QUANTITATIVE FINANCE
RESEARCH CENTRE

UNIVERSITY OF
TECHNOLOGY SYDNEY

QUANTITATIVE FINANCE
RESEARCH CENTRE

UTS

THINK.CHANGE.DO

QUANTITATIVE FINANCE RESEARCH CENTRE

Research Paper 118 **July 2004**

A Survey of the Integral Representation of American Option Prices

Carl Chiarella and Andrew Ziogas

ISSN 1441-8010 www.qfrc.uts.edu.au

A SURVEY OF THE INTEGRAL REPRESENTATION OF AMERICAN OPTION PRICES

CARL CHIARELLA*, ADAM KUCERA* AND ANDREW ZIOGAS†

Abstract. This paper surveys some of the literature on American option pricing, in particular the representations of McKean (1965), Kim (1990), Jamshidian (1992) and Carr, Jarrow & Myneni (1992). In particular the paper seeks to demonstrate that the approach regarding the problem as a free boundary value problem, and its solution via incomplete Fourier transforms, is the most robust for further theoretical and applied developments involving more complex payoff structures, and higher dimensional problems such as multi-asset American options. Some comparison of different numerical solution methods is also provided.

Keywords: American options, Volterra integral equation, incomplete Fourier transform, free-boundary problem.
JEL Classification: C61, D11.

Date: Current Version October 5, 2005.

* carl.chiarella@uts.edu.au; School of Finance and Economics, University of Technology, Sydney, and Graduate School of Economics, University of Kyoto.
* Integral Energy, Sydney, Australia.
† Corresponding author: andrew.ziogas@uts.edu.au; School of Finance and Economics, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007, Australia.

1. Introduction

The evaluation of American options under the geometric Brownian motion framework of Black & Scholes (1973) and Merton (1973) has been investigated using a vast array of analytical techniques and numerical methods. Unlike European options, American options can be exercised at any time during the term of the derivative contract. As a result there exists some value of the underlying asset at which it is optimal to exercise the option, and this value varies with time to maturity. There are numerous ways this optimal stopping problem can be explored within the Black-Scholes-Merton framework. In an appendix to Samuelson (1965) on the pricing of warrants, McKean (1965) (who seems to have been the first to consider the problem) treats the American call as a free boundary value problem for the Black-Scholes partial differential equation (PDE). In order to cope with the free (early exercise) boundary he uses an incomplete Fourier transform, and obtains an integral expression for the American call price that involves the free boundary. Evaluating this expression at the early exercise boundary produces an integral equation for the free boundary. Although applied to the American call case, in principle this approach should be applicable to any general payoff function.

Parkinson (1977) considers the American put problem by taking series expansions of the solution in transform-space. For numerical implementation, he uses a binomial approximation of the continuous log-normal density for the stock price process. By assuming that the option can only be exercised at a discrete number of time points, Geske & Johnson (1984) treat the American put as a Bermudan option. They solve the problem using compound options and offer the method as a discrete approximation for continuous American put prices. Kim (1990) takes the limit of the Geske-Johnson solution and shows that it yields an integral expression for the American put price, along with an integral equation for its free boundary, however this representation differs from that of McKean (1965). A similar result is found for the American call in the case where the underlying asset pays a continuous dividend yield. In both the call and put examples, the compound option method relies on explicit knowledge of the option payoff in order to proceed. Kim (1990) also shows that the limit of the Geske-Johnson method is equivalent to McKean's solution, as he converts McKean's integral equations into his own representation. Furthermore, Kim (1990) gives a clear economic interpretation to his solution for the American call and put, something that is extremely difficult to deduce from McKean's representation of the solution.

Jacka (1991) shows that the American put is equivalent to an optimal stopping problem. He confirms Kim's results, and adds to the earlier literature by showing that the solution is unique. Carr et al. (1992) contribute to the American put analysis by obtaining an alternative representation. They decompose the American put into its intrinsic value and time value components, and provide lower and upper bounds for the American put solution. Jamshidian (1992) demonstrates that solving the homogeneous PDE in a restricted domain for American calls and puts is equivalent to solving an inhomogeneous PDE in an unrestricted domain.

Establishing a suitable hedging argument, Karatzas (1988) develops an expectation operator expression for the fair price of American contingent claims under geometric Brownian motion. In this way he finds the Snell envelope representation for the American put. Jaillet, Lamberton & Lapeyre (1990) use variational inequalities to evaluate American options. Little, Pant & Hou (2000) derive an alternative representation for the free boundary of an American put. They take advantage of information about the stopping region to find an integral equation that is faster to evaluate numerically.

This paper seeks to consolidate and give some perspective on some of these various contributions. In particular, we shall focus this survey on the results of McKean (1965), Kim (1990), Jamshidian (1992) and Carr et al. (1992). Taking the free boundary value problem approach, we use McKean's Fourier transform method to solve the Black-Scholes PDE for an American contingent claim with a general monotonic payoff function. In the process, we derive all the relevant properties of the incomplete Fourier transform. We demonstrate how to go from McKean's representation to that of Kim, and vice versa. We also demonstrate how Jamshidian's representation of the free boundary value problem can be solved using Fourier transforms to rapidly derive Kim's integral equations. The Carr-Jarrow-Myneni representation is also reproduced, and we consider the economic interpretations of the various representations. We draw out the fact that certain methods can cater for payoffs of a fairly general form whilst others (e.g. Kim (1990)) are tied rather strongly to the particular form of the payoff being considered.

A brief comparison of numerical methods for calculating the American option price is provided, along with free boundary estimates where applicable. The numerical solutions presented include numerical integration of Kim's integral equations (Kallast & Kivinukk 2003), the method of lines (Meyer & van der Hoek 1997), finite differences (Brennan & Schwartz 1977) and binomial trees (Cox, Ross & Rubinstein 1977). Comprehensive studies comparing various numerical methods for American option pricing have also been conducted by AitSahlia & Carr (1997), and Broadie & Detemple (1996).

The remainder of this paper is structured as follows. Section 2 details the American option pricing problem under consideration. Section 3 proceeds to solve this problem using the incomplete Fourier transform method, and we invert the transform of the solution in Section 4. In Section 5 we outline an alternative representation of the American option pricing problem due to Jamshidian (1992). We then consider the specific example of an American call option in Section 6, detailing the various ways in which the integral equations can be represented. Section 7 uses the compound option solution method as applied to an American call, and Section 8 contains a comparison of the price and, where appropriate, early exercise boundary, as found using a range of numerical techniques. Conclusions follow in Section 9, with the various mathematical proofs provided in appendices.

2. Problem Statement - Monotonic Payoff

Let $C_b(S,\tau)$ be the price of an American option written on an underlying asset with price S at time τ prior to maturity, written at time $\tau = T$. The underlying pays a continuously compounded dividend at the rate q. Let the payoff function for the option be given by $c(S)$. We assume that $c(S)$ is a non-negative, monotonic increasing function of S (strictly so for all S such that $c(S) > 0$), and that $c(S) \to 0$ as $S \to 0$. The early exercise boundary for this American option is denoted by $b(\tau)$. Figure 1 demonstrates the payoff and continuation region for $C_b(S,\tau)$ in the case where C_b is an American call option, for which $c(S) = \max(S - K, 0)$.

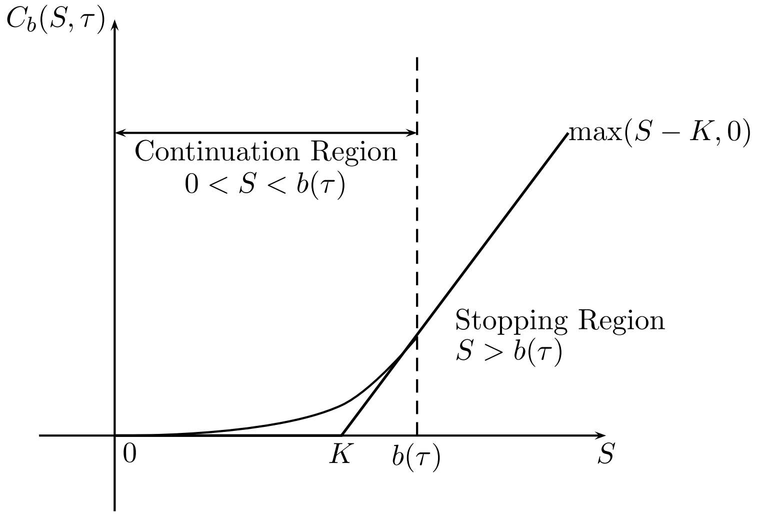


FIGURE 1. Continuation region in S-space for an American call option

Under the assumption that the price, S, of the underlying asset is driven by the geometric Brownian motion process

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

where t denotes the current time[1], with drift μ, volatility σ and Wiener process increments dW, it is known (for example using the standard hedging argument) that C_b satisfies the Black-Scholes PDE

$$\frac{\partial C_b}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_b}{\partial S^2} + (r-q)S\frac{\partial C_b}{\partial S} - rC_b, \quad 0 \le \tau \le T, \tag{2}$$

in the region $0 < S < b(\tau)$, where r is the risk-free rate, subject to the following initial and boundary conditions:

$$C_b(S,0) = c(S), \quad 0 < S < \infty, \tag{3}$$

$$C_b(0,\tau) = 0, \quad \tau \ge 0, \tag{4}$$

$$C_b(b(\tau),\tau) = c(b(\tau)), \quad \tau \ge 0, \tag{5}$$

$$\lim_{S\to b(\tau)} \frac{\partial C_b}{\partial S} = \lim_{S\to b(\tau)} \frac{dc(S)}{dS} = c'(b(t)), \quad \tau \ge 0. \tag{6}$$

---

[1] Note that $\tau = T - t$.

Condition (3) is the payoff function for the option at expiry, while condition (5) is referred to as the "value matching" condition, and it ensures that the option price, $C_b(S,\tau)$, is continuous with respect to S. This is necessary to maintain the Black-Scholes assumption of an arbitrage-free market. Equation (6) is known as "smooth-pasting" condition. This condition is imposed to maximise the value of the American option[2]. A further consequence of condition (6) is that the delta of the American option will be continuous at the free boundary $b(\tau)$. Note that the continuity of delta at the free boundary only applies for the class of payoff functions $c(S)$ under consideration, and the given price process (1). For more on the justification of the smooth pasting condition, we refer the reader to Peskir (1998).

It is convenient to first transform the PDE (2) to an equation with constant coefficients. Setting $S = e^x$, we define the transformed function V_b by

$$C_b(S,\tau) = C_b(e^x,\tau) \equiv V_b(x,\tau). \tag{7}$$

The transformed PDE for V_b is then

$$\frac{\partial V_b}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V_b}{\partial x^2} + k\frac{\partial V_b}{\partial x} - rV_b, \quad 0 \le \tau \le T, \tag{8}$$

to be solved in the region $-\infty < x < \ln b(\tau)$ where $k = r - q - \frac{1}{2}\sigma^2$. The payoff is now $v(x) \equiv c(e^x)$, and the transformed initial and boundary conditions are

$$\begin{aligned}
V_b(x,0) &= v(x), \ -\infty < x < \infty, && (9)\\
\lim_{x\to-\infty} V_b(x,\tau) &= 0, \quad \tau \ge 0, && (10)\\
V_b(\ln b(\tau),\tau) &= v(\ln b(\tau)), \quad \tau \ge 0, && (11)\\
\lim_{x\to\ln b(\tau)} \frac{\partial V_b}{\partial x} &= \lim_{x\to\ln b(\tau)} \frac{dv(x)}{dx} = v'(\ln b(\tau)), \quad \tau \ge 0. && (12)
\end{aligned}$$

In what follows, we will use the notation $b \equiv b(\tau)$ for simplicity, unless there is a particular reason to highlight the maturity dependence of b.

---

[2] It can be readily shown that condition (6) maximises C_b for the perpetual American call (i.e. where $T \to \infty$). We demonstrate this result in Section 6.5.

In order to be able to apply integral transform methods to solve this PDE for $V_b(x, \tau)$, the x-domain shall be extended to $-\infty < x < \infty$ by expressing the PDE as

$$H(\ln b - x)\left(\frac{\partial V_b}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V_b}{\partial x^2} - k\frac{\partial V_b}{\partial x} + rV_b\right) = 0, \tag{13}$$

where $H(x)$ is the Heaviside step function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \tag{14}$$

The reason for the appearance of the factor of $\frac{1}{2}$ at the point of discontinuity is explained below. The initial and boundary conditions remain unchanged.

3. Applying the Fourier Transform

We propose to solve the free boundary value problem defined by equations (8)-(12) by using the Fourier transform technique to reduce the PDE to an ordinary differential equation (ODE), whose solution is readily obtainable. This is the same method used by McKean (1965). Given that the payoff function for the option is well-known to have a "binding" influence on the price and sensitivities of the associated option contract, we can safely assume the function V_b and its first two derivatives with respect to x can be treated as zero when $x \to -\infty$. This assumption is subsequently justified by virtue of the fact that the general payoff function under consideration will have both a delta and gamma of zero for some large negative value of x, given the option price behaviour specified in (10). Further justification arises in that the solution obtained satisfies the PDE and associated boundary conditions, and that the solution is unique[3].

Since the x-domain is now $-\infty < x < \infty$, the Fourier transform can be applied to the PDE. Using i to denote the complex number $\sqrt{-1}$, the Fourier transform of V_b, $\mathcal{F}\{V_b(x, \tau)\}$, is defined as

$$\mathcal{F}\{V_b(x, \tau)\} = \int_{-\infty}^{\infty} e^{i\eta x} V_b(x, \tau) dx. \tag{15}$$

---

[3] This is a standard procedure in the solution of PDEs by integral transform methods; see for example Debnath (1995).

Thus applying the Fourier transform to the PDE (13) we obtain

$$\begin{aligned}\mathcal{F}\{H(\ln b - x)\frac{\partial V_b}{\partial \tau}\} &= \frac{1}{2}\sigma^2 \mathcal{F}\{H(\ln b - x)\frac{\partial^2 V_b}{\partial x^2}\} \\ &\quad + k\mathcal{F}\{H(\ln b - x)\frac{\partial V_b}{\partial x}\} - r\mathcal{F}\{H(\ln b - x)V_b\}.\end{aligned}$$

By definition

$$\begin{aligned}\mathcal{F}\{H(\ln b - x)V_b(x,\tau)\} &= \int_{-\infty}^{\infty} e^{i\eta x} H(\ln b - x)V_b(x,\tau)dx \\ &= \int_{-\infty}^{\ln b} e^{i\eta x} V_b(x,\tau)dx \\ &\equiv \mathcal{F}^b\{V_b(x,\tau)\} \equiv \hat{V}_b(\eta,\tau), \qquad (16)\end{aligned}$$

where for convenience we introduce the notation $\hat{V}_b(\eta,\tau)$ to also denote the transform. We note that, $\mathcal{F}^b$ is an incomplete Fourier transform, since it is equivalent to a standard Fourier transform applied to $V_b(x,\tau)$ in the x-domain of $-\infty < x < \ln b$. The inversion formula for this incomplete Fourier transform is given in Proposition 3.1.

Proposition 3.1. *The inverse of the Fourier transform of the function* $H(a-x)g(x,\tau)$, *with* $a \equiv a(\tau)$, *is given by*

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(x,\tau)e^{i\eta x}dx\right]e^{-i\eta x}d\eta, \quad -\infty < x < a. \qquad (17)$$

Proof: Refer to Appendix 2.

□

Equation (17) of Proposition 3.1 provides the basis for the inversion of the incomplete Fourier transform $\mathcal{F}^b$. Three specific properties of $\mathcal{F}^b$, given in Proposition 3.2, will allow us to convert equation (13) into an ODE for $\mathcal{F}^b$.

Proposition 3.2. *Given the definition of* $\mathcal{F}^b$ *in equation (16), the following identities exist for* $\mathcal{F}^b$*:*

$$\begin{aligned}
\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} &= v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b; && (18)\\
\mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} &= e^{i\eta\ln b}(v'(\ln b) - i\eta v(\ln b)) - \eta^2\hat{V}_b; && (19)\\
\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} &= \frac{\partial\hat{V}_b}{\partial\tau} - \frac{b'}{b}e^{i\eta\ln b}v(\ln b), && (20)
\end{aligned}$$

where $b' \equiv db(\tau)/d\tau$.

Proof: Refer to Appendix A3.1.

□

Note that in deriving the above results, we make use of the so-called "value matching" and "smooth-pasting" conditions given in equations (11)-(12). We also must assume that V_b, $\partial V_b/\partial x$ and $\partial^2 V_b/\partial x^2$ all tend to zero as $x \to -\infty$, which is valid for the class of payoffs under consideration. Applying the results of Proposition 3.2 to equation (13) we have:

Proposition 3.3. *The incomplete Fourier transform of the PDE (13) with respect to* x *satisfies the ODE*

$$\frac{d\hat{V}_b}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V}_b = F(\eta,\tau), \tag{21}$$

where

$$F(\eta,\tau) = e^{i\eta\ln b}\left[\frac{\sigma^2 v'(\ln b)}{2} + \left(\frac{b'}{b} - \frac{\sigma^2 i\eta}{2} + k\right)v(\ln b)\right]. \tag{22}$$

The initial condition

$$\mathcal{F}\{V_b(x,0)\} \equiv \hat{V}_b(\eta,0)$$

may be calculated from equation (9).

Proof: Refer to Appendix A3.2.

□

Instead of solving a PDE for the function $V_b(x,\tau)$, we are now faced with the simpler task of solving the ODE (21) for the function $\hat{V}_b(\eta,\tau)$. This can then be inverted via the Fourier inversion theorem as stated in Proposition 3.1 (see Appendix 2) to recover the desired function $V_b(x, \tau)$. Before concluding this section, we obtain the solution to (21).

Proposition 3.4. *The solution $\hat{V}_b(\eta, \tau)$ to the ODE (21) in Proposition 3.3 is given by*

$$\hat{V}_b(\eta, \tau) = \hat{V}_b(\eta, 0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau} + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)(\tau - s)}F(\eta, s)ds. \tag{23}$$

Proof: Recalling that b is a function of τ, the ODE (21) is of the form

$$\frac{d\hat{V}_b}{d\tau} + \alpha(\eta)\hat{V}_b = F(\eta, \tau),$$

where

$$\alpha(\eta) \equiv \frac{1}{2}\sigma^2\eta^2 + ki\eta + r.$$

Using the integrating factor $e^{\alpha(\eta)\tau}$, the solution to the ODE may be expressed as

$$\hat{V}_b(\eta, \tau)e^{\alpha(\eta)\tau} - \hat{V}_b(\eta, 0) = \int_0^\tau F(\eta, s)e^{\alpha(\eta)s}ds,$$

which is readily reduced to equation (23).

□

4. Inverting the Fourier Transform

Having found $\hat{V}_b(\eta, \tau)$, the next step is to recover $V_b(x, \tau)$, the American option price in the x-τ plane. Taking the inverse (complete) Fourier transform of (23) gives

$$\begin{aligned} H(\ln b - x)V_b(x, \tau) &= \mathcal{F}^{-1}\{\hat{V}_b(\eta, 0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}\} \\ &\quad + \mathcal{F}^{-1}\left\{\int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)(\tau - s)}F(\eta, s)ds\right\}. \end{aligned}$$

Applying the definition of the Heaviside function, this last equation may be expressed as

$$V_b(x, \tau) \equiv V_b^{(1)}(x, \tau) + V_b^{(2)}(x, \tau), \quad -\infty < x < \ln b(\tau). \tag{24}$$

We now determine explicit expressions for $V_b^{(1)}(x, \tau)$ and $V_b^{(2)}(x, \tau)$.

Proposition 4.1. *The function* $V_b^{(1)}(x,\tau)$ *of the representation in equation (24) is given by*

$$V_b^{(1)}(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln b(0^+)} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} v(u)du. \tag{25}$$

Proof: Refer to Appendix A4.1.

□

The proof of Proposition 4.1 follows from a standard application of the convolution theorem for Fourier transforms.

Proposition 4.2. *The function* $V_b^{(2)}(x,\tau)$ *of the representation in equation (24) is given by*

$$V_b^{(2)}(x,\tau) = \int_0^{\tau} \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}}[e^{-h(x,s)}Q(x,s)]ds, \tag{26}$$

where

$$h(x,s) = \frac{(x-\ln b(s)+k(\tau-s))^2}{2\sigma^2(\tau-s)}, \tag{27}$$

and

$$Q(x,s) = \frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + \frac{1}{2}\left[k - \frac{(x-\ln b(s))}{(\tau-s)}\right]\right) v(\ln b(s)) \tag{28}$$

for $-\infty < x < \ln b(\tau)$.

Proof: Refer to Appendix A4.2.

□

To arrive at Proposition 4.2 we apply the inverse transform directly, which subsequently involves evaluating integrals of the exponential of a quadratic function.

With the values of $V_b^{(1)}(x,\tau)$ and $V_b^{(2)}(x,\tau)$ given by Propositions 4.1 and 4.2, we may use equation (24) to write the value of the American option in the x-τ plane as

$$V_b(x,\tau) = V_b^{(1)}(x,\tau) + V_b^{(2)}(x,\tau), \tag{29}$$

for $0 \leq \tau \leq T$ and $-\infty < x < \ln b(\tau)$. Equation (29) expresses the value of the American option in terms of the early exercise boundary, $b(\tau)$. At present this remains unknown, but we are able to obtain an integral equation that determines the free boundary by requiring the expression for $V_b(x,\tau)$ to satisfy the early exercise boundary condition (11). Recalling our definition for the Heaviside function, the free boundary is thus found to satisfy the integral equation

$$\frac{v(\ln b(\tau))}{2} = V_b(\ln b(\tau), \tau), \tag{30}$$

where $V_b(x,\tau)$ is given by equation (29) in conjunction with (25)-(28). The factor of $\frac{1}{2}$ appears in the left hand side of (30) due to properties of the Fourier transform. Recall that the complete Fourier transform was applied to a discontinuous function of the form $H(\ln b - x)f(x,\tau)$. As proved in Dettman (1965), the inverted Fourier transform of a discontinuous function will converge to the midpoint of the discontinuity, as illustrated in Figure 2 for the American call option example. Thus in equation (30), when V_b is evaluated at $\ln b(\tau)$, the factor of $\frac{1}{2}$ must be introduced into the left hand side. This is accounted for by our Heaviside function definition in equation (14).

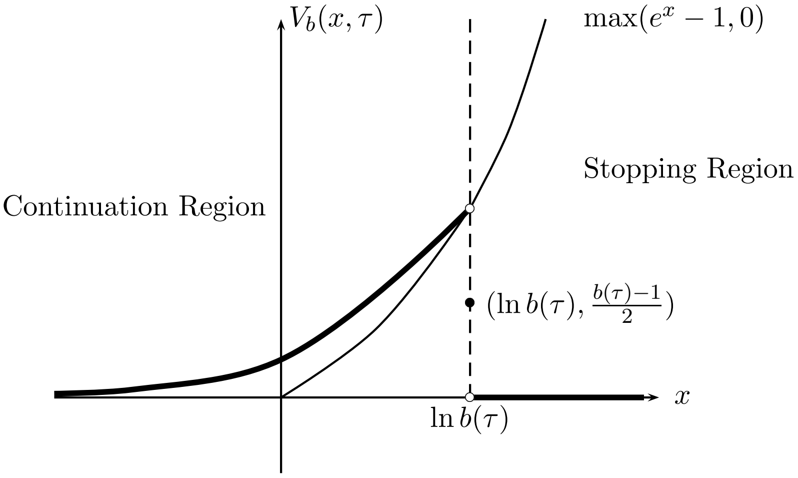


FIGURE 2. Behaviour of $V_b(x,\tau)$ (shown in bold) at $x = \ln b(\tau)$ in the case of an American call option.

It should also be noted that by using the Fourier transform method, we have been able to derive equations (29)-(30) without specifying the exact form of the payoff $v(x)$, beyond a few basic properties. Such generality cannot be easily attained when using Kim's (1990) approach based on compound options, and demonstrates one of the significant advantages obtained from using integral transform solution techniques. Thus to price an American option with monotonic payoff $v(x)$, one must first solve the integral equation (30) using numerical methods to find $b(\tau)$ since an analytical solution seems impossible. Once this is found, it is a simple matter to evaluate $V_b(x, \tau)$ from equation (29) via numerical integration.

One comment that should be made about the solution (29) is that it seems to lack any obvious economic interpretation. This will turn out to not be the case for some of the alternative representations to be discussed below.

5. Jamshidian's Representation and Solution - Monotonic Payoff

The free boundary value problem given by (2)-(6) involves a homogeneous PDE for the American option price C_b, to be solved in the restricted asset-price domain of $0 < S < b(\tau)$. In Section 4 we have shown how McKean's incomplete Fourier transform can be applied to solve this problem, and the solution is given by equation (29) in conjunction with propositions 4.1 and 4.2. While it is clear how one can extract the European component of the option price by manipulating equation (25), it is difficult to interpret the economic meaning of the early exercise premium components that arise from both (25) and (26).

For the class of payoffs under consideration, there is an alternative representation for the free boundary value problem (2)-(6) that was first demonstrated by Jamshidian (1992). By evaluating the PDE (2) in the stopping region, Jamshidian transformed the homogeneous PDE, to be solved in the restricted asset-price domain of $0 < S < b(\tau)$, into an inhomogeneous PDE to be solved in the unrestricted asset-price domain of $0 < S < \infty$. Here we extend his results to the general class of monotonic payoff functions $c(S)$ from Section 2. We begin by deriving the inhomogeneous PDE for C_b.

Proposition 5.1. *Solving the homogeneous PDE (2) for C_b in the domain $0 < S < b(\tau)$ subject to the initial and boundary conditions (3)-(6) is equivalent to solving the* *inhomogeneous PDE*

$$\begin{aligned}\frac{\partial C_b}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_b}{\partial S^2} + (r-q)S\frac{\partial C_b}{\partial S} - rC_b \\ &\quad - H(S-b(\tau))\left(\frac{1}{2}\sigma^2 S^2 \frac{d^2 c(S)}{dS^2} + (r-q)S\frac{dc(S)}{dS} - rc(S)\right), \quad 0 \le \tau \le T, \end{aligned} \tag{31}$$

in the region $0 < S < \infty$, *where* $H(x)$ *is the Heaviside step function defined by (14), subject to the initial condition* $C_b(S,0) = c(S)$.

In terms of the transformed variables x *and* τ, *the inhomogeneous PDE becomes*

$$\begin{aligned}\frac{\partial V_b}{\partial \tau} =&\frac{1}{2}\sigma^2 \frac{\partial^2 V_b}{\partial x^2} + k\frac{\partial V_b}{\partial x} - rV_b \\ &- H(x-\ln b)\left(\frac{1}{2}\sigma^2 \frac{d^2 v(x)}{dx^2} + k\frac{dv(x)}{dx} - rv(x)\right), \quad 0 \le \tau \le T, \end{aligned} \tag{32}$$

to be solved in the region $-\infty < x < \infty$, *subject to the initial condition* $V_b(x,0) = v(x)$.

Proof: Refer to Appendix A5.1.

□

It should be noted that this technique of transforming a homogeneous PDE into an inhomogeneous PDE seems only useful in the case where the value matching and smooth pasting conditions are valid.[4] For example, this approach cannot be used to establish a valid inhomogeneous PDE for an American cash binary option (e.g. option paying one dollar for S above/below a given strike price K, and paying zero otherwise). Whenever the smooth pasting conditions fail, so too does Jamshidian's transformation. Likewise, McKean's method fails to produce a result in the absence of smooth pasting and value matching as well. In these cases one must instead use the method of images, a proven technique in the valuation of barrier options (see for example Wilmott, Dewynne & Howison (1993) and Buchen (2004)).

---

[4] The reasons for this need to be investigated further. It probably has something to do with the amount of information one is given about the sought function at the boundary. Jamshidian (1992) notes that the continuous differentiability of $C(S,\tau)$ across the boundary ensures that the inhomogeneous PDE does not involve the Dirac delta function, $\delta(S-b(\tau))$. Jamshidian provides no proof for this. For European barrier options the boundary conditions are not sufficient for one to determine the value of the option delta at the barrier in advance. In the American option case, choosing an exercise strategy that maximises the value of the option imposes that the first derivative of $C(S,\tau)$ be continuous at $S = b(\tau)$.

An economic interpretation of the inhomogeneous term in (31) and (32) is somewhat difficult when the payoff function remains unspecified. In Section 6 we consider the specific example of an American call, and it becomes clear that the inhomogeneous term represents the net cash flows incurred by an investor holding the portfolio $c(S)$ in the stopping region $S > b(\tau)$.

Given that the solution to the homogeneous PDE (8) is well known, one can then use Duhamel's principle[5] to find the solution to the inhomogeneous PDE. Thus the inhomogeneous PDE (32) is relatively easier to solve than our original free boundary value problem, given that the Black-Scholes-Merton solution for European options is already well known.

Proposition 5.2. *Using the Fourier transform method from sections 3 and 4, the solution to 32 subject to $V(x,0) = v(x)$ is given by*

$$\begin{aligned} V_b(x,\tau) = & \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} v(u) e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} du \\ & + \int_0^{\tau} \int_{\ln b(s)}^{\infty} \left[rv(u) - k\frac{dv(u)}{du} - \frac{1}{2}\sigma^2 \frac{d^2v(u)}{du^2} \right] \\ & \times \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{(x-u+k(\tau-s))^2}{2\sigma^2(\tau-s)}} du ds. \end{aligned} \tag{33}$$

Furthermore, the free boundary $b(\tau)$ is the solution to the Volterra integral equation

$$\begin{aligned} b(\tau) - K = & \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} v(u) e^{-\frac{(\ln b(\tau)-u+k\tau)^2}{2\sigma^2\tau}} du \\ & + \int_0^{\tau} \int_{\ln b(s)}^{\infty} \left[rv(u) - k\frac{dv(u)}{du} - \frac{1}{2}\sigma^2 \frac{d^2v(u)}{du^2} \right] \\ & \times \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}} e^{-\frac{(\ln b(\tau)-u+k(\tau-s))^2}{2\sigma^2(\tau-s)}} du ds. \end{aligned} \tag{34}$$

Proof: Refer to Appendix A5.2.

□

By the theorems of existence and uniqueness, McKean's solution, given by (29), (25) and (26), and Jamshidian's solution (33) are equivalent. It remains unclear how this

---

[5] See for example Logan (2004).

equivalence between the two forms of the solution can be proven in the case of general payoff functions, although in Appendix A6.2 we are able to prove the equivalence in the case of an American call option.

The first integral term in (33) represents the price of a European call option with payoff $v(x)$. Thus the second integral term can be interpreted as the early exercise premium for V_b. Unlike equation (26), it is easier to interpret the nature of the integral terms for the early exercise premium in (33). The kernel is readily compared with that for the European component, allowing us to interpret the early exercise premium as a discounted expected value taken in the stopping region, $x > \ln b(\tau)$, over all possible times until maturity τ. The portfolio being valued is again difficult interpret while $v(x)$ remains unspecified, but as mentioned previously, this is shown to be the net cash flows from holding the portfolio $v(x)$ in the case where $v(x)$ is the payoff for an American call option.

There are several advantages to solving Jamshidian's formulation for the American option pricing problem, rather than using that of McKean. Firstly, the Fourier transform can be applied directly to solve (32) without the need to introduce the concept of incomplete Fourier transforms. The solution to (32) is also more easily obtained, and requires far fewer steps than McKean's approach. Furthermore, the solution (33) is already decomposed into the sum of a corresponding European option plus an early exercise premium without the need for further simplifications. Jamshidian's formulation also leads us directly to a solution that does not involve the derivative of the free boundary, $b'(\tau)$. While the solutions (25)-(29) and (33) are equivalent, it appears non-trivial to prove this for the general monotonic payoff $v(x)$. In the case where $v(x)$ is known to be piecewise affine, such as the American call option, this equivalence has been proven by Kim (1990).

6. Alternative Representations of the American Call Value

The Fourier transform approach is capable of handling a broad class of payoff types in a general form, as evidenced by the general price for an American option with monotonic payoff given by equations (29) and (33). While alternative methods, such as Kim (1990) and Carr et al. (1992) are more restrictive in that they require the explicit form of the payoff function, the results they obtain are far easier to interpret in an economic sense. Thus to demonstrate the connections between McKean's method, Jamshdian's solution and the approaches of Kim and Carr-Jarrow-Myneni, we shall consider the example of an American call option. This also allows us to derive the limit of the early exercise boundary at expiry, a task that is far more tractable using Kim's integral equations.

6.1. **McKean's Representation.** Equation (29) provides us with an integral expression for the price of the American call option in the $x-\tau$ plane. While it is convenient to define the price in terms of time until maturity, the log-transformation has little economic interpretation. Furthermore, we are unable to draw explicit links between the solutions of McKean (1965) and Jamshidian (1992) while the form of the payoff function $c(S)$ remains unspecified[6]. For the sake of definiteness we shall consider an American call option with strike price K, for which $c(S) = \max(S-K, 0)$, and hence $v(x) = \max(e^x - K, 0)$. By substituting this expression for $v(x)$ into (25) and (26), simplifying and transforming back to the original variable, S, we obtain McKean's representation for the American call option.

Proposition 6.1. ***(McKean's Representation)*** *The integral expression for the price of an American call may be written as*

$$\begin{aligned} C_b(S,\tau) &= C_E(S,\tau) - Se^{-q\tau}N(d_1(S,\tau;b(0^+))) + Ke^{-r\tau}N(d_2(S,\tau;b(0^+))) \\ &\quad + \int_0^\tau \frac{e^{-r(\tau-\xi)-\hat{h}(S,\tau)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2}\left[k - \frac{(\ln\frac{S}{b(\xi)})}{\tau-\xi}\right]\right)(b(\xi)-K)\right] d\xi, \end{aligned} \tag{35}$$

where

$$\begin{aligned} C_E(S,\tau) &= Se^{-q\tau}N(d_1(S,\tau;K)) - Ke^{-r\tau}N(d_2(S,\tau;K)), \\ \hat{h}(S,\xi) &= \frac{(\ln\frac{S}{b(\xi)} + k(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}, \end{aligned}$$

---

[6] It is possible to carry forward the analysis by considering an affine payoff, where after the first non-zero value the payoff function is piecewise linear. The complication of this approach is that it introduces time-dependent structural breaks into the early exercise boundary $b(\tau)$ at times $\tau = t_1^*, t_2^*, \ldots$. These t^* values are determined by maintaining continuity in the free boundary at each t^*, in the same manner as described by Broadie & Detemple (1995) for capped American call options. We instead consider the simplest case of an American call option to keep the results presented both clear and concise.

with

$$\begin{aligned} d_1(x,\tau;\beta) &= \frac{\ln(x/\beta)+(k+\sigma^2)\tau}{\sigma\sqrt{\tau}}, \\ d_2(x,\tau;\beta) &= d_1(x,\tau;\beta)-\sigma\sqrt{\tau}, \\ b(0^+) &= \lim_{\tau\to 0^+} b(\tau), \end{aligned}$$

and

$$N(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{\alpha^2}{2}}\,d\alpha.$$

The integral equation for the free boundary is given by

$$\begin{aligned} b(\tau)-K &= C_E(b(\tau),\tau)-b(\tau)e^{-q\tau}N(d_1(b(\tau),\tau;b(0^+)))+Ke^{-r\tau}N(d_2(b(\tau),\tau;b(0^+))) \quad (36) \\ &+\int_0^{\tau}\frac{e^{-r(\tau-\xi)-\hat{h}(b(\tau),\tau)}}{\sigma\sqrt{2\pi(\tau-\xi)}}\left[\frac{\sigma^2 b(\xi)}{2}+\left(\frac{b'(\xi)}{b(\xi)}+\frac{1}{2}\left[k-\frac{(\ln\frac{b(\tau)}{b(\xi)})}{\tau-\xi}\right]\right)(b(\xi)-K)\right]d\xi, \end{aligned}$$

found by evaluating equation (35) at $S=b(\tau)$.

Proof: Refer to Appendix A6.1.

□

Since C_E is in fact the Black-Scholes price for a European call option written on S, equation (35) now represents a clear decomposition of the American call price into its European value, given by C_E, and the premium paid for early exercise, determined by the remaining terms. This is the solution form presented by McKean (1965), and we henceforth refer to this as McKean's representation for the price of an American call option. While this form is a valid mathematical representation, it seems impossible to develop from it any economic meaning for the early exercise premium. The presence of the derivative of the free boundary, $b'(\tau)$, in the integral equation (36) is also undesirable for the purposes of solving for $b(\tau)$ numerically, since it creates numerical difficulties due to the infinite slope of $b(\tau)$ at maturity[7].

---

[7] It should be noted that Buchen, Kelly & Rodolfo (2000) consider a numerical scheme to solve equation (35), whereby the free boundary is approximated using a cubic spline approximation. This appears to provide an efficient method for dealing with the behaviour of $b'(\tau)$ near expiry.

6.2. **Kim's Representation.** An alternative representation of the American call option price was first found by an approach due to Kim (1990). Kim arrived at a simplified form of equation (35) by taking the limit of the compound option approach to American option pricing (see Section 7). By manipulating (35) and applying integration by parts, Kim showed how the $b'(\tau)$ term could be removed. It is important to note that unless the payoff is given explicitly, further simplification in this manner appears impossible. Thus the techniques of Kim (1990) are closely tied to the particular payoff function being considered. When the payoff is given explicitly, an important consequence of this re-expression of the American option price is that the early exercise premium becomes more readily interpreted.

Jamshidian (1992) also arrived at the same representation as Kim for the American call, using Duhamel's principle as outlined in Section 5. It is interesting to note that Jamshidian's approach, which also permits a Fourier transform solution, yields a more economically intuitive form for the general American option price, which in the case of an American call option, is equivalent to Kim's solution. The result obtained by these two approach may be stated as in Proposition 6.2.

Proposition 6.2. ***(Kim's Representation)*** *Using integration by parts, the American call price $C_b(S,\tau)$ in equation (35) can be expressed as*

$$\begin{aligned} C_b(S,\tau) &= C_E(S,\tau) + \int_0^\tau qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi \qquad (37) \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi, \end{aligned}$$

where $0 < S < b(\tau)$. Furthermore, the free boundary, $b(\tau)$, is given by the integral equation

$$\begin{aligned} b(\tau)-K &= C_E(b(\tau),\tau) + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi. \qquad (38) \end{aligned}$$

Proof: Refer to Appendix A6.2.

□

Note that the factor of $\frac{1}{2}$ in McKean's form is no longer required when using Kim's integral equation for the American call price, nor does it arise under the Jamshidian approach. While Kim does not discuss this detail in his original paper, we provide a more complete explanation in Appendix A6.2. Furthermore, by following the steps outlined in Appendix A6.2 in the reverse order, it is possible to go from the representation of Proposition 6.2 to McKean's representation of Proposition 6.1. The manipulations are not as intuitively obvious when going from (37) to (35), but they are certainly achievable nonetheless.

With Kim's representation it is now possible to give a complete economic interpretation to the early exercise premium. This premium is comprised of two integral components on the right-hand side of (37). Should the holder of the call exercise early, borrowing an amount K to purchase the underlying S, then the portfolio held will be of the form $(S - K)$. Thus the early exercise premium is the expected dividend earnings received by holding S, less the expected interest to be repaid on the loan of K. This represents the expected value of the cash flows that the holder of the American call can realise via the early exercise feature.

6.3. **The Carr-Jarrow-Myneni Representation.** There exists a third representation for the American call, first derived by Carr et al. (1992), that focuses on the time value of the American option. This is found by decomposing the value of a European call option into its intrinsic value and delayed exercise value.

Proposition 6.3. ***(Carr-Jarrow-Myneni Representation)*** *By first decomposing $C_E(S,\tau)$ in equation (37), the American call price $C_b(S,\tau)$ can be expressed as*

$$\begin{aligned} C_b(S,\tau) \;=\; & \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau e^{-q(\tau-\xi)}N'(d_1(S,\tau-\xi;K))d\xi \qquad (39) \\ & + \int_0^\tau qSe^{-q(\tau-\xi)}[N(d_1(S,\tau-\xi;b(\xi))) - N(d_1(S,\tau-\xi;K))]d\xi \\ & - \int_0^\tau rKe^{-r(\tau-\xi)}[N(d_2(S,\tau-\xi;b(\xi))) - N(d_2(S,\tau-\xi;K))]d\xi, \end{aligned}$$

where $0 < S < b(\tau)$, and

$$N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.$$

The early exercise boundary, $b(\tau)$, is found by solving the integral equation

$$\begin{aligned} &\frac{b(\tau)\sigma^2}{2}\int_0^\tau e^{-q(\tau-\xi)}N'(d_1(b(\tau),\tau-\xi;K))d\xi \qquad (40)\\ &+\int_0^\tau qb(\tau)e^{-q(\tau-\xi)}[N(d_1(b(\tau),\tau-\xi;b(\xi)))-N(d_1(b(\tau),\tau-\xi;K))]d\xi\\ &-\int_0^\tau rKe^{-r(\tau-\xi)}[N(d_2(b(\tau),\tau-\xi;b(\xi)))-N(d_2(b(\tau),\tau-\xi;K))]d\xi=0, \end{aligned}$$

which is equation (39) evaluated at $S = b(\tau)$.

Proof: Refer to Appendix A6.3.

□

The intrinsic value component of the American call price is given by the present value of the payoff, $\max(S-K,0)$, equivalent to the immediate exercise value of the call. The additional terms represent the added value gained by delaying the exercise, which can also be interpreted as the time value of the American call option.

To better understand the economic meaning behind the integral terms in (39), consider a portfolio of $C_b(S,\tau) - \max(S-K,0)$ held in the continuation region for the American call. The payoff component is achieved by investing K dollars in government bonds and shorting one unit of the underlying asset, S, only when the call is in-the-money. Note that this portfolio will have a net value of zero at expiry, or upon the underlying price entering the stopping region. Given this portfolio, the integral terms in (39) are the expected present value of the cash flows incurred during the life of this portfolio. The first integral term is the sum of movements in the underlying asset's price about the strike. The last two terms measure the dividends earned and interest rate expense incurred on the portfolio $S-K$ while the call is in-the-money but remaining unexercised i.e. $K \le S \le b(\xi)$, where $0 \le \xi \le \tau$.

6.4. **The Free Boundary at Expiry.** Next we shall present one additional result regarding the free boundary of the American call option, as found by Kim (1990). Using Kim's representation of the integral equation for the free boundary, it is possible to find the limit of $b(\tau)$ at expiry (i.e. as $\tau \to 0^+$). An alternative derivation is possible. Wilmott et al. (1993) show that we can also derive the limiting value of $b(\tau)$ by analysing the PDE (32) for sufficiently small values of τ. This result for the free boundary is important when trying to solve equation (38) numerically for $b(\tau)$.

Proposition 6.4. *Taking the limit as* τ *tends to* 0^+ *in equation (38), the value of the free boundary,* $b(\tau)$*, at expiry is given by*

$$b(0^+) = K \max\left(1, \frac{r}{q}\right). \tag{41}$$

This result also follows by analysing the PDE (32) for small values of τ*.*

Proof: Refer to Appendix 7.

□

Thus the value of $b(0^+)$ depends entirely on the relative parameter values of the risk-free rate, r, and the continuously compounded dividend yield, q. Note that when q is reduced to zero, the value for $b(0^+)$ becomes infinite, which coincides with the well-known result that it is never optimal to exercise an American call option early in the absence of dividends.

When $\tau = 0$, the decision whether or not to exercise the call depends entirely on the value of the underlying, S, when compared with the strike, K. As such, the early exercise boundary at expiry is simply given by $b(0) = K$. It is therefore important to note that a consequence of equation (41) is that $b(\tau)$ can be discontinuous at $b(0)$, and this occurs specifically when $r > q$. There has been some confusion regarding this detail in the literature, where many have defined $b(0)$ to be the result in equation (41), rather than $b(0^+) \equiv \lim_{\tau\to 0^+} b(\tau)$. Throughout the paper we shall adopt this more explicit notation for the limit of $b(\tau)$ to avoid confusion. Figure 3 illustrates the behaviour of the free boundary at expiry.

6.5. **The Perpetual American Call.** Before concluding this section, we shall derive the price and early exercise boundary of the perpetual American call option. Kim (1990) derives this result by noting that $C_b(S, \tau)$ and $b(\tau)$ become time-invariant as τ increases without bound, and letting $\tau \to \infty$ in equations (37) and (38). An alternative approach

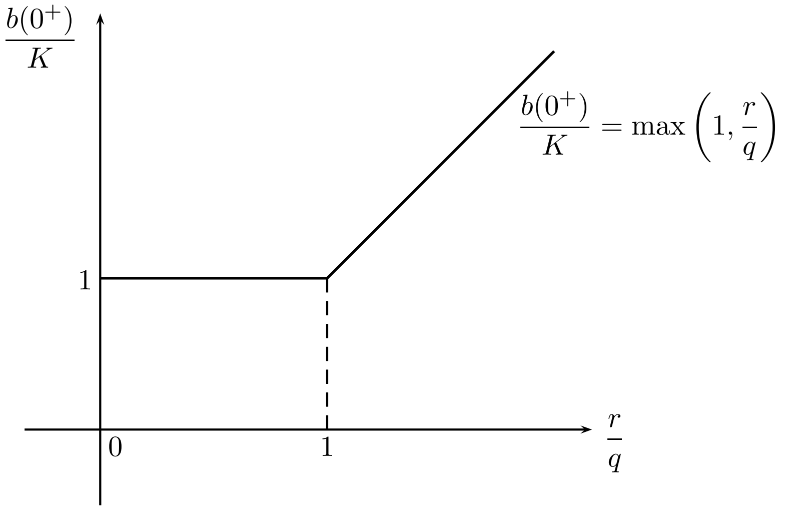


FIGURE 3. The free boundary at expiry (41), as a function of r/q.

involves solving the time-invariant form of (2), and selecting the early exercise boundary such that the option value is maximised.

Proposition 6.5. *The price of the perpetual American call option, $C_b(S,\infty)$, with early exercise boundary $b(\infty)$ and strike price K, satisfies the ODE*

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 C_b}{dS^2} + (r-q)S\frac{dC_b}{dS} - rC_b = 0, \tag{42}$$

in the region $0 < S < b(\infty)$, subject to the boundary conditions

$$C_b(0,\infty) = 0, \tag{43}$$

$$C_a(b(\infty),\infty) = b(\infty) - K. \tag{44}$$

If we select the optimal stopping strategy $b(\infty) = b^*(\infty)$ such that C_b is maximised, then the solution to the free boundary value problem (42) - (44) is given by

$$C_{b^*}(S,\infty) = \frac{K}{\alpha_+ - 1}\left(\frac{(\alpha_+ - 1)S}{\alpha_+ K}\right)^{\alpha_+}, \tag{45}$$

where

$$\alpha_+ = \frac{-k + \sqrt{k^2 + 2\sigma^2 r}}{\sigma^2}, \tag{46}$$

with $k = r - q - \frac{1}{2}\sigma^2$. *The optimal early exercise boundary,* $b^*(\infty)$, *is*

$$b^*(\infty) = \frac{\alpha_+ K}{\alpha_+ - 1}, \tag{47}$$

and in addition

$$\lim_{S \to b^*} \frac{dC_{b^*}}{dS} = 1, \tag{48}$$

which is the smooth-pasting condition.

Proof: Refer to Appendix 8.

□

It is important to note that in the perpetual case we can readily show that the smooth-pasting condition is a consequence of optimally exercising the American call. Thus we are able to determine the early exercise strategy by requiring that the the decision to exercise early will maximise the value of the option, which happens to coincide with the option having a continuous delta at the free boundary. The perpetual American call price and free boundary are also useful results when exploring numerical methods for the American call with finite time until maturity, since the perpetual price is an upper bound for the finite American call, and likewise the perpetual exercise boundary is an upper bound for the early exercise boundary of the finite American call.

7. American Call as a Compound Option

Kim (1990) was one of the first to confirm equations (37)-(38) for the American call using McKean's method, however his primary derivation was based on the compound option approach of Geske & Johnson (1984). Here we replicate this alternative approach, both to contrast the methodology with the incomplete Fourier transform, and to reiterate the equivalence of the results obtained by the two solution techniques.

For the American call, $\hat{C}_b(S, \tau)$, assume that we can only exercise at a finite number of time points $\tau_k, k = n, n-1, ..., 1, 0$ with $\tau_k - \tau_{k-1} = \Delta\tau$ for all k, and expiry occurs at τ_0 (i.e. $\tau = 0$). Let $p(S_{k-1}, \tau_{k-1} | S_k, \tau_k)$ be the transition density for S_{k-1} at time to maturity τ_{k-1}, given that price S_k was observed at time to maturity τ_k.

Let $U(S_k, k\Delta\tau; b_{k-1})$ denote the value of the unexercised call at time to maturity $k\Delta\tau$, where $b_{k-1} \equiv b((k-1)\Delta\tau)$. Since the holder of the call will not be able to exercise early until time $\Delta\tau$ in the discrete case, we find that at time $\Delta\tau$ prior to expiry, the unexercised call has value

$$\begin{aligned} U(S_1, \Delta\tau; K) = C_E(S_1, \Delta\tau) &= S_1 e^{-q\Delta\tau} N(d_1(S_1, \Delta\tau; K)) \\ &\quad - K e^{-r\Delta\tau} N(d_2(S_1, \Delta\tau; K)), \end{aligned} \tag{49}$$

which is simply a European call option with $\Delta\tau$ remaining until maturity. The early exercise boundary, b_1, is given by the solution of

$$b_1 - K = b_1 e^{-q\Delta\tau} N(d_1(b_1, \Delta\tau; K)) - K e^{-r\Delta\tau} N(d_2(b_1, \Delta\tau; K)). \tag{50}$$

With the starting case of $U(S_1, \Delta t; K)$ completed, we now develop an induction proof to find $U(S_n, n\Delta\tau; b_{n-1})$.

Proposition 7.1. *The price of the unexercised call, U, at time to expiry $2\Delta\tau$ is*

$$\begin{aligned} U(S_2, 2\Delta\tau; b_1) &= C_E(S_2, 2\Delta\tau) + o(\Delta\tau) \qquad (51) \\ &\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} \left[(1 - e^{-q\Delta\tau})S_1 - (1 - e^{-r\Delta\tau})K\right] p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1. \end{aligned}$$

The early exercise price, b_2, at time to expiry $2\Delta\tau$ is defined implicitly by

$$b_2 - K = U(b_2, 2\Delta\tau; b_1).$$

Proof: Refer to Appendix A9.1.

□

Having derived an expression for U when $n = 2$, we proceed to find the value of U for a general non-zero integer value of n.

Proposition 7.2. *The price of the unexercised call, U, at a general time until maturity $n\Delta\tau$ is*

$$\begin{aligned} U(S_n, n\Delta\tau, b_{n-1}) &= \sum_{k=1}^{n-1} e^{-(n-k)r\Delta\tau} \int_{b_k}^{\infty} \left[(1-e^{-q\Delta\tau})S_k - (1-e^{-r\Delta\tau})K\right] \\ &\qquad \times p(S_k, k\Delta\tau | S_n, n\Delta\tau) dS_k \\ &\quad + C_E(S_n, n\Delta\tau) + o(n\Delta\tau). \end{aligned} \tag{52}$$

This equation is satisfied for $n = 2$, as shown in Proposition 7.1, and for $n = m$ and $n = m + 1$, where m is a non-negative integer.

Proof: Refer to Appendix A9.2.

□

Equation (52) provides us with the value of an unexercised American call option with discrete, equally spaced early exercise dates. The integral term calculates the expected present value of the cash flows incurred when holding the portfolio $S-K$ in the stopping region. Up until this point, the solution method is equivalent to that of Geske & Johnson (1984). Kim's contribution was to take the limit of U as $\Delta\tau \to 0$, thereby returning to the continuous American call case.

Proposition 7.3. *The price of the unexercised American call, $\hat{C}_b$, at a general time until maturity $n\Delta\tau$ is*

$$\begin{aligned} \hat{C}_b(S_n, n\Delta\tau) &= C_E(S_n, n\Delta\tau) + o(n\Delta\tau) \qquad (53) \\ &\quad + \sum_{k=1}^{n-1} e^{-(n-k)\Delta\tau} \left(\int_{b_k}^{\infty} [qS_k - rK] p(S_k, k\Delta\tau | S_n, n\Delta\tau) dS_k \right) \Delta\tau. \end{aligned}$$

Taking the limit of (53) as $\Delta\tau \to 0$, this becomes equation (37) of Proposition 6.2.

Proof: Refer to Appendix A9.3.

□

There are several important observations one can make regarding this method. Firstly, it is important to note that the compound option approach requires that the payoff function be known explicitly before the analysis can be carried out. In particular, demonstrating that some of the terms are of order $\Delta\tau$ requires that the payoff be given in an explicit form. The initial steps of the Fourier transform method are not restricted by the need for an explicit payoff function, though we note that some information on the limits and derivative of the payoff are still required for meaningful analysis.

The second detail to note is that the compound option method requires that we first consider a discrete time situation, and then apply limit analysis to find the continuous case. When using Fourier transforms we are able to remain in continuous time at no significant increase in the mathematical complexity of the solution. In this sense the Fourier transform approach is a more natural extension of the PDE solution methods applied to European options. Using the compound option approach for American options introduces an additional level of theoretical complexity that can otherwise be avoided.

8. Numerical Examples

Equation (37) is an explicit expression for the price of an American call option, but it requires the free boundary, $b(\tau)$, to be known before it can be used. While $b(\tau)$ can be found by solving equation (38), there exists no known closed-form solution for the free boundary, and one must use numerical methods in order to estimate the price and free boundary of the American call option. In this section we apply simple implementations of four existing numerical methods for pricing American options, comparing their pricing efficiency, and also the quality of their free boundary estimates for two sets of parameters. The analysis in this section is necessarily brief, as much of the paper is given over to outlining analytic methods for deriving the integral representations of the early exercise premium. For a more thorough comparison of numerical techniques, we refer the reader to other surveys on this topic, including AitSahlia & Carr (1997) and Broadie & Detemple (1996).

We consider a 3-month American call ($T-t=0.25$), with strike $K=100$ and volatility $\sigma=20\%$. The first call under consideration has risk-free rate $r=8\%$ and dividend yield $q=12\%$. For the second call, we take $r=12\%$ and $q=8\%$. This allows us to explore the differences between the individual cases of $r<q$ and $r>q$. For each method we generated five option prices at $S = 80, 90, 100, 110$ and 120, representing a range of moneyness values centred around the strike price.

The first method we use is the binomial tree procedure of Cox et al. (1977). We calculate the risk-neutral transition probabilities as detailed in de Jager (1995) (p.251-252)[8], and structure the tree such that the nodes at expiry are centred about the strike. We use $n = 1000, 2000, \ldots, 20000$ layers for a total of 20 trees ($\Delta t = 2.5 \times 10^{-4}, \ldots, 1.25 \times 10^{-5}$). Note that a new tree is generated for each value of S, and thus five trees are needed to find the five option prices required. We make no attempt to estimate the free boundary using this approach, noting that the form of the tree does not always provide sufficient information to extract such an estimate for all time steps.

Next we consider the finite difference solution for the PDE (2), subject to the boundary and final conditions (3)-(6). The method was first suggested in option pricing by Brennan & Schwartz (1977). The Crank-Nicolson scheme is used in the standard way, with LU-decomposition used to solve the resulting tri-diagonal system at each time step. Time step values numbering $n = 500, 1000, \ldots, 10000$ are taken for 20 applications of the method and ($\Delta t = 5 \times 10^{-4}, \ldots, 2.5 \times 10^{-5}$). The state-space nodes were set at $m = 2n$, spaced evenly between $S = 0$ and $S = 200$. We also approximate the early exercise boundary at each time step by fitting a cubic spline through the solution in the region near the free boundary, and then finding the intercept between this solution and the payoff function using the bisection method.

The third technique is the method of lines, as given by Meyer & van der Hoek (1997). We implement their method as written, save that we use an evenly spaced discretisation for both τ and S, and use cubic splines for any interpolation required. Cubic splines appear particularly well-suited for the method of lines, as the derivatives of the functions being interpolated are known in closed-form, leading to an excellent fit for the spline.

---

[8] Given a time-step size of $\Delta\tau$, the underlying asset price, S_j, at the current time step j rises to $S_{j+1} = US_j$ at the next time step with probability p, and falls to $S_{j+1} = DS_j$ with probability $(1-p)$ where

$$U = e^{(k+x)\Delta\tau + w\sigma\sqrt{\Delta\tau}}, \quad D = e^{(k+x)\Delta\tau - w\sigma\sqrt{\Delta\tau}},$$

with

$$p = \frac{1}{2} - \frac{x}{2w\sigma}\sqrt{\Delta\tau},$$

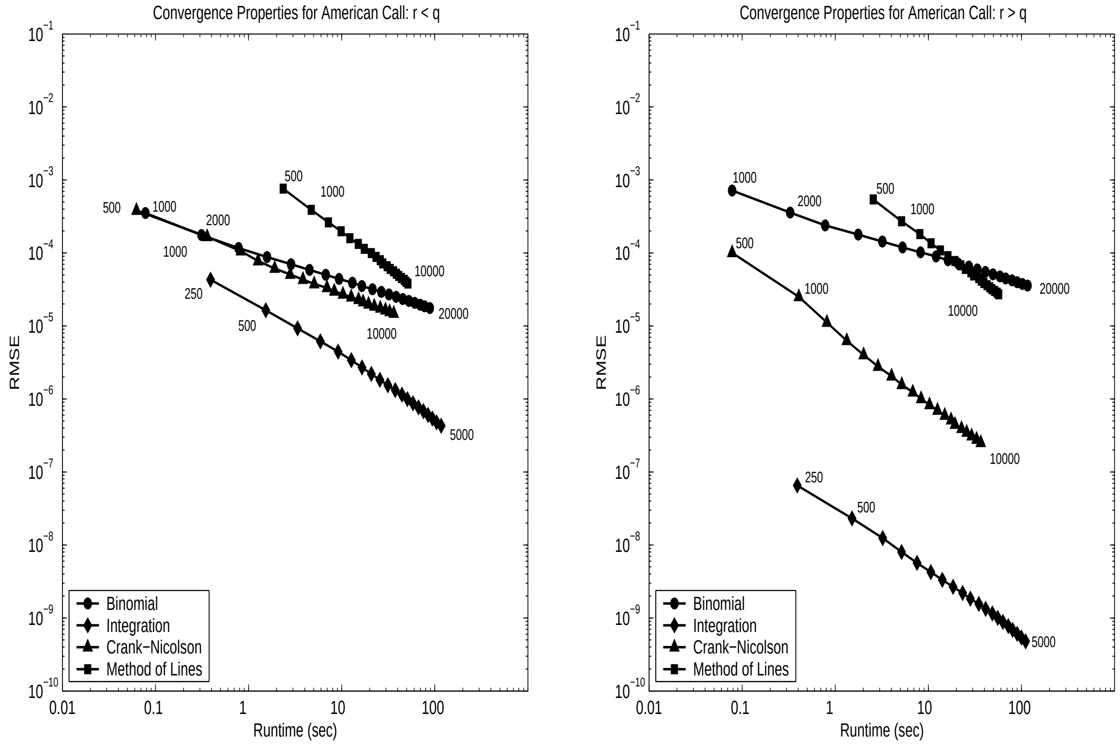
where $x = \ln(K/S_0)/\Delta\tau - k$, $w = \sqrt{1 + \Delta\tau x^2/\sigma^2}$, and S_0 is the value of S at time T prior to expiry.

We use 20,000 state-space nodes in the region $0 < S < 200$, and take a range of time steps, specifically $n = 500, 1000, \ldots, 10000$ for a total of 20 applications of the method. Note that the method of lines produces an estimate of the free boundary as part of the solution.

Finally, using techniques frequently applied to Volterra integral equations, we numerically integrate equation (38) to obtain the free boundary, as suggested by Kim (1990), and carried out in full by Kallast & Kivinukk (2003). The resulting free boundary estimate is then used to perform a simple numerical integration to solve (37) for the call price. As in Kallast & Kivinukk (2003) we conduct all numerical integration using the trapezoidal rule, and find the free boundary at each time step using Newton's method. We apply the method 20 times with increasing time step values of $n = 250, 500, \ldots, 5000$ ($\Delta t = 1.0 \times 10^{-3}, \ldots, 5.0 \times 10^{-5}$). The time steps are uniformly spaced, as in the other three methods. We do not use any techniques to accelerate the convergence of the numerical integration, such as Richardson extrapolation.

To compare the relative efficiency of these numerical methods, we provide an overview of the computation time and accuracy for each. The code for all five methods was implemented using LAHEY$^{\mathrm{TM}}$FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system. We assume that the true solution is given by the numerical integration method using 10000 time steps, and use this to compute the root mean square error (RMSE) for each method across the five moneyness levels stated previously. The runtime, measured in seconds, includes the time required to find the five required option values, along with the free boundary estimate (where applicable).

In figures 4(a) and 4(b) we present the convergence rates for the four methods. We plot log(RMSE) against log(runtime). In each case we also include some of the time step sizes used for to generate some of the plotted points. Figure 4(a) presents the findings for $r < q$, with $r > q$ shown in 4(b). In both cases it is clear that the most efficient method is the numerical integration approach, followed by the Crank-Nicolson scheme. The least efficient is usually the method of lines, although when $r > q$ we see that the method does do better than the binomial tree for large time step sizes. Even in the



(a) (b)

FIGURE 4. Comparing the efficiency of numerical methods for pricing American call options with $K = 100$, $\sigma = 20\%$ and $T - t = 0.25$ for $S = 80, 90, 100, 110$ and 120. In Figure 4(a) $r = 8\%$, $q = 12\%$, and in Figure 4(b) $r = 12\%$, $q = 8\%$. Numbers on the plot indicate the time steps associated with a given point. Both axes are given in log-scale.

$r < q$ case there is evidence that this relationship would hold for large values of n. It is also interesting to see that when $r < q$ the binomial tree and finite difference method have very similar efficiency levels, but when $r > q$ the Crank-Nicolson scheme becomes far more favourable.

It is particularly interesting to note the differences between each method for the relative values of r and q. The integration method performs considerably better in the $r > q$ case, and the Crank-Nicolson scheme displays similar behaviour, including a noticeable increase in its rate of convergence (as indicated by the slope of the line). The binomial tree, on the other hand, performs better when $r < q$, whereas the method of lines appears relatively unchanged between the two cases. Given the significant differences in the free boundary for relative values of r and q, it is interesting to note that the impact of these parameters on the efficiency of various numerical methods is non-uniform.

To complete the comparison, we also examine the free boundary estimates produced by the numerical integration approach, Crank-Nicolson scheme and method of lines. The $r < q$ case is given in Figure 5. There is no significant difference between these estimates, where the number of time steps used is indicated in the figure. When $r > q$, as shown in Figure 6, a slight difference can be seen. If we consider the integration result as being closest to the true solution (an assumption supported by the convergence shown in Figure 4), we see that the method of lines solution appears to converge to the true free boundary from above, whereas the Crank-Nicolson estimate converges from below. Note that we only used the method of lines free boundary estimate for 5000 time steps because larger values of n produced non-monotonic free boundaries. Such errors can be rectified by using a finer state-space grid, although we do not demonstrate this attribute here. Note however that these non-monotonic errors in the free boundary were small, and had no impact on the convergence of the prices, as shown in Figure 4(b). Despite these differences in the $r > q$ case, there appears little reason to favour any one method in particular for estimating the early exercise boundary.

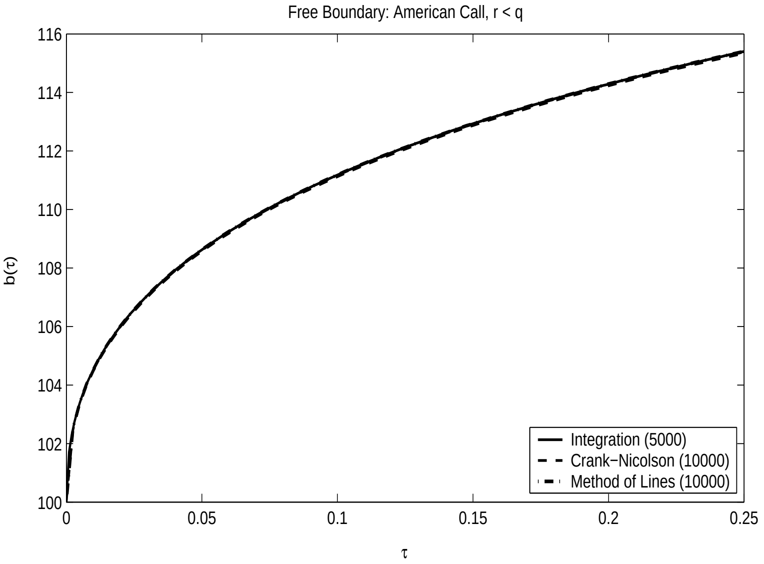


FIGURE 5. Comparing free boundary estimates for an American call option with $K = 100$, $\sigma = 20\%$; $r = 8\%$, $q = 12\%$, and $T - t = 0.25$.

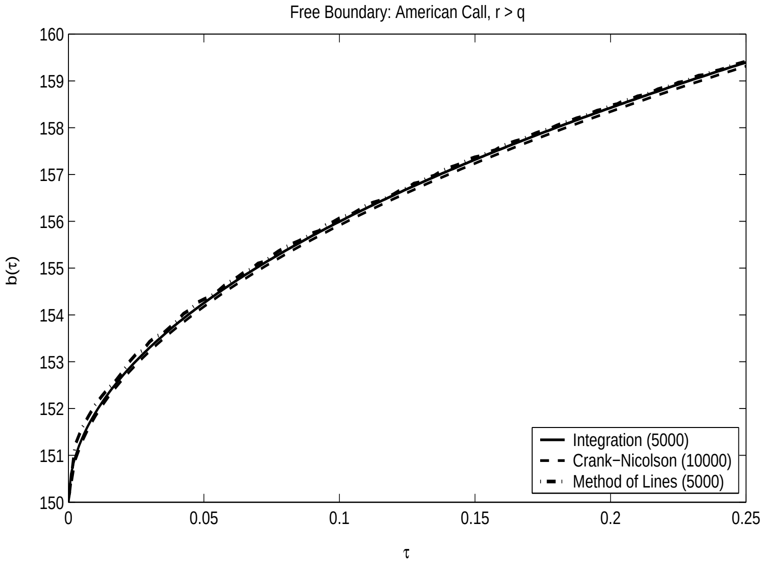


FIGURE 6. Comparing free boundary estimates for an American call option with $K = 100$, $\sigma = 20\%$; $r = 12\%$, $q = 8\%$, and $T - t = 0.25$.

Overall these results demonstrate that the numerical integration method proposed by Kallast & Kivinukk (2003) is an excellent way to numerically find the price and free boundary for American call options. We must emphasise, however, that the results here represent a mild analysis at best. We only consider a single time to maturity and volatility, and two pairs of values for r and q. The range of moneyness values is also quite small. We do not employ any advanced techniques when implementing the numerical methods here. For example, Kallast & Kivinukk (2003) use Richardson extrapolation to improve the rate of convergence for the numerical integration approach. There are also many alternatives for approximating the solution to the integral equation (38) beyond what we have considered here, and Kallast & Kivinukk (2003) provides a good survey of these alternative approaches.

For the method of lines, Meyer & van der Hoek (1997) recommend the use of state-space grids with non-uniform spacing to better focus the computation efforts around critical S values. The Crank-Nicolson scheme is also known to benefit from non-uniform state-space grids. The binomial tree could also be implemented more efficiently than has been done here when one seeks to find prices for a range of moneyness values simultaneously. Furthermore, the binomial tree can be used to estimate the free boundary, although our implementation is not well-suited to this purpose. Thus our results are at best a simple demonstration of the efficiency of some of the more simple implementations for these four numerical methods.

9. Conclusion

In this paper we have presented a survey of the methods for deriving the various integral representations of American option prices, with particular focus on the American call. We revisited McKean's (1965) incomplete Fourier transform method, and demonstrated how his results reconcile with the early exercise premium representation of Kim (1990), the inhomogeneous PDE representation of Jamshidian (1992), and the intrinsic/time value decomposition of Carr et al. (1992). We reviewed the compound option solution technique used by Kim for the American call option, and highlighted the fact that the method relies upon explicit knowledge of the payoff function to produce the final integral expression for the American option price. The Fourier transform solution, on the other hand, is able to produce an integral expression based only on knowing that the function is monotonic. In this respect the Fourier transform displays a higher degree of flexibility when considering a broader class of payoff functions within a single framework. Figure 7 shows the relation between the different approaches to the American option pricing problem[9].

Given that there exists no closed-form solution for the American call option, we compared four existing numerical techniques, using simple implementations for each. We found that direct numerical integration was the most efficient method, followed by, the Crank-Nicolson finite difference scheme. The method of lines outperformed binomial trees when the times steps were large, but the binomial tree method was not well suited for approximating the early exercise boundary along with the option price. We also found that the relative values of the risk-free rate and continuous dividend yield can have a significant impact on the efficiency of all methods, with the exception of the method of lines. For numerical integration, the method of lines and the Crank-Nicolson

---

[9] The functions F^M and G^M are implied by the results in Proposition 6.1, referring to McKean's form of the solution. Similarly the functions F^K and G^K are implied by the results in Proposition 6.2, referring to Kim's form.

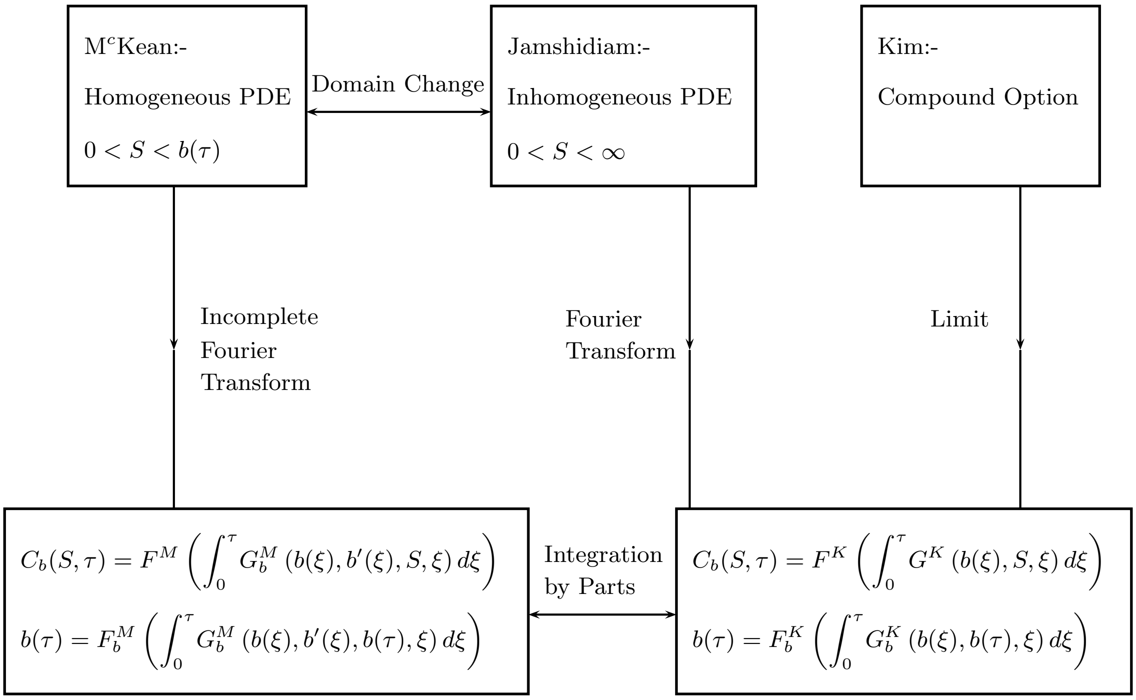


Figure 7. The different approaches to American option pricing.

scheme we were also able to compare the free boundary estimates produced. We found no significant differences between these estimates, although it was interesting to note that the finite difference approximation appeared to converge to the true free boundary from below, while the method of lines converged from above.

This survey implies several directions for further research. Given that the Fourier transform is well-suited to general monotonic payoff functions, it should be possible to extend the methodology to consider American options with convex or concave payoffs. Examples include American portfolios such as strangles (Chiarella & Ziogas 2005) and butterflies. The method can also be applied to evaluate American options with more complex price dynamics, such as jump-diffusion models (Chiarella & Ziogas 2004). Two-dimensional extensions could also be considered, including American options under stochastic volatility, and American options on multiple underlying assets, such as an American basket option. When the asset dynamics are more complicated, direct numerical integration may become less efficient than in the simple case considered in this chapter. Under these circumstances, alternative methods such as the method of lines may provide a more optimal accuracy-efficiency tradeoff.

Appendix 1. Fundamental Results

Here we present a collection of fundamental results that are frequently required throughout the paper. These results relate to the inversion of Fourier transforms and the evaluation of commonly recurring integrals.

A1.1. **Convolution Theorem for Fourier Transforms.** The Fourier transform of the convolution integral is given by

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f((x-u),\tau_1)g(u,\tau_2)du\right\} = \hat{F}(\eta,\tau_1)\hat{G}(\eta,\tau_2), \tag{54}$$

where $\hat{F}$ and $\hat{G}$ are the Fourier transforms, with respect to x, of $f(x,\tau_1)$ and $g(x,\tau_2)$ respectively.

A1.2. **Integrals of the Exponential-Quadratic Function.** Let $\hat{p}$ and $\hat{q}$ be any complex functions not involving the integration variable η, with $\mathrm{Re}(\hat{p}) \geq 0$. Furthermore, let n be any non-negative integer. Then the integral of the exponential-quadratic function with respect to η is given by

$$\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}\eta}\eta^n d\eta = (-1)^n\sqrt{\frac{\pi}{\hat{p}}}\frac{\partial^n}{\partial\hat{q}^n}e^{\frac{\hat{q}^2}{4\hat{p}}}. \tag{55}$$

In addition, we shall consider a more general form of this integral where the limits are finite with $n=0$. Let $\hat{a}$, α_1 and α_2 be any real functions not involving η, along with $\hat{p}$ and $\hat{q}$ as before. Then the finite integral of the more general exponential-quadratic function with respect to η is given by

$$\int_{\alpha_1}^{\alpha_2} e^{\hat{a}\eta}e^{-\frac{(\hat{q}-\eta)^2}{\hat{p}}}d\eta = \sqrt{\hat{p}\pi}\exp\left\{\frac{(4\hat{q}+\hat{a}\hat{p})\hat{a}}{4}\right\}\left\{N[f(\alpha_2)]-N[f(\alpha_1)]\right\}, \tag{56}$$

where

$$f(u) = \sqrt{\frac{2}{\hat{p}}}\left(\frac{2u-(2\hat{q}+\hat{a}\hat{p})}{2}\right).$$

Another useful exponential integral result arises when the exponent involves a sum of perfect squares. Define $\hat{p}$, $\hat{q}$, α_1 and α_2 to be real functions not involving η, with $\alpha_1, \alpha_2 > 0$. Then we can readily show that

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{[\eta+\hat{p}]^2}{\alpha_1} - \frac{[\eta+\hat{q}]^2}{\alpha_2}\right\} d\eta = \sqrt{\frac{\pi\alpha_1\alpha_2}{\alpha_1+\alpha_2}} \exp\left\{-\frac{(\hat{p}-\hat{q})^2}{\alpha_1+\alpha_2}\right\}. \tag{57}$$

Appendix 2. The Incomplete Fourier Transform

Our aim is to prove that if $f(x,\tau) = H(a-x)g(x,\tau)$, $a \equiv a(\tau)$, and $H(x)$ is the Heaviside function, then application of the standard Fourier inversion theorem

$$f(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < \infty,$$

yields

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < a,$$

which may be regarded as an inversion theorem for the incomplete Fourier transform. With regard to the last equation consider first

$$\begin{aligned} RHS &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(a-\xi)g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta. \end{aligned}$$

Next consider

$$LHS = H(a-x)g(x,\tau) = \begin{cases} g(x,\tau), & -\infty < x < a \\ \frac{g(x,\tau)}{2}, & x = a \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$H(x-a)g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < \infty$$

or alternatively,

$$g(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad -\infty < x < a$$

and

$$\frac{g(x,\tau)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{a} g(\xi,\tau)e^{i\eta\xi}d\xi\right]e^{-i\eta x}d\eta, \quad x = a.$$

Refer to Section 4 for an explanation regarding the factor of $\frac{1}{2}$ on the left hand side.

Appendix 3. Properties of the Incomplete Fourier Transform

A3.1. **Proof of Proposition 3.2.** Firstly consider

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} = V_b(\ln b,\tau)e^{i\eta\ln b} - i\eta\hat{V}_b(\eta,\tau).$$

By use of the boundary condition (11),

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} = v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b.$$

Next consider

$$\begin{aligned}\mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} &= \left.\frac{\partial V_b(x,\tau)}{\partial x}\right|_{x=\ln b}\cdot e^{i\eta\ln b} - i\eta\mathcal{F}^b\left\{\frac{\partial V_b}{\partial x}\right\} \\ &= v'(\ln b)e^{i\eta\ln b} - i\eta[v(\ln b)e^{i\eta\ln b} - i\eta\hat{V}_b],\end{aligned}$$

where the last equality follows by use of the boundary condition (12), and the transform result (18). This simplifies to

$$\mathcal{F}^b\left\{\frac{\partial^2 V_b}{\partial x^2}\right\} = e^{i\eta\ln b}(v'(\ln b) - i\eta v(\ln b)) - \eta^2\hat{V}_b.$$

Finally consider

$$\begin{aligned}\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} &= \frac{\partial}{\partial\tau}\left[\int_{-\infty}^{\ln b} e^{i\eta x}V_b(x,\tau)dx\right] - \frac{b'}{b}e^{i\eta\ln b}V_b(\ln b,\tau) \\ &= \frac{\partial}{\partial\tau}\left[\mathcal{F}^b\{V_b\}\right] - \frac{b'}{b}e^{i\eta\ln b}V_b(\ln b,\tau),\end{aligned}$$

where $b' \equiv db(\tau)/d\tau$. Applying the boundary condition (11) we have

$$\mathcal{F}^b\left\{\frac{\partial V_b}{\partial \tau}\right\} = \frac{\partial \hat{V}_b}{\partial \tau} - \frac{b'}{b}e^{i\eta \ln b}v(\ln b).$$

A3.2. **Proof of Proposition 3.3.** Taking the incomplete Fourier transform of equation (8) with respect to x and using (18) - (20), we obtain

$$\begin{aligned}&\frac{d\hat{V}_b}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V}_b \\ &\quad = e^{i\eta \ln b}\left[\frac{b'}{b}v(\ln b) + \frac{1}{2}\sigma^2(v'(\ln b) - i\eta v(\ln b)) + kv(\ln b)\right].\end{aligned}$$

It is a simple matter to rewrite this in terms of $F(\eta, \tau)$ to produce equations (21)-(22), and the initial condition is obtained by definition.

Appendix 4. Derivation of the American Call Integral Expression

A4.1. **Proof of Proposition 4.1.** Recall the definition of $V_b^{(1)}(x,\tau)$, namely

$$V_b^{(1)}(x,\tau) = \mathcal{F}^{-1}\left\{\hat{V}_b(x,0)e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}\right\}.$$

We shall evaluate this inverse Fourier transform using the standard Fourier convolution result given in equation (54). To apply this convolution we first let

$$F(\eta,\tau_1) = e^{-(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r)\tau}.$$

Hence

$$f(x,\tau_1) = \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\eta^2\tau - i\eta(x+k\tau)}d\eta = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{\frac{-(x+k\tau)^2}{2\sigma^2\tau}}$$

by use of equation (55) with $\hat{p} = \frac{1}{2}\sigma^2\tau$, $\hat{q} = i(x+k\tau)$ and $n = 0$.

Next we let $G(\eta,\tau_2) = \hat{V}_b(\eta, 0)$. Hence we have

$$\begin{aligned}g(x,\tau_2) &= H(\ln b(0^+) - x)V_b(x,0) \\ &= H(\ln b(0^+) - x)v(x).\end{aligned}$$

Thus

$$\begin{aligned} V_b^{(1)}(x,\tau) &= \int_{-\infty}^{\infty} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} H(\ln b(0^+)-u)v(u)du \\ &= \int_{-\infty}^{\ln b(0^+)} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}} v(u)du. \end{aligned}$$

A4.2. **Proof of Proposition 4.2.** We recall first that

$$V_b^{(2)}(x,\tau) = \mathcal{F}^{-1}\left\{\int_0^\tau F(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\},$$

so that

$$V_b^{(2)}(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}\left\{\int_0^\tau F(\eta,s)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}ds\right\}d\eta,$$

where from equation (22),

$$F(\eta,s) = e^{i\eta\ln b(s)}\left[\frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} - \frac{\sigma^2 i\eta}{2} + k\right)v(\ln b(s))\right].$$

We can rewrite the function $F(\eta,s)$ as

$$F(\eta,s) = e^{i\eta\ln b(s)}\{f_1(s) - \eta f_2(s)\},$$

where we set

$$f_1(s) = \frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + k\right)v(\ln b(s)),$$

and

$$f_2(s) = \frac{\sigma^2 i}{2}v(\ln b(s)).$$

Thus the inverse transformation of $V_b^{(2)}(x,\tau)$ becomes

$$V_b^{(2)}(x,\tau) = \frac{1}{2\pi}\int_0^\tau e^{-r(\tau-s)}\left[\int_{-\infty}^{\infty} e^{-\hat{p}\eta^2-\hat{q}\eta}\{f_1(s)-\eta f_2(s)\}d\eta\right]ds,$$

where $\hat{p} = \sigma^2(\tau - s)/2$, and $\hat{q} = i(x + k(\tau - s) - \ln b)$. Using the result in equation (55) with $n = 0$ and $n = 1$ we have

$$\begin{aligned} V_b^{(2)}(x,\tau) &= \frac{1}{2\pi}\int_0^\tau e^{-r(\tau-s)}\left[f_1(s)\sqrt{\frac{\pi}{\hat{p}}}e^{\frac{\hat{q}^2}{4\hat{p}}} + f_2(s)\sqrt{\frac{\pi}{\hat{p}}}e^{\frac{\hat{q}^2}{4\hat{p}}}\frac{\hat{q}}{2\hat{p}}\right]ds \\ &= \int_0^\tau \frac{e^{-r(\tau-s)+\frac{\hat{q}^2}{4\hat{p}}}}{2\sqrt{\pi\hat{p}}} \\ &\qquad \times\left[\frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + k\right)v(\ln b(s)) + \frac{\sigma^2 i v(\ln b(s))\hat{q}}{2\sigma^2(\tau - s)}\right]ds. \end{aligned}$$

Substituting for $\hat{p}$ and $\hat{q}$, we find that

$$\begin{aligned} V_b^{(2)}(x,\tau) &= \int_0^\tau \frac{e^{-r(\tau-s)-h(x,s)}}{\sigma\sqrt{2\pi(\tau-s)}} \\ &\qquad \times\left[\frac{\sigma^2 v'(\ln b(s))}{2} + \left(\frac{b'(s)}{b(s)} + \frac{1}{2}\left[k - \frac{(x - \ln b(s))}{(\tau - s)}\right]\right)v(\ln b(s))\right]ds, \end{aligned} \tag{58}$$

where we set

$$h(x,s) \equiv \frac{(x - \ln b(s) + k(\tau - s))^2}{2\sigma^2(\tau - s)}.$$

With a simple change of notation, equation (58) may be written as it appears in equations (26)-(28).

Appendix 5. Jamshidian's Results for American Options

A5.1. **Proof of Proposition 5.1.** Here we present an intuitive method for deriving Jamshidian's inhomogeneous PDE (31) for C_b. In Section 3 we extend the domain of the PDE for V_b by multiplying the PDE by the Heaviside step function $H(\ln b - x)$. This effectively sets the solution $V_b(x,\tau) = 0$ for all $x > \ln b(\tau)$. We do not lose anything from the solution since we already know the value of V_b in the stopping region, and the boundary conditions (11)-(12) guarantee that the option price and delta will be continuous at the free boundary. Jamshidian (1992) also extends the domain of the problem, but rather than setting the solution to be zero in the stopping region, he replaces the homogeneous PDE with an inhomogeneous one that will be satisfied by the payoff function $c(S)$ whenever $S > b(\tau)$.

To derive the inhomogeneous term, we first evaluate (2) for $S > b(\tau)$. In this case, using boundary condition (5), we readily find that

$$-\frac{\partial C_b}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_b}{\partial S^2} + (r-q)S\frac{\partial C_b}{\partial S} - rC_b = \frac{1}{2}\sigma^2 S^2 \frac{d^2 c(S)}{dS^2} + (r-q)S\frac{dc(S)}{dS} - rc(S),$$

whenever $S > b(\tau)$. Given that the PDE (2) is only satisfied by C_b when $S < b(\tau)$, we introduce the inhomogeneous term

$$H(S - b(\tau))\left(\frac{1}{2}\sigma^2 S^2 \frac{d^2 c(S)}{dS^2} + (r-q)S\frac{dc(S)}{dS} - rc(S)\right),$$

to the right-hand side of (2). Thus the inhomogeneous PDE for C_b is given by

$$\begin{aligned}\frac{\partial C_b}{\partial \tau} =& \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_b}{\partial S^2} + (r-q)S\frac{\partial C_b}{\partial S} - rC_b \\ & - H(S - b(\tau))\left(\frac{1}{2}\sigma^2 S^2 \frac{d^2 c(S)}{dS^2} + (r-q)S\frac{dc(S)}{dS} - rc(S)\right),\end{aligned}$$

which is equation (31) of the Proposition (5.1). It is easy to verify that C_b satisfies (31) for all values of S via direct substitution. Finally, equation (32) follows in the same manner as (8) in Section 2.

A5.2. **Proof of Proposition 5.2.** To solve (32), it is possible to simply apply Duhamel's principle (see Logan (2004)). Rather than apply Duhamel's principle directly, here we shall again make use of the Fourier transform method to derive the result. Given the definition for the transform in equation (15), the Fourier transform of the PDE (32) with respect to x is given by

$$\frac{d\hat{V}_b}{d\tau} + \left(\frac{1}{2}\sigma^2\eta^2 + ki\eta + r\right)\hat{V}_b = \hat{\Phi}(\eta, \tau), \tag{59}$$

where

$$\hat{\Phi}(\eta, \tau) \quad \equiv \quad \mathcal{F}\left\{H(x - \ln b(\tau))\left[rv(x) - k\frac{dv(x)}{dx}\frac{1}{2}\sigma^2\frac{d^2 v(x)}{dx^2}\right]\right\}. \tag{60}$$

The ODE (59) is solved subject to the initial condition $\mathcal{F}\{V_b(x,0)\} \equiv \hat{V}_b(\eta, 0)$.

From Proposition 3.4, we know that the solution to (59) is

$$\hat{V}_b(\eta,\tau) = \hat{V}_b(\eta,0)e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)\tau} + \int_0^\tau e^{-(\frac{1}{2}\sigma^2\eta^2+ki\eta+r)(\tau-s)}\hat{\Phi}(\eta,s)ds. \tag{61}$$

The key differences between equations (23) and (61) are the functional forms of $F(\eta,s)$ and $\hat{\Phi}(\eta,s)$, and the fact that the solution in (23) is only valid for $x < \ln b(\tau)$, whereas (61) is valid for all real values of x.

Applying the inverse transform to (61), we can apply the results from Proposition 4.1 to show that

$$\begin{aligned} V_b(x,\tau) =& \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} v(u)e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du \\ &+ \int_0^\tau \int_{\ln b(s)}^{\infty}\left[rv(u) - k\frac{dv(u)}{du} - \frac{1}{2}\sigma^2\frac{d^2v(u)}{du^2}\right] \\ &\times \frac{e^{-r(\tau-s)}}{\sigma\sqrt{2\pi(\tau-s)}}e^{-\frac{(x-u+k(\tau-s))^2}{2\sigma^2(\tau-s)}}duds, \end{aligned} \tag{62}$$

which is the result given in Proposition 5.2. Given that (62) is the solution to an inhomogeneous PDE, we can identify the first integral term in (62) as the complimentary function component of the solution V_b, since this is the value of the corresponding European option, which we know satisfies the PDE (8) for all real x. The second integral term is therefore the particular integral component of solution V_b, generated by the inhomogeneous term in (31). Economic interpretations of these terms are provided in Section 5.

Appendix 6. Alternative Representations of the American Call Price

A6.1. **McKean's Representation for an American Call.** If we set $c(S) = \max(S - K, 0)$, this implies that $v(x) = \max(e^x - K, 0)$, and equation (25) becomes

$$\begin{aligned} V_b^{(1)}(x,\tau) &= \int_{\ln K}^{\ln b(0^+)} \frac{e^u e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du - K\int_{\ln K}^{\ln b(0^+)} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-u+k\tau)^2}{2\sigma^2\tau}}du \\ &\equiv I_1 - KI_2. \end{aligned}$$

To simplify $V_b^{(1)}(x,\tau)$ further, we shall re-express it in terms of the cumulative standard normal distribution, $N(y)$. For the first term, I_1, we can evaluate this using equation

(56) with $\alpha_1 = \ln K$, $\alpha_2 = \ln b(0^+)$, $\hat{a} = 1$, $\hat{q} = x + k\tau$ and $\hat{p} = 2\sigma^2\tau$. Recalling that $k = r - q - \frac{1}{2}\sigma^2$, and defining $d_1(x, \tau; \beta) \equiv (\ln(x/\beta) + (k + \sigma^2)\tau)/\sigma\sqrt{\tau}$, I_1 then becomes

$$I_1 \quad = \quad e^x e^{-q\tau}[N(d_1(e^x, \tau; K)) - N(d_1(e^x, \tau; b(0^+)))].$$

For the second term, I_2, by defining $d_2(x, \tau; \beta) \equiv (\ln(x/\beta) + k\tau)/\sigma\sqrt{\tau}$, the integral becomes

$$I_2 \quad = \quad e^{-r\tau}[N(d_2(e^x, \tau; K)) - N(d_2(e^x, \tau; b(0^+)))].$$

Thus it is concluded that

$$\begin{aligned} V_b^{(1)}(x, \tau) \quad &= \quad [e^x e^{-q\tau} N(d_1(e^x, \tau; K)) - Ke^{-r\tau} N(d_2(e^x, \tau; K))] \\ &\quad\quad -[e^x e^{-q\tau} N(d_1(e^x, \tau; b(0^+))) - Ke^{-r\tau} N(d_2(e^x, \tau; b(0^+)))] \end{aligned}$$

It is worth noting that in the case where $r \leq q$, $b(0) = K$, as proven by Kim (1990), and this in turn implies that $V_b^{(1)}(x, \tau) = 0$.

Having evaluated $V_b^{(1)}(x, \tau)$, it is a simple matter to evaluate $V_b^{(2)}(x, \tau)$ when $v(x) = \max(e^x - K, 0)$, and reverting back to the original underlying asset variable via $S = e^x$ we obtain (35).

A6.2. **Proof of Proposition 6.2.** There are two methods by which one can derive equation (37). The first, and most simple, is to set $v(x) = e^x - K$ in Jamshidian's general solution from (33). All that is then required is to apply the results from Appendix A1.2 and return to the original underlying variable S.

The second method involves manipulating McKean's solution from equation (35) as demonstrated by Kim (1990). We shall now provide the details of this approach, and point out that these manipulations do not readily extend to the class of general monotonic payoff functions $c(S)$ presented in Section 2. It remains unclear how the equivalence between the solutions of McKean (1965) and Jamshidian (1992) can be shown in the case of general payoff functions.

We begin by expressing equation (35) as

$$C_b(S,\tau) = C_E(S,\tau) - Se^{-q\tau}N(d_1(S,\tau;b(0^+))) + Ke^{-r\tau}N(d_2(S,\tau;b(0^+))) + R(S,\tau),$$

where

$$\begin{aligned} R(S,\tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} e^{-\hat{h}(S,\xi)} \\ &\qquad \times \left[\frac{\sigma^2 b(\xi)}{2} + \left(\frac{b'(\xi)}{b(\xi)} + \frac{1}{2}\left[k - \frac{\ln\frac{S}{b(\xi)}}{\tau-\xi} \right] \right)(b(\xi)-K) \right] d\xi. \end{aligned}$$

Following Kim (1990), we aim to remove the $b'(\xi)$ term from the integral $R(S,\tau)$. We begin by expressing $\hat{h}(S,\xi)$ as

$$\begin{aligned} \hat{h}(S,\xi) &= \frac{[\ln S - \ln b(\xi) + (r-q-\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)} \\ &= \frac{1}{2(\tau-\xi)}\left(\frac{\ln S + (r-q-\frac{1}{2}\sigma^2)\tau}{\sigma} - \frac{\ln b(\xi) + (r-q-\frac{1}{2}\sigma^2)\xi}{\sigma} \right)^2 \\ &= \frac{[x-P(\xi)]^2}{2(\tau-\xi)}, \end{aligned}$$

where $x \equiv [\ln S + (r-q-\frac{1}{2}\sigma^2)\tau]/\sigma$ and $P(\xi) \equiv [\ln b(\xi) + (r-q-\frac{1}{2}\sigma^2)\xi]/\sigma$. Note also that $P'(\xi) = \left(\frac{b'(\xi)}{b(\xi)} + (r-q-\frac{1}{2}\sigma^2) \right)/\sigma$. Thus $R(S,\tau)$ may be rewritten as

$$\begin{aligned} R(S,\tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi) - \frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma b(\xi)}{2} + \frac{1}{\sigma}\left(\frac{b'(\xi)}{b(\xi)} + (r-q-\frac{1}{2}\sigma^2) - (r-q-\frac{1}{2}\sigma^2) \right.\right. \\ &\qquad \left.\left. + \frac{1}{2}\left[(r-q-\frac{1}{2}\sigma^2) - \frac{\ln\frac{S}{b(\xi)}}{\tau-\xi} \right] \right)(b(\xi)-K) \right] d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} R(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma b(\xi)}{2} + \left(P'(\xi) - \frac{1}{\sigma}(r-q-\frac{1}{2}\sigma^2) \right.\right. \\ &\qquad \left.\left. - \frac{1}{2\sigma}\left[\frac{\ln S - \ln b(\xi) - (r-q-\frac{1}{2}\sigma^2)(\tau-\xi)}{\tau-\xi} \right] \right)(b(\xi)-K) \right] d\xi, \end{aligned}$$

which implies the linear decomposition

$$\begin{aligned} R(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma b(\xi)}{2} + \left(P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)} \right) (b(\xi)-K) \right] d\xi \\ &\equiv R_1(S,\tau) - KR_2(S,\tau), \end{aligned}$$

where

$$R_1(S,\tau) = \int_0^\tau e^{-r(\tau-\xi)} b(\xi) \frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[\frac{\sigma}{2} + P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)} \right] d\xi,$$

and

$$R_2(S,\tau) = \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \left[P'(\xi) - \frac{x-P(\xi)}{2(\tau-\xi)} \right] d\xi.$$

Beginning with $R_1(S,\tau)$ we have

$$\begin{aligned} R_1(S,\tau) &= \int_0^\tau e^{-r(\tau-\xi)} \frac{b(\xi)}{\sqrt{\tau-\xi}} \left[\frac{\sigma(\tau-\xi) + 2P'(\xi)(\tau-\xi) - x + P(\xi)}{2(\tau-\xi)} \right] \\ &\qquad \times \frac{1}{\sqrt{2\pi}} e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} e^{(x-P(\xi))\sigma + \frac{\sigma^2}{2}(\tau-\xi)} d\xi \\ &= -\int_0^\tau e^{-q(\tau-\xi)} S \frac{1}{\sqrt{2\pi}} e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\ &\qquad \times \left[\frac{\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}(x-P(\xi)+\sigma(\tau-\xi)) - (P'(\xi)+\sigma)\sqrt{(\tau-\xi)}}{(\sqrt{\tau-\xi})^2} \right] d\xi \\ &= -\int_0^\tau e^{-q(\tau-\xi)} S \frac{1}{\sqrt{2\pi}} e^{-\frac{[x-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \frac{\partial}{\partial\xi} \left[\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}} \right] d\xi \\ &= -\int_0^\tau e^{-q(\tau-\xi)} S \frac{\partial}{\partial\xi} N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}} \right) d\xi. \end{aligned}$$

Repeating this process for $R_2(S,\tau)$, we produce

$$\begin{aligned} R_2(S,\tau) &= -\int_0^\tau e^{-r(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}} \left[\frac{-\sqrt{\tau-\xi}P'(\xi) + (x-P(\xi))\frac{1}{2}\frac{1}{\sqrt{\tau-\xi}}}{(\sqrt{\tau-\xi})^2} \right] d\xi \\ &= -\int_0^\tau e^{-r(\tau-\xi)} \frac{1}{\sqrt{2\pi}} e^{-\frac{[x-P(\xi)]^2}{2(\tau-\xi)}} \frac{\partial}{\partial\xi} \left[\frac{x-P(\xi)}{\sqrt{\tau-\xi}} \right] d\xi \\ &= -\int_0^\tau e^{-r(\tau-\xi)} \frac{\partial}{\partial\xi} N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}} \right) d\xi. \end{aligned}$$

Substitution of R_1 and R_2 into $R(S,\tau)$ gives

$$\begin{aligned} R(S,\tau) &= -\int_0^\tau e^{-q(\tau-\xi)} S \frac{\partial}{\partial \xi} N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\ &\quad +K\int_0^\tau e^{-r(\tau-\xi)} \frac{\partial}{\partial \xi} N\left(\frac{x-P(\xi)}{2\sqrt{\tau-\xi}}\right) d\xi \\ &= -S\Bigg\{ \left[e^{-q(\tau-\xi)} N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)\right]_0^\tau \\ &\qquad\qquad - \int_0^\tau q e^{-q(\tau-\xi)} N\left(\frac{x-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \Bigg\} \\ &\quad +K\left\{ \left[e^{-r(\tau-\xi)} N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right)\right]_0^\tau - \int_0^\tau r e^{-r(\tau-\xi)} N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right) d\xi \right\}, \end{aligned}$$

where the manipulations follow from an application of integration by parts. In order to further evaluate $R(S,\tau)$, we require the limit result that

$$\lim_{\xi\to\tau} \frac{\ln \frac{S}{b(\xi)}}{\sqrt{\tau-\xi}} = \begin{cases} -\infty, & S < b(\tau), \\ 0, & S = b(\tau), \end{cases}$$

since the equation for $C_b(S,\tau)$ must be satisfied for $0 < S \leq b(\tau)$ for a live American call. If we introduce a special point-indicator function defined by

$$\mathbf{1}_{S=b(\tau)} \equiv \begin{cases} \frac{1}{2}, & S = b(\tau), \\ 0, & \text{otherwise}, \end{cases}$$

then further evaluation of $R(S,\tau)$ produces

$$\begin{aligned} R(S,\tau) &= -S\left\{ \mathbf{1}_{S=b(\tau)} - e^{-q\tau} N\left(\frac{x-P(0)+\sigma\tau}{\sqrt{\tau}}\right) \right. \\ &\qquad\qquad \left. - \int_0^\tau q e^{-q(\tau-\xi)} N\left(\frac{x-P(\xi)+(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \right\} \\ &\quad +K\left\{ \mathbf{1}_{S=b(\tau)} - e^{-r\tau} N\left(\frac{x-P(0)}{\sqrt{\tau}}\right) - \int_0^\tau r e^{-r(\tau-\xi)} N\left(\frac{x-P(\xi)}{\sqrt{\tau-\xi}}\right) d\xi \right\} \\ &= Se^{-q\tau} N(d_1(S,\tau;b(0^+))) - Ke^{-r\tau} N(d_2(S,\tau;b(0^+))) - \mathbf{1}_{S=b(\tau)}(S-K) \\ &\quad + \int_0^\tau \Big[qSe^{-q(\tau-\xi)} N(d_1(S,\tau-\xi;b(\xi))) \\ &\qquad\qquad - rKe^{-r(\tau-\xi)} N(d_2(S,\tau-\xi;b(\xi))) \Big] d\xi. \end{aligned}$$

If we then substitute $R(S,\tau)$ into the expression for $C_b(S,\tau)$, the American call price is

$$\begin{aligned} H(b(\tau)-S)C_b(S,\tau) &= C_E(S,\tau) - \mathbf{1}_{S=b(\tau)}(S-K) \\ &\quad + \int_0^\tau qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi, \end{aligned}$$

where $0 < S \leq b(\tau)$. When S is strictly less than $b(\tau)$, this can be written more simply as equation (37) in Proposition 4.2. Furthermore, if we evaluate C_b at $S = b(\tau)$, we find that

$$\begin{aligned} \frac{1}{2}(b(\tau)-K) &= C_E(b(\tau),\tau) - \frac{1}{2}(b(\tau)-K) \\ &\quad + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi, \end{aligned}$$

which simplifies to

$$\begin{aligned} b(\tau)-K &= C_E(b(\tau),\tau) + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi, \end{aligned}$$

which is equation (38), and allows us to see why the factor of $\frac{1}{2}$ is no longer present when evaluating Kim's representation at the free boundary.

A6.3. **Proof of Proposition 6.3.** Here we present an alternative method of deriving the American call option representation given in equation (39), based on the appendix of Carr et al. (1992). In particular, our derivation of this result demonstrates how one can reproduce the Carr-Jarrow-Myneni representation directly from Kim's (1990) form. Taking the European call price, $C_E(S,\tau)$, we can write

$$C_E(S,\tau) = SH(S-K) - SH(S-K) + Se^{-q\tau}N(d_1(S,\tau;K)) - Ke^{-r\tau}N(d_2(S,\tau;K)).$$

Given the limit result that

$$\lim_{\tau\to 0} d_1(S,\tau;K) = \lim_{\tau\to 0} d_2(S,\tau;K) = \begin{cases} \infty, & S > K \\ 0, & S = K \\ -\infty, & S < K \end{cases}$$

we can show that

$$SH(S-K) = [Se^{qs}N(d_1(S,s;K)]_{s=0},$$

and thus we can express $C_E(S,\tau)$ as

$$\begin{aligned}
C_E(S,\tau) &= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) \\
&\quad +[Se^{-qs}N(d_1(S,s;K))]_{s=\tau} - [Se^{-qs}N(d_1(S,s;K))]_{s=0} \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) \\
&\quad +S\int_0^\tau \left[N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_1(S,s;K)]e^{-qs} - qN(d_1(S,s;K))e^{-qs}\right] ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)+\sigma\sqrt{s}]ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\left[\frac{\partial}{\partial s}[d_2(S,s;K)] + \frac{\sigma}{2\sqrt{s}}\right] ds \\
&= SH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) - qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds \\
&\quad +S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\sigma}{2\sqrt{s}}ds.
\end{aligned}$$

Noting that $N'(d_1(S,s;K)) = Ke^{-(r-q)s}N'(d_2(S,s;K))/S$, we have

$$\begin{aligned}
C_E(S,\tau) &= (S-K)H(S-K) + KH(S-K) - Ke^{-r\tau}N(d_2(S,\tau;K)) \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds + S\int_0^\tau e^{-qs}N'(d_1(S,s;K))\frac{\sigma}{2\sqrt{s}}ds \\
&\quad +K\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds \\
&= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau \frac{e^{-qs}}{\sigma\sqrt{s}}N'(d_1(S,s;K))ds \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad -K\Bigg\{e^{-r\tau}N(d_2(S,\tau;K)) - H(S-K) \\
&\qquad -\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds\Bigg\} \\
&= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau \frac{e^{-qs}}{\sigma\sqrt{s}}N'(d_1(S,s;K))ds \\
&\quad -qS\int_0^\tau e^{-qs}N(d_1(S,s;K))ds \\
&\quad -K\Bigg\{\left[e^{-rs}N(d_2(S,s;K))\right]_0^\tau \\
&\qquad -\int_0^\tau e^{-rs}N'(d_2(S,s;K))\frac{\partial}{\partial s}[d_2(S,s;K)]ds\Bigg\},
\end{aligned}$$

where the last line follows by use of the previous limit result for d_2. After changing the integration variable to $s = \tau - \xi$, we can represent the European call price as

$$\begin{aligned}
C_E(S,\tau) &= \max(S-K,0) + \frac{S\sigma^2}{2}\int_0^\tau e^{-q(\tau-\xi)}N'(d_1(S,\tau-\xi;K))d\xi \\
&\quad -qS\int_0^\tau e^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;K))d\xi \\
&\qquad +rK\int_0^\tau e^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;K))d\xi,
\end{aligned}$$

and substituting this into (37) will yield equation (39) of Proposition 6.3, following a simple rearrangement of terms.

Appendix 7. Value of the American Call Free Boundary at Expiry

We shall consider two methods for deriving equation (41). The first is based on the analysis of Wilmott et al. (1993). They show that by considering the behaviour of a transformed version of the free boundary value problem given by (8)-(12) for sufficiently small values of τ, one can develop a thorough understanding of the value of $b(\tau)$ as $\tau \to 0^+$. Here we present an intuitive means by which (41) can be determined based on their findings.

Consider the Jamshidian representation of the free boundary value problem for V_b, given by (32). Wilmott et al. (1993) showed that for sufficiently small values of τ, one can determine the value of the early exercise boundary by finding the value of x for which the inhomogeneous term is zero in the PDE for $U_b(x,\tau) = V_b(x,\tau) - e^x + 1$. By virtue of their analysis, a quick way to find the value of $b(0^+)$ is to evaluate the inhomogeneous term in PDE (32) at $x = \ln b(0^+)$ and set this equal to zero. Thus we have

$$H(0)(qb(0^+) - rK) = 0,$$

which simplifies to $b(0^+) = Kr/q$. Since it is never optimal to exercise the American call when $S < K$, we also know that $b(0^+) > K$ must always be true. Combining these leads us to equation (41).

The second derivation method was presented by Kim (1990). For this approach it is necessary to analyse the limit of equation (38) as τ tends to 0^+. Using the method outlined by Kim (1990), we begin by considering

$$\begin{aligned} b(\tau) - K \quad = \quad & b(\tau)e^{-q\tau}N(d_1(b(\tau),\tau;K)) - Ke^{-r\tau}N(d_2(b(\tau),\tau;K)) \\ & + \int_0^\tau qb(\tau)e^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi \\ & - \int_0^\tau rKe^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi. \end{aligned}$$

This equation can be factorised to produce

$$\begin{aligned} b(\tau)&\left\{1-e^{-q\tau}[N(d_1(b(\tau),\tau;K))-\int_0^\tau qe^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\right\} \\ &=K\left\{1-e^{-r\tau}N(d_2(b(\tau),\tau;K))-\int_0^\tau re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))]d\xi\right\}, \end{aligned}$$

which then yields the following implicit equation for $b(\tau)$:

$$\begin{aligned} \frac{b(\tau)}{K}=&\left(1-e^{-r\tau}N(d_2(b(\tau),\tau;K))-\int_0^\tau re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi\right) \\ &\times\left(1-e^{-q\tau}N(d_1(b(\tau),\tau;K))-\int_0^\tau qe^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi\right)^{-1}. \end{aligned} \tag{63}$$

Before proceeding further, we again note that $b(\tau)\geq K$. To find the value of $b(0^+)$, we take the limit of equation (63) as τ tends to 0^+. In order to evaluate this limit, we need to find two limits involving d_1 and d_2. The first to consider is

$$\lim_{\tau\to0^+}d_2(b(\tau),\tau;K)=\lim_{\tau\to0^+}\frac{\ln\frac{b(\tau)}{K}}{\sigma\sqrt{\tau}}=\begin{cases}0, & b(0^+)=K\\ \infty, & b(0^+)>K.\end{cases} \tag{64}$$

Similarly the following limit for d_1 can be shown to be

$$\lim_{\tau\to0^+}d_1(b(\tau),\tau;K)=\begin{cases}0, & b(0^+)=K\\ \infty, & b(0^+)>K.\end{cases} \tag{65}$$

Note also that $N(0)=0.5$ and $N(\infty)=1$. Given that the limits (64) and (65) depend on the value of $b(0^+)$ relative to K, there are two cases to consider when finding the limit of equation (63). Consider the first case where $b(0^+)=K$. Taking the limit of equation (63) as τ tends to 0^+, and using the results from equations (64)-(65), we obtain

$$\lim_{\tau\to0^+}\frac{b(\tau)}{K}=1, \tag{66}$$

and thus $b(0^+)=K$ is one possible solution for $b(0^+)$.

Now consider the second case, where $b(0^+)>K$. The limit as τ tends to zero of equation (63) is now of the form $\frac{0}{0}$, and therefore L'Hopital's rule can be applied. Firstly, let

$$\lim_{\tau\to0^+}\frac{b(\tau)}{K}=\lim_{\tau\to0^+}\frac{\hat{N}(\tau)}{\hat{D}(\tau)},$$

where

$$\begin{aligned}\hat{N}(\tau) &\equiv 1-e^{-r\tau}N(d_2(b(\tau),\tau;K))\\ &\quad -\int_0^\tau re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

and

$$\begin{aligned}\hat{D}(\tau) &\equiv 1-e^{-q\tau}N(d_1(b(\tau),\tau;K))\\ &\quad -\int_0^\tau qe^{-q(\tau-\xi)}N(d_1(b(\tau),\tau-\xi;b(\xi)))d\xi.\end{aligned}$$

To apply L'Hopital's rule, we must differentiate both $\hat{N}(\tau)$ and $\hat{D}(\tau)$ with respect to τ, and take their limits as τ tends to 0^+. For $\hat{N}(\tau)$ we have

$$\begin{aligned}\hat{N}'(\tau) &= re^{-r\tau}N(d_2(b(\tau),\tau;K))-e^{-r\tau}N'(d_2(b(\tau),\tau;K))\frac{\partial}{\partial\tau}[d_2(b(\tau),\tau;K)]\\ &\quad -rN(d_2(b(\tau),0;b(\tau)))\\ &\quad -r\int_0^\tau\{-re^{-r(\tau-\xi)}N(d_2(b(\tau),\tau-\xi;b(\xi)))\\ &\quad\quad +e^{-r(\tau-\xi)}N'(d_2(b(\tau),\tau-\xi;b(\xi)))\frac{\partial}{\partial\tau}[d_2(b(\tau),\tau-\xi;b(\xi))]\}d\xi,\end{aligned}$$

Note that as $x\to\infty$, $N'(x)\to 0$ at a faster rate than any other terms observed in $\hat{N}'(\tau)$ (see Kim, 1990). We also note that

$$\lim_{\xi\to\tau} d_2(b(\tau),\tau-\xi;b(\xi))=0.$$

Combining all these limit results, it is concluded that

$$\lim_{\tau\to 0^+}\hat{N}'(\tau)=-\frac{r}{2}. \tag{67}$$

Similarly for $\hat{D}'(\tau)$ it can be shown that

$$\lim_{\tau\to 0^+}\hat{D}(\tau)=-\frac{q}{2}. \tag{68}$$

Thus it is concluded that

$$\lim_{\tau\to 0^+}\frac{b(\tau)}{K}=\frac{r}{q}. \tag{69}$$

Recalling that this result only holds when $b(0^+) > K$, it follows that we must have $r > q$. Finally, combining the results from equations (66) and (69) gives

$$\lim_{\tau \to 0^+} b(\tau) = K \max\left(1, \frac{r}{q}\right),$$

which is equation (41) of Proposition 6.4.

Appendix 8. Value of the Perpetual American Call

We begin by considering a solution to the PDE (42) of the form $C_b(S, \infty) = S^\alpha$. Substituting this into (42) we have

$$\frac{1}{2}\sigma^2\alpha^2 + k\alpha - r = 0, \tag{70}$$

where $k = r - q - \frac{1}{2}\sigma^2$. The quadratic equation (70) has solution

$$\alpha_\pm = \frac{-k \pm \sqrt{k^2 + 2\sigma^2 r}}{\sigma^2}, \tag{71}$$

and thus the general solution to (42) is

$$C_b(S, \infty) = AS^{\alpha_+} + BS^{\alpha_-}, \tag{72}$$

where A and B are constants.

Since $2\sigma^2 r \geq 0$, we have the relationship that

$$\sqrt{k^2 + 2\sigma^2 r} \geq |k|,$$

and it follows that $\alpha_+ \geq 0$ and $\alpha_- \leq 0$. Thus in order to satisfy boundary condition (43), we must set $B = 0$. To determine A we make use of boundary condition (44) and find that $A = (b - K)/b^{\alpha_+}$. Thus the solution for a general early exercise strategy $b(\infty)$ is

$$C_b(S, \infty) = (b - K)\left(\frac{S}{b}\right)^{\alpha_+}. \tag{73}$$

At this stage we still have not determined the early exercise strategy. The optimal strategy is to select b such that $C_b(S, \infty)$ is maximised. Setting $dC_b/db = 0$ we find that the optimal exercise strategy is

$$b^* = \frac{\alpha_+ K}{\alpha_+ - 1}, \tag{74}$$

and thus the optimal value of the perpetual call becomes

$$C_{b^*}(S, \infty) = \frac{K}{\alpha_+ - 1}\left(\frac{(\alpha_+ - 1)S}{\alpha_+ K}\right)^{\alpha_+}. \tag{75}$$

Finally, to derive the smooth-pasting condition, we evaluate dC_{b^*}/dS at $S = b^*$ which gives

$$\begin{aligned}
\lim_{S \to b^*} \frac{dC_{b^*}}{dS} &= \left(\frac{\alpha_+ - 1}{\alpha_+ K}\right)^{\alpha_+ - 1} (b^*)^{\alpha_+ - 1} \\
&= \left(\frac{\alpha_+ - 1}{\alpha_+ K}\right)^{\alpha_+ - 1} \left(\frac{\alpha_+ K}{\alpha_+ - 1}\right)^{\alpha_+ - 1} \\
&= 1.
\end{aligned}$$

Appendix 9. Induction Proof for the American Call Option Price

The details of this appendix are drawn from the proofs presented in Kim (1990).

A9.1. **Proof of Proposition 7.1.**

Given that

$$\hat{C}_b(S_1, \Delta\tau) = \begin{cases} U(S_1, \Delta\tau; K), S_1 < b_1, \\ S_1 - K, \qquad S_1 \geq b_1, \end{cases}$$

the value of $U(S_2, 2\Delta\tau; b_1)$ is

$$\begin{aligned}
U(S_2, 2\Delta\tau; b_1) &= \int_0^{b_1} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} (S_1 - K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\
&= \int_0^{\infty} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\
&\quad - \int_{b_1}^{\infty} e^{-r\Delta\tau} U(S_1, \Delta\tau; K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau} (S_1, -K) p(S_1, \Delta\tau | S_2, 2\Delta\tau) dS_1 \\
&= C_E(S_2, 2\Delta\tau) + I(b_1),
\end{aligned}$$

where

$$I(b_1) \equiv \int_{b_1}^{\infty} e^{-r\Delta\tau}[S_1 - K - C_E(S_1, \Delta\tau)]p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1.$$

Consider $I(b_1)$, which can be manipulated to produce

$$\begin{aligned}
I(b_1) &= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[\int_{K}^{\infty} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0\right] \\
&\qquad \times p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty} \left\{e^{-r\Delta\tau}\left[\int_{0}^{\infty} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0\right.\right. \\
&\qquad \left.\left. - \int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0\right] p(S_1, \Delta\tau|S_2, 2\Delta\tau)\right\} dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}(S_1 - K)p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&\quad - \int_{b_1}^{\infty} \left\{e^{-r\Delta\tau}\left[e^{-r\Delta\tau}(S_1 e^{(r-q)\Delta\tau} - K)\right.\right. \\
&\qquad \left.\left. - \int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0\right] p(S_1, \Delta\tau|S_2, 2\Delta\tau)\right\} dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[(S_1 - K) - S_1 e^{-q\Delta\tau} + Ke^{-r\Delta\tau}\right] p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[\int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0)\right] \\
&\qquad \times p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&= \int_{b_1}^{\infty} e^{-r\Delta\tau}\left[(1 - e^{-q\Delta\tau})S_1 - (1 - e^{-r\Delta\tau})K\right] p(S_1, \Delta\tau|S_2, 2\Delta\tau)dS_1 \\
&\quad + \int_{b_1}^{\infty} e^{-r\Delta\tau}p(S_1, \Delta\tau|S_2, 2\Delta\tau) \\
&\qquad \times \left[\int_{0}^{K} e^{-r\Delta\tau}(S_0 - K)p(S_0, 0|S_1, \Delta\tau)dS_0)\right] dS_1.
\end{aligned}$$

Let L_1 be defined as

$$\begin{aligned} L_1 &\equiv \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_1, \Delta\tau | S_2, 2\Delta\tau) \left[\int_0^K e^{-r\Delta\tau}(K - S_0) p(S_0, 0 | S_1, \Delta\tau) dS_0 \right] dS_1 \\ &= \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_1, \Delta\tau | S_2, 2\Delta\tau) P_E(S_1, \Delta\tau) dS_1, \end{aligned}$$

where $P_E(S_1, \Delta\tau) = Ke^{-r\Delta\tau} N(-d_2(S_1, \Delta\tau; K)) - S_1 e^{-q\Delta\tau} N(-d_1(S_1, \Delta\tau; K))$, which is the price of a European put written on S with strike K. Since $P_E(S_1, \Delta\tau)$ is a decreasing function of S_1, an upper bound for L_1 is

$$L_1 < \int_{b_1}^{\infty} e^{-r\Delta\tau} p(S_2, 2\Delta\tau | S_1, \Delta\tau) P_E(b_1, \Delta\tau) dS_1,$$

which evaluates to

$$L_1 < e^{-r\Delta\tau} N(d_2(S_2, \Delta\tau; b_1)) P_E(b_1, \Delta\tau).$$

Note that as $(\tau_2 - \tau_1) \to 0$, $b_2 \to b_1$. Since $S_2 < b_2$ for an unexercised call we find that

$$\lim_{\Delta\tau \to 0} N(d_2(S_2, \Delta\tau; b_1)) = 0.$$

For $P_E(b_1, \Delta\tau)$, since $b_1 \geq K$, we have

$$\lim_{\Delta\tau \to 0} \frac{\ln \frac{K}{b_1}}{\sqrt{\Delta\tau}} = \begin{cases} -\infty, & b_1 > K, \\ 0, & b_1 = K. \end{cases}$$

In either case $\lim_{\Delta\tau \to 0} P_E(b_1, \Delta\tau) = 0$, and hence the term L_1 is of $o(\Delta\tau)$.

The price of the unexercised call at time to maturity $2\Delta\tau$ is therefore

$$\begin{aligned} U(S_2, 2\Delta\tau; b_1) = C_E(S_2, 2\Delta\tau) + o(\Delta\tau) \\ + \int_{b_1}^{\infty} e^{-r\Delta\tau} \left[(1 - e^{-q\Delta\tau}) S_1 - (1 - e^{-r\Delta\tau}) K \right] p(S_1, \Delta\tau | S_2, \Delta\tau) dS_1, \end{aligned}$$

as given in equation (51).

A9.2. **Proof of Proposition 7.2.** To find the price of the unexercised call at a general time step $n\Delta\tau$, Kim (1990) uses an indication proof. Assume that the unexercised call price at time $m\Delta\tau$ is given by

$$\begin{aligned} U(S_m, m\Delta\tau, b_{m-1}) &= \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} \left[(1-e^{-q\Delta\tau})S_k - (1-e^{-r\Delta\tau})K\right] \\ &\qquad\qquad \times p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\ &\quad + C_E(S_m, m\Delta\tau) + o(m\Delta\tau), \end{aligned}$$

as stated in equation (52). It is simple to show that this holds for $m = 2$ (see Appendix A9.1). We must now prove that this relationship holds for $m + 1$.

Given that

$$b_m - K = U(b_m, m\Delta\tau; b_{m-1}),$$

the value of the unexercised call at $(m + 1)\Delta\tau$ is

$$\begin{aligned} &U(S_{m+1}, (m+1)\Delta\tau; b_m) \\ &\quad = \int_0^{b_m} e^{-r\Delta\tau} U(S_m, m\Delta\tau; b_{m-1}) p(S_m, m\Delta\tau | S_{m+1}(m+1)\Delta\tau) dS_m \\ &\qquad + \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m \\ &\quad = \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K - U(S_m, m\Delta\tau; b_{m-1})) \\ &\qquad\qquad \times p(S_m, m\Delta\tau)|S_{m+1}, (m+1)\Delta\tau) dS_m \\ &\qquad + \int_0^{\infty} e^{-r\Delta\tau} U(S_m, m\Delta\tau, b_{m-1}) \\ &\qquad\qquad \times p(S_m, m\Delta\tau | S_{m+1}(m+1)\Delta\tau) dS_m \\ &\quad \equiv U_1 + U_2. \end{aligned}$$

Consider firstly the term U_2, which can be simplified as

$$\begin{aligned}
U_2 &= \int_0^\infty e^{-r\Delta\tau} C_E(S_m, m\Delta\tau) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m + o(m\Delta\tau) \\
&\quad + \int_0^\infty e^{-r\Delta\tau} \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^\infty \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K \right] \\
&\qquad \times p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\
&\qquad\quad \times p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m \\
&= C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) \\
&\quad + \sum_{k=1}^{m-1} \int_{b_k}^\infty e^{-(m-k+1)r\Delta\tau} \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K \right] \\
&\qquad \times \int_0^\infty p(S_k, k\Delta\tau | S_m, m\Delta\tau) p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m dS_k \\
&= C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) \\
&\quad + \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_k}^\infty \left[(1 - e^{-q\Delta\tau}) S_k - (1 - e^{-r\Delta\tau}) K \right] \\
&\qquad\qquad \times p(S_k, k\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_k
\end{aligned}$$

Next consider the term U_1. Extensive manipulations yield

$$\begin{aligned}
U_1 &= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}C_E(S_m, m\Delta\tau)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + \int_{b_m}^{\infty} e^{-r\Delta\tau}\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\Bigg[\int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right) \\
&\quad \times p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k\Bigg]p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\int_{b_k}^{\infty}(S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k \\
&\quad \times p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m + o(m\Delta\tau) \\
&= \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad - \int_{b_m}^{\infty} e^{-r\Delta\tau}C_E(S_m, m\Delta\tau)p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau)dS_m \\
&\quad + \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - \sum_{k=1}^{m-2} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}(S_k - K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m \\
&\quad - e^{2r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_{m-1}}^{\infty}(S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}dS_m \\
&\quad + o(m\Delta\tau).
\end{aligned}$$

If we set

$$\begin{aligned}
L_m^{(1)} &\equiv \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_m}^{\infty} p(S_m, m\Delta\tau|S_{m+1}, (m+1)\Delta\tau) \\
&\quad \times \int_{b_k}^{\infty}\left(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K\right)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k dS_m,
\end{aligned}$$

then further manipulations yield

$$
\begin{aligned}
U_1 \;=\;& \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau)dS_m \\
& -\int_{b_m}^{\infty} e^{-r\Delta\tau} \int_{K}^{\infty} e^{-rm\Delta\tau}(S_0 - K)p(S_0, 0 | S_m, m\Delta\tau)dS_0 \\
& \quad \times p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau)dS_m \\
& + L_m^{(1)} - \sum_{k=1}^{m-2} e^{-(m-k+1)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\
& \quad \times \int_{b_k}^{\infty} (S_k - K)p(S_k, k\Delta\tau | S_m, m\Delta\tau)dS_k dS_m \\
& - e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\
& \quad \times \int_{0}^{\infty} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau)dS_{m-1}dS_m \\
& + e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\
& \quad \times \int_{0}^{b_{m-1}} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau)dS_{m-1}dS_m \\
& \qquad + o(m\Delta\tau) \\
=\;& \int_{b_m}^{\infty} e^{-r\Delta\tau}(S_m - K)p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau)dS_m \\
& - \sum_{k=0}^{m-2} e^{-(m-k+1)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\
& \quad \times \int_{b_k}^{\infty} (S_k - K)p(S_k, k\Delta\tau | S_m, m\Delta\tau)dS_k dS_m \\
& - e^{-2r\Delta\tau} \int_{b_m}^{\infty} (S_m e^{(r-q)\Delta\tau} - K)p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau)dS_m \\
& \quad + L_m^{(1)} + L_m^{(2)} + o(m\Delta\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
L_m^{(2)} \;\equiv\;& e^{-2r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\
& \qquad \times \int_{0}^{b_{m-1}} (S_{m-1} - K)p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau)dS_{m-1}dS_m,
\end{aligned}
$$

and we use the notation $b_0 \equiv b(0) = K$. Thus if we set

$$\begin{aligned} L_m^{(3)} \equiv & -\sum_{k=1}^{m-1} e^{-(m-k+2)r\Delta\tau} \int_{b_m}^{\infty} p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) \\ & \times \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} dS_m, \end{aligned}$$

U_1 becomes

$$\begin{aligned} U_1 = & \int_{b_m}^{\infty} e^{-r\Delta\tau}[(1 - e^{-q\Delta\tau})S_m - (1 - e^{-r\Delta\tau})K] p(S_m, m\Delta\tau | S_{m+1}, (m+1)\Delta\tau) dS_m \\ & + L_m^{(1)} + L_m^{(2)} + L_m^{(3)} + o(m\Delta\tau), \end{aligned}$$

and defining $L_m \equiv L_m^{(1)} + L_m^{(2)} + L_m^{(3)}$, $U(S_{m+1}, (m+1)\Delta\tau; b_m)$ reduces to

$$\begin{aligned} U(S_{m+1}, (m+1)\Delta\tau; b_m) = & C_E(S_{m+1}, (m+1)\Delta\tau) + o(m\Delta\tau) + L_m \\ & + \sum_{k=1}^{m} e^{-(m-k+1)r\Delta\tau} \int_{b_k}^{\infty} [(1 - e^{-q\Delta\tau})S_k - (1 - e^{-r\Delta\tau})K] \\ & \times p(S_k, k\Delta\tau | S_{m+1}, (m+1)\Delta\tau) S_k. \end{aligned}$$

All that remains is to prove that L_m is of $o(\Delta\tau)$. We begin by noting that when $S_m = b_m$, $U(S_m, m\Delta\tau; b_{m-1})$ becomes

$$\begin{aligned} b_m - K = & \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} [(1 - e^{-q\Delta\tau})S_k - (1 - e^{-r\Delta\tau})K] \\ & \times p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k \\ & + C_E(b_m, m\Delta\tau) + o(m\Delta\tau) \\ = & \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (S_k - K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k \\ & - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k \\ & + e^{-rm\Delta\tau} \int_{K}^{\infty} (S_0 - K) p(S_0, 0 | b_m, m\Delta\tau) dS_0 + o(m\Delta\tau), \end{aligned}$$

and thus

$$\begin{aligned}
b_m - K \quad = \quad & \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | b_m, m\Delta\tau) dS_{k-1} \\
& - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | b_m, m\Delta\tau) dS_k \\
& + e^{-r\Delta\tau} \int_{b_{m-1}}^{\infty} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | b_m, m\Delta\tau) dS_{m-1} \\
& + o(m\Delta\tau),
\end{aligned}$$

where again $b_0 \equiv b(0) = K$. If we take $S_m > b_m$, we have

$$\begin{aligned}
S_m - K \quad > \quad & \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} \\
& - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\
& + e^{-r\Delta\tau} \int_{0}^{\infty} (S_{m-1} - K) p(S_{m-1}(m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1} \\
& - e^{-r\Delta\tau} \int_{0}^{b_{m-1}} (S_{m-1} - K) p(S_{m-1}(m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1}.
\end{aligned}$$

Thus we arrive at the inequality

$$\begin{aligned}
& S_m - K - e^{-r\Delta\tau} \int_{0}^{\infty} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1} \\
& \quad > \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau} \int_{b_{k-1}}^{\infty} (S_{k-1} - K) p(S_{k-1}, (k-1)\Delta\tau | S_m, m\Delta\tau) dS_{k-1} \\
& \quad - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau} \int_{b_k}^{\infty} (e^{-q\Delta\tau} S_k - e^{-r\Delta\tau} K) p(S_k, k\Delta\tau | S_m, m\Delta\tau) dS_k \\
& \quad - e^{-r\Delta\tau} \int_{0}^{b_{m-1}} (S_{m-1} - K) p(S_{m-1}, (m-1)\Delta\tau | S_m, m\Delta\tau) dS_{m-1}.
\end{aligned}$$

Rewriting L_m gives

$$\begin{aligned}L_m &= e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{\sum_{k=1}^{m-1} e^{-(m-k)r\Delta\tau}\int_{b_k}^{\infty}(e^{-q\Delta\tau}S_k - e^{-r\Delta\tau}K)p(S_k, k\Delta\tau|S_m, m\Delta\tau)dS_k\right.\\ &\quad + \int_0^{b_{m-1}} e^{-r\Delta\tau}(S_{m-1}-K)p(S_{m-1}(m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}\\ &\quad \left.-\sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta\tau}\int_{b_{k-1}}^{\infty}(S_{k-1}-K)p(S_{k-1},(k-1)\Delta\tau|S_m, m\Delta\tau)dS_{k-1}\right\}\\ &\quad \times p(S_m, m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m,\end{aligned}$$

which, by use of the previous inequality, becomes

$$\begin{aligned}-L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{S_m - K - e^{-r\Delta\tau}\int_0^{\infty}(S_{m-1}-K)\right.\\ &\qquad \left.\times p(S_{m-1},(m-1)\Delta\tau|S_m, m\Delta\tau)dS_{m-1}\right\}\\ &\qquad \times p(S_m, m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m,\\ -L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{S_m - K - e^{-r\Delta\tau}(S_m e^{(r-q)\Delta\tau} - K)\right.\\ &\qquad \left.\times p(S_m, m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m\right\}\\ -L_m &< e^{-r\Delta\tau}\int_{b_m}^{\infty}\left\{(1-e^{-q\Delta\tau})S_m - (1-e^{-r\Delta\tau})K\right\},\\ &\qquad \times p(S_m, m\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_m,\\ -L_m &< (1-e^{-q\Delta\tau})S_{m+1}e^{-q\Delta\tau}N(d_1(S_{m+1},\Delta\tau;b_m)),\\ &\qquad -(1-e^{-r\Delta\tau})Ke^{-r\Delta\tau}N(d_2(S_{m+1},\Delta\tau;b_m)).\end{aligned}$$

We now consider the limit of this bound for $|L_m|$ as $\Delta\tau \to 0$. Firstly we note that

$$\lim_{\Delta\tau\to 0}(1-e^{-q\Delta\tau}) = \lim_{\Delta\tau\to 0}(1-e^{-r\Delta\tau}) = 0.$$

As $\Delta\tau \to 0$, $b_m \to b_{m+1}$, and since $S_{m+1} < b_{m+1}$ for an unexercised American call, we have

$$\lim_{\Delta\tau\to 0} N(d_1(S_{m+1},\Delta\tau;b_m)) = \lim_{\Delta\tau\to 0} N(d_2(S_{m+1},\Delta\tau;b_m)) = 0,$$

and thus $|L_m| \to 0$ as $\Delta\tau \to 0$. Hence L_m is of $o(\Delta\tau)$ and $U(S_{m+1},(m+1)\Delta\tau;b_m)$ is given by

$$\begin{aligned} U(S_{m+1},(m+1)\Delta\tau;b_m) &= C_E(S_{m+1},(m+1)\Delta\tau)+o((m+1)\Delta\tau) \\ &\quad +\sum_{k=1}^{m} e^{-(m-k+1)r\Delta\tau}\int_{b_k}^{\infty}[(1-e^{-q\Delta\tau})S_k-(1-e^{-r\Delta\tau})K] \\ &\quad \times p(S_k,k\Delta\tau|S_{m+1},(m+1)\Delta\tau)dS_k, \end{aligned}$$

and equation (52) is satisfied for $n = m+1$, completing the induction proof.

A9.3. **Proof of Proposition 7.3.** An obvious use of equivalent notation in (52) produces

$$\begin{aligned} \hat{C}_b(S_n,n\Delta\tau) &= C_E(S_n,n\Delta\tau)+o(n\Delta\tau) \\ &\quad +\sum_{k=1}^{n-1} e^{-(n-k)r\Delta\tau}\int_{b_k}^{\infty}[(1-e^{-q\Delta\tau})S_k-(1-e^{-r\Delta\tau})K] \\ &\quad \times p(S_k,k\Delta\tau|S_n,n\Delta\tau)dS_k, \end{aligned}$$

which is the unexercised American call option price with n discrete early exercise dates, occurring after every time step $\Delta\tau$. Using Taylor series we have

$$(1-e^{-\alpha\Delta\tau}) = \alpha\Delta\tau + o(\Delta\tau),$$

where α is a constant, and thus the price becomes

$$\begin{aligned} \hat{C}_b(S_n,n\Delta\tau) &= C_E(S_n,n\Delta\tau)+o(n\Delta\tau) \\ &\quad +\sum_{k=1}^{n-1} e^{-(n-k)r\Delta\tau}\int_{b_k}^{\infty}[qS_k-rK]p(S_k,k\Delta\tau|S_n,n\Delta\tau)dS_k\Delta\tau. \end{aligned}$$

Finally, to find the value of the continuous American call, we set $n\Delta\tau = \tau, S_n = S$, and take the limit as $\Delta\tau \to 0$ to produce

$$\begin{aligned}\hat{C}_b(S,\tau) &= C_E(S,\tau) + \int_0^{\tau} e^{-r(\tau-\xi)} \int_{b(\xi)}^{\infty} (qS_\xi - rK)p(S_\xi,\xi|S,\tau)dS_\xi d\xi \\ &= C_E(S,\tau) + \int_0^{\tau} qSe^{-q(\tau-\xi)}N(d_1(S,\tau-\xi;b(\xi)))d\xi \\ &\quad - \int_0^{\tau} rKe^{-r(\tau-\xi)}N(d_2(S,\tau-\xi;b(\xi)))d\xi,\end{aligned}$$

which is equation (37) of Proposition 6.2.

REFERENCES

AitSahlia, F. & Carr, P. (1997), American Options: A Comparison of Numemerical Methods, *in* L. Rogers & D. Talay, eds, 'Numerical Methods in Finance', Cambridge University Press, pp. 67–87.

Black, F. & Scholes, M. (1973), 'The Pricing of Corporate Liabilites', *Journal of Political Economy* **81**, 637–659.

Brennan, M. J. & Schwartz, E. S. (1977), 'The Valuation of American Put Options', *Journal of Finance* **32**, 449–462.

Broadie, M. & Detemple, J. (1995), 'American Capped Call Options on Dividend-Paying Assets', *Review of Financial Studies* **8**, 161–191.

Broadie, M. & Detemple, J. (1996), 'American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods', *Review of Financial Studies* **9**, 1211–1250.

Buchen, P. W. (2004), 'Pricing European Barrier Options', *University of Sydney* . Working Paper.

Buchen, P. W., Kelly, M. F. & Rodolfo, K. (2000), 'American Options in the Black-Scholes Framework', *University of Sydney* . Working Paper.

Carr, P., Jarrow, R. & Myneni, R. (1992), 'Alternative Characterizations of American Put Options', *Mathematical Finance* **2**, 87–106.

Chiarella, C. & Ziogas, A. (2004), 'McKean's Method applied to American Call Options on Jump-Diffusion Processes', *Quantitative Finance Research Centre, University of Technology Sydney* . Research Paper, No. 117.

Chiarella, C. & Ziogas, A. (2005), 'Evaluation of American Strangles', *Journal of Economic Dynamics and Control* **29**, 31–62.

Cox, J. C., Ross, S. A. & Rubinstein, M. (1977), 'Option Pricing: a Simplified Approach', *Journal of Financial Economics* **3**, 229–263.

de Jager, G. (1995), Option Pricing Properties and Techniques using Binomial and Multinomial Lattices, PhD thesis, University of Technology Sydney.

Debnath, L. (1995), *Integral Transforms and their Applications*, CRC Press, Baton Rouge.

Dettman, J. W. (1965), *Applied Complex Variables*, Dover, New York.

Geske, R. & Johnson, H. E. (1984), 'The American Put Option Valued Analytically', *Journal of Finance* **39**, 1511–1524.

Jacka, S. D. (1991), 'Optimal Stopping and the American Put', *Mathematical Finance* **1**, 1–14.

Jaillet, P., Lamberton, D. & Lapeyre, B. (1990), 'Variational Inequalities and the Pricing of American Options', *Acta Mathematica* **21**, 263–289.

Jamshidian, F. (1992), 'An Analysis of American Options', *Review of Futures Markets* **11**, 72–80.

Kallast, S. & Kivinukk, A. (2003), 'Pricing and Hedging American Options using Approximations by Kim Integral Equations', *European Finance Review* **7**, 361–383.

Karatzas, I. (1988), 'On the Pricing of American Options', *Applied Mathematics and Optimization* **17**, 37–60.

Kim, I. J. (1990), 'The Analytic Valuation of American Options', *Review of Financial Studies* **3**, 547–572.

Little, T., Pant, V. & Hou, C. (2000), 'A New Integral Representation of the Early Exercise Boundary for American Put Options', *Journal of Computational Finance* **3**, 73–96.

Logan, J. D. (2004), *Applied Partial Differential Equations*, Springer, New York.

McKean, H. P. (1965), 'Appendix: A Free Boundary Value Problem for the Heat Equation Arising from a Problem in Mathematical Economics', *Industrial Management Review* **6**, 32–39.

Merton, R. C. (1973), 'Theory of Rational Option Pricing', *Bell Journal of Economics and Management Science* **4**, 141–183.

Meyer, G. H. & van der Hoek, J. (1997), 'The Evaluation of American Options with the Method of Lines', *Advances in Futures and Options Research* **9**, 265–285.

Parkinson, M. (1977), 'Option Pricing: The American Put', *Journal of Business* **50**, 21–36.

Peskir, G. (1998), 'Optimal Stopping of the Maximal Process: The Maximality Principle', *The Annals of Probability* **26**, 1614–1640.

Samuelson, P. A. (1965), 'Rational Theory of Warrant Pricing', *Industrial Management Review* **6**, 13–31.

Wilmott, P., Dewynne, J. & Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.