_\tau(\xi) - \bar{C}_t(\xi). \tag{28}$$

Note that $R_t^\tau(\xi)$ represents in some sense the cost incurred in ensuring that the value of the strategy equals the value of the process, Z_t^I, being hedged over the interval $[t, \tau]$. For the motivation for, and the discussion of, the need to localise the notion of R_t^τ, refer to Föllmer and Sondermann (1986) and Schweizer (1991).

Now, in determining a measure for the deviation of the TEV-minimising strategy from being locally risk-minimising, the next proposition is needed.

Proposition 5 *If r_t, μ_t^i and $\sigma_t^{i,j}$ are deterministic, then $\widehat{E}[Z_\tau^I|\mathscr{F}_t] = e^{\int_t^\tau \alpha_s ds} Z_t^I$ for all $\tau \in [0, T]$.*

Proof From the proof of Lemma 1, the process $X_t \triangleq e^{-\int_0^t \alpha_s ds} Z_t^I$ is a $\widehat{\mathbb{P}}$-martingale, and so we have $\widehat{\mathbb{E}}[X_\tau|\mathscr{F}_t] = X_t$. That is, $\widehat{\mathbb{E}}[e^{-\int_0^\tau \alpha_s ds} Z_\tau^I|\mathscr{F}_t] = e^{-\int_0^t \alpha_s ds} Z_t^I$. Since $\int_0^\tau \alpha_s ds$ is deterministic, the result follows. □

In this case, it is possible to compute explicitly the mean and the variance of the process $R_t^\tau(\xi)$. As expected, these quantities are intimately connected to the mean and the variance of the tracking error ε_t.

Proposition 6 *Let $0 \le t \le \tau \le T$, and let $R_t^\tau(\xi)$ be as defined in (28). If μ_t^i, $\sigma_t^{i,j}$ and r_t are deterministic, then*

$$\mathbb{E}[R_t^\tau(\xi)|\mathscr{F}_t] = Z_t^I \int_t^\tau \mu_{\varepsilon_u} e^{\int_t^u (\mu_s^I - r_s)\,dt}\,du, \tag{29}$$

$$\mathbb{E}[\langle R^\tau(\xi)\rangle_t|\mathscr{F}_t] = (Z_t^I)^2 \int_t^\tau \sigma_{\varepsilon_u}^2 e^{\int_t^u 2\left(\mu_s^I - r_s + \frac{1}{2}\varsigma_s^{I,I}\right)ds}\,du, \tag{30}$$

where $\mu_{\varepsilon_t} = \alpha_t$ and σ_{ε_t} are as defined in (23) and (22).

Proof Let $\rho_t^2 = \varsigma_t^{I,I}$. Then we have

$$Z_\tau^I = Z_t^I \exp\left[\int_t^\tau \left[(\mu_u^I - r_u) - \frac{1}{2}\rho_u^2\right] du + \int_t^\tau \rho_u\, dw_u^*\right]$$

for all $\tau \in [t, T]$, where $w_t^* = (\sigma_t^I \cdot w_t)/\rho_t$ is a standard 1-dimensional $\mathbb{P}$-Wiener process. Now, substituting for ξ_t in the expression for $R_t^\tau(\xi)$ gives

$$R_t^\tau(\xi) = \int_t^\tau Z_u^I \left[\alpha_u\, du + \left((\sigma_u^I)' - (\Phi_u^{1,2})'(\Phi_u^{1,1})^{-1}\sigma_u^A\right) dw_u\right].$$

Since α_t is deterministic in this case, applying Fubini's theorem gives

$$\mathbb{E}[R_t^\tau(\xi)|\mathscr{F}_t] = \mathbb{E}\left[\int_t^\tau Z_u^I \alpha_u\, du \middle| \mathscr{F}_t\right] = \int_t^\tau \mathbb{E}\left[Z_u^I \middle| \mathscr{F}_t\right] \alpha_u\, du.$$

Substituting the expression for Z_τ^I from above, and noting that μ_t^I and r_t are deterministic and $e^{-\int_0^t \frac{1}{2}\rho_s^2\,ds + \int_0^t \rho_s\, dw_s^*}$ is a $\mathbb{P}$-martingale gives

$$\mathbb{E}[R_t^\tau(\xi)|\mathscr{F}_t] = \int_t^\tau Z_t^I e^{\int_t^u (\mu_s^I - r_s)ds} \cdot 1 \cdot \alpha_u\, du = Z_t^I \int_t^\tau \alpha_u e^{\int_t^u (\mu_s^I - r_s)dt}\, du$$

as required. Similarly,

$$\begin{aligned}\langle R^\tau(\xi)\rangle_t &= \int_t^\tau (Z_u^I)^2 \left((\sigma_u^I)' - (\Phi_u^{1,2})'(\Phi_u^{1,1})^{-1}\sigma_u^A\right)\left(\sigma_u^I - (\sigma_u^A)'(\Phi_u^{1,1})^{-1}\Phi_u^{1,2}\right) du \\ &= \int_t^\tau (Z_u^I)^2 \left(\varsigma_u^{I,I} - (\Phi_u^{1,2})'(\Phi_u^{1,1})^{-1}\Phi_u^{1,2}\right) du,\end{aligned}$$

and so applying the above arguments gives

$$\mathbb{E}[\langle R^\tau(\xi)\rangle_t|\mathscr{F}_t] = \int_t^\tau \left(\varsigma_u^{I,I} - (\Phi_u^{1,2})'(\Phi_u^{1,1})^{-1}\Phi_u^{1,2}\right) \mathbb{E}[(Z_u^I)^2|\mathscr{F}_t]\, du.$$

Since $e^{-\int_0^t \frac{1}{2}(2\rho_s)^2\,ds + \int_0^t 2\sigma_s^I\, dw_s^*}$ is a $\mathbb{P}$-martingale, we have

$$\begin{aligned}\mathbb{E}[(Z_u^I)^2|\mathscr{F}_t] &= (Z_t^I)^2 e^{\int_t^u 2\left(\mu_s^I - r_s + \frac{1}{2}\rho_s^2\right)ds}\, \mathbb{E}\left[e^{-\int_t^u \frac{1}{2}(2\rho_s)^2\,ds + \int_t^u 2\rho_s\, dw_s^*} \middle| \mathscr{F}_t\right] \\ &= (Z_t^I)^2 e^{\int_t^u 2\left(\mu_s^I - r_s + \frac{1}{2}\rho_s^2\right)ds},\end{aligned}$$

and so the second identity follows. □

In the special case where the coefficients are constant, the integrals can be computed explicitly.

Corollary 4 *If $\mu_t^i \equiv \mu^i$, $\sigma_t^{i,j} \equiv \sigma^{i,j}$ and $r_t \equiv r$ are constant, then*

$$\mathbb{E}[R_\tau^\tau(\xi)|\mathscr{F}_t] = \frac{Z_t^I \mu_\epsilon}{\mu^I - r}\left[e^{(\mu^I - r)(\tau - t)} - 1\right], \tag{31}$$

$$\mathbb{E}[\langle R^\tau(\xi)\rangle_\tau|\mathscr{F}_t] = \frac{(Z_t^I)^2\sigma_\varepsilon^2}{2(\mu^I - r + \frac{1}{2}\varsigma^{I,I})}\left[e^{2(\mu^I - r + \frac{1}{2}\varsigma^{I,I})(\tau - t)} - 1\right]. \tag{32}$$

Interpreting $R_t^\tau(\xi)$ as the remaining discounted cost for tracking the index using the strategy ξ_t, which may not necessarily be locally risk-minimising, the above formulae give the mean and variance of this remaining cost. Consequently, in analogy with the notion of *value-at-risk* we may define the *cost-at-risk, κ_t^τ*, in the case of constant coefficients by

$$\kappa_t^\tau = Z_t^I\left[\mu_\varepsilon\left(\frac{e^{(\mu^I - r)(\tau - t)} - 1}{\mu^I - r}\right) + \sigma_\varepsilon\sqrt{\frac{e^{2(\mu^I - r + \frac{1}{2}\varsigma^{I,I})(\tau - t)} - 1}{2(\mu^I - r + \frac{1}{2}\varsigma^{I,I})}}\right], \tag{33}$$

which is the remaining cost that is one standard deviation from the mean. Since κ_t^τ is completely determined by the properties of the assets in $\mathcal{S}(A)$ and the index, this leads to the following improved criterion for the optimal set of assets for forming tracker portfolios.

Criterion (Selection of Assets for Index Tracking II) *Suppose r_t, μ_t^i and $\sigma_t^{i,j}$ are constant. Then for a minimal cost-at-risk strategy over the interval $[t, \tau]$, select assets for which κ_t^τ is minimal.*

An illustration of κ_t^τ as a function of μ_ε and σ_ε is shown in Figure 1. Although the surface is linear as expected, an interesting point is that κ_t^τ may take negative values.

Next, indicative values for μ_ε, σ_ε^2 and κ_t^τ for all possible 9 asset portfolios that can be constructed from the 10 largest stocks in the S&P500 index are given in Table 2. The daily closing prices[1] for the GE, MSFT, WMT, XOM, PFE, C, INTC, BP, AIG and JNJ and the S&P500 index over the period January 2, 2003 to June 30, 2003 were used to estimate μ^i and $\varsigma^{i,j}$. It is worth noting that the difference between the maximum and minimum values of κ_t^τ reported in the table is approximately 1.89% of the initial value of the portfolio. Since this difference grows exponentially as a function of $\tau - t$, it represents an important quantity for consideration in selecting the assets for index tracking.

Note that since $\mu_\varepsilon \neq 0$ for the set of assets considered in the table, it is *not* sufficient to consider only the variance of the remaining cost $R_t^\tau(\xi)$. This was the motivation for introducing the cost-at-risk measure κ_t^τ. If only the variance of $R_t^\tau(\xi)$ was considered, then one would incorrectly consider the portfolio in which XOM was excluded as being optimal.

[1] These were obtained from YAHOO! finance.

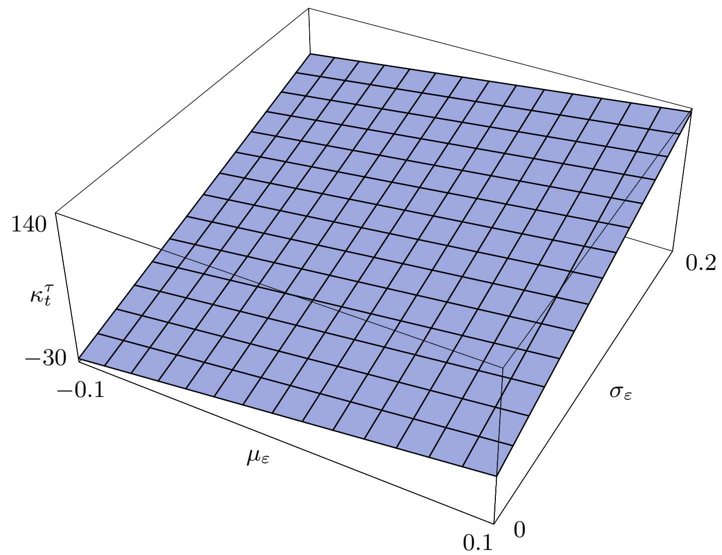

Fig. 1 The cost-at-risk, κ_t^τ, as a function of μ_ε and σ_ε, with $\mu^I = 10\%$, $r = 5\%$, $\sqrt{\varsigma^{I,I}} = 20\%$, $Z_t^I = 1,000$ and $\tau - t = 0.25$.

Omitted Stock	μ_ε	σ_ε^2	κ_t^τ
GE	0.0555	0.00268	15.1213
MSFT	0.0206	0.00295	6.5987
WMT	0.0369	0.00289	10.6025
XOM	0.0424	0.00265	11.8532
PFE	0.0635	0.00300	17.2814
C	0.0755	0.00299	20.2473
INTC	0.0645	0.00288	17.4584
BP	0.0521	0.00279	14.3233
AIG	0.0014	0.00300	1.8406
JNJ	0.0375	0.00269	10.6634

Table 2 Indicative values of κ_t^τ. The risk free rate was set to 2% for convenience and we used $\tau - t = 0.25$. The value of the S&P500 index on June 30, 2003 was used as the initial value so that $Z_t^n = 974.51$.

5 Conclusion

This paper considered the problem of hedging diffusion processes by local risk-minimisiation and determined the necessary and sufficient conditions under which this is possible. The results were applied to the problem of index tracking to establish close connections between the notions of local risk-minimisation and tracking error variance minimisation, with the latter being the market preferred method for determining the portfolio weights for index tracking. By exploiting the connections, simple criteria for the selection of optimal set of assets for index tracking were obtained, and in the case of constant coefficients a closed form expression for a value-at-risk type measure was obtained for evaluating a given set of assets for the purposes of index tracking.

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