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ON TAIL DISTRIBUTIONS OF SUPREMUM AND QUADRATIC VARIATION OF LOCAL MARTINGALES

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ABSTRACT. We extend some known results on a relation between the distribution tails of the continuous local martingale supremum and its quadratic variation to the case of locally square integrable martingale with bounded jumps. The predictable and optional quadratic variations are involved in the main result.

1. INTRODUCTION AND MAIN RESULT

Denote $\mathscr{M}(\mathscr{M}_{\rm loc})$ and $\mathscr{M}^2(\mathscr{M}^2_{\rm loc}, \mathscr{M}^c_{\rm loc})$ the classes of all martingales (local martingale) and square integrable (locally square integrable, continuous local martingales) $M = (M_t)_{t\ge 0}$, $M_0 = 0$ (with paths in the Skorokhod space $\mathbb{D}_{[0,\infty)}$) defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, P)$ the stochastic basis with standard general conditions etc. Recall that any random X with with paths in the Skorokhod space and defined on the above-mentioned stochastic basis belongs to the class D if the family $(X_\tau, \tau \in \mathcal{T})$, where $\mathcal{T}$ is the set of stopping times τ, is uniformly integrable.

Henceforth $\triangle M_t := M_t - M_{t-}$, $\langle M\rangle_t$ and $[M,M]_t$ are jumps, predictable quadratic variation and optional quadratic variation processes of M respectively.

It is well-known (see e.g. corresponding results and references in [9], [7]) that for local martingales from $\mathscr{M}^2_{\rm loc}$:

$$\langle M\rangle_\infty < \infty, \text{ a.s.} \Rightarrow \begin{cases} [M,M]_\infty < \infty, \text{ a.s.} \\ \lim\limits_{t\to\infty} M_t = M_\infty \in \mathbb{R}, \text{ a.s.} \end{cases}$$

There are many other well-known relations between M_∞ and $\langle M\rangle_\infty$ (e.g. Burkholder - Gundy - Davis's inequalities, law of large numbers for martingales etc) which are valid for local martingales with jumps.

If $M \in \mathscr{M} \cap D$, then M satisfies the Wald equality:

$$EM_\infty = 0$$

which plays a fundamental role in many applications of the stochastic analysis. Often, the direct checking of the uniform integrability would be difficult. In this connection, we mention one result from [10] establishing the relation between the tail distributions of $\langle M\rangle_\infty$ and EM_∞ (a similar result is also proved in [2] under slightly different conditions than in [10]).

Theorem*. *Let* $M \in \mathscr{M}^c_{\rm loc}$ and $\langle M\rangle_\infty < \infty$ *a.s. If* $\sup_{t>0} Ee^{\varepsilon M_t} < \infty$ *for some positive* ε*, then*[1] $0 \le EM_\infty \le EM^+_\infty < \infty$ *and*

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}} EM_\infty.$$

So, one of our goal is a generalization of Theorem* for local martingales with bounded jumps.

Theorem 1.1. *Let* $M \in \mathscr{M}^2_{\rm loc}, \langle M\rangle_\infty < \infty$ *a.s. and* $M^+ \in D$*. Then*
(i)

$$0 \le EM_\infty \le EM^+_\infty < \infty.$$

Besides,
(ii) *if* $|\triangle M| \in D$*, then*

$$\lim_{\lambda\to\infty} \lambda P\big(\sup_{t\ge 0}(M^-_t) > \lambda\big) = EM_\infty;$$

(iii) *if* $|\triangle M| \le K$ *and*

$$Ee^{\varepsilon M_\infty} < \infty \tag{1.1}$$

for some positive constants K *and* ε*, then*

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = \lim_{\lambda\to\infty} \lambda P\big([M,M]^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}} EM_\infty.$$

If $M^+ \in D$, Theorem 1.1 provides the necessary and sufficient conditions for $M \in D$ expressed in terms of $\sup_{t\ge 0} M^-_t$, $\langle M\rangle_\infty$, and $[M,M]_\infty$ which are handy in some applications (see e.g. [8]).

Corollary 1. *Under the assumptions of Theorem 1.1,*$M \in D$ *iff any of the following conditions holds:*

$$\lim_{\lambda\to\infty} \lambda P\big(\sup_{t\ge 0}(M^-_t) > \lambda\big) = 0,$$

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = 0,$$

$$\lim_{\lambda\to\infty} \lambda P\big([M,M]^{1/2}_\infty > \lambda\big) = 0.$$

A few publication preceded [10] and [2]. The first characterization of the uniform integrability for continuous martingales in terms of their distribution tails and quadratic variation processes is known from Azema-Gundy-Yor [1]. For the discrete time martingales, Gundy, [5] and Galtchouk-Novikov, [6], considered the related problem on a validity of the Wald identity. Takaoka, [13] presented the result similar to Theorem* without using the condition (1.1).

The proofs of parts (i) and (ii) of Theorem 1.1 are obvious and even might be known. The proof of (iii) exploits a combination of techniques

"Stochastic exponential + Tauberian theorem"

[1] $a^+ = \max(a,0), a^- = \max(-a,0)$

which seems to be firstly used by Novikov, [11], for obtaining the asymptotics of the first passage times for Brownian motion (see also [10]) and for random walks ([12]). Some necessary facts on the stochastic exponential are gathered in Section 2. The proofs are given in Section 3.

The uniform boundedness assumption for ΔM might be weakened by applying a standard "truncation" technique with some additional assumptions on the tails distribution of ΔM. We show in Theorem 3.1 that the uniform boundedness assumption for ΔM might be avoided if the stochastic exponential possesses possesses an evaluation in terms of $\langle M\rangle_\infty$. This condition is borrowed from [10] where it was effectively applied for analyzing of asymptotic characteristics for the discrete-time martingales in some popular gambling strategies.

Some numerical results, giving rates of convergence in $\lambda \to \infty$ for corresponding relations of Theorem 1.1, will be presented in a forthcoming publication.

2. Preliminaries

2.1. **Stochastic exponential.** For discontinuous martingales, the stochastic exponential has an "intricate" structure. So, we start with refreshing of necessary notions and objects especially concerning to the setting from (ii) (for more details, see e.g. [9] or [7]).

For $M \in \mathscr{M}^2_{\rm loc}$, $M_0 = 0$, it is well known the decomposition $M = M^c + M^d$ with $M^c, M^d \in \mathscr{M}^2_{\rm loc}$, where M^c is continuous and M^d is purely discontinuous martingales. Moreover, $\langle M\rangle = \langle M^c\rangle + \langle M^d\rangle$, so that the assumption $\langle M\rangle_\infty < \infty$ provides $\langle M^c\rangle_\infty < \infty$, $\langle M^d\rangle_\infty < \infty$. The measure μ is associated with the jump process $\triangle M \equiv \triangle M^d$ in a sense that for any measurable set A and $t > 0$ $\mu((0,t]\times A) = \sum_{s\le t} I(\triangle M_s \in A)$. Denote by $\nu = \nu(dt, dx)$ its compensator. The condition $|\triangle M| \le K$ provides $\int_0^\infty \int_{|z|>K} \mu(dt, dz) = 0$, so a version of ν can be chosen such that $I(|z| \le K)\nu = \nu$. The purely discontinuous martingale M^d is defined as the Itô integral with respect to $\mu - \nu$:

$$M^d_t = \int_0^t \int_{|z|\le K} z\big(\mu(ds, dz) - \nu(ds, dz)\big).$$

Recall also that $\int_{|z|\le K} z\nu(\{t\}, dz) = 0$ a.s. and

$$\langle M^d\rangle_t = \int_0^t \int_{|z|\le K} z^2 \nu(ds, dz) < \infty \text{ a.s., } t > 0.$$

Hence, $\langle M^d\rangle_t < \infty$ a.s. provides

$$\int_0^\infty \int_{|z|\le K} z^2 \nu(ds, dz) < \infty \text{ a.s.} \tag{2.1}$$

This fact is important for the further considerations as long as we will deal with the cumulant process $G(\lambda) = (G_t(\lambda))_{t\ge 0}$ where

$$G_t(\lambda) = \int_0^t \int_{|z|\le K} \big(e^{\lambda z} - 1 - \lambda z\big)\nu(ds, dz), \ \lambda \in \mathbb{R}.$$

The boundedness of jumps and (2.1) provide the existence of $G_t(\lambda)$ and $G_\infty(\lambda) := \lim_{t\to\infty} G_t(\lambda) < \infty$. The cumulant process $G(\lambda)$, being increasing, possesses nonnegative jumps process

$$\triangle G_t(\lambda) := \int_{|z|\le K} \big(e^{\lambda z} - 1 - \lambda z\big)\nu(\{t\}, dz).$$

A random process $\mathscr{E}(\lambda) = (\mathscr{E}_t(\lambda))_{t\ge 0}$ with

$$\mathscr{E}_t(\lambda) = \exp\Big(\frac{\lambda^2}{2}\langle M^c\rangle_t + G_t(\lambda)\Big) \prod_{0<s\le t} \big(1 + \triangle G_s(\lambda)\big)e^{-\triangle G_s(\lambda)} \tag{2.2}$$

is known as "stochastic exponential". Note that $\mathscr{E}_t > 0$, since $\triangle G(\lambda) \ge 0$.

A remarkable property of the stochastic exponential is that the process $\mathfrak{z}(\lambda) = (\mathfrak{z}_t(\lambda))_{t\ge 0}$ with

$$\mathfrak{z}_t(\lambda) = e^{\lambda M_t - \log \mathscr{E}_t(\lambda)} \tag{2.3}$$

is the positive local martingale. Indeed, applying the Itô formula to (2.3), we get

$$\mathfrak{z}_t(\lambda) = 1 + \lambda \int_0^t \mathfrak{z}_s(\lambda) dM^c_s + \int_0^t \int_{|z|\le K} \mathfrak{z}_{s-}(\lambda) \frac{\big(e^{\lambda z} - 1\big)}{1 + \triangle G_s(\lambda)} (\mu - \nu)(ds, dz)$$

where the right-hand side is a sum of local martingales. As any nonnegative local martingale, $\mathfrak{z}(\lambda)$ is the supermartingale too (see e.g. Problem 1.4.4 in [9]). The latter provides the existence of

$$\mathfrak{z}_\infty(\lambda) := \lim_{t\to\infty} \mathfrak{z}_t(\lambda) \in \mathbb{R}_+ \text{ a.s.}$$

with $E\mathfrak{z}_\tau(\lambda) \le 1$ for any Markov time τ; particularly $E\mathfrak{z}_\infty \le 1$.

Proposition 2.1. *Let $|\triangle M| \le K$ and condition (1.1) hold. Then, with ε from (1.1) and any $\lambda \in (0, \varepsilon]$,*
1) $E\mathfrak{z}_\infty(\lambda) = 1$.
2) $\mathscr{E}_\infty(\lambda) = \lim_{t\to\infty} \mathscr{E}_t(\lambda) \in \mathbb{R}_+$ a.s. and $\mathscr{E}_\infty(\lambda) > 0$ a.s.

Proof. 1) Let $(\tau_n)_{n\ge 1}$ be a localizing sequence for both M and $\mathfrak{z}(\lambda)$, i.e. $(M_{t\wedge\tau_n})_{t\ge 0}$ and $(\mathfrak{z}_{t\wedge\tau_n}(\lambda))_{t\ge 0} \in D$for any n. Then

$$E\mathfrak{z}_{\tau_n}(\lambda) \equiv 1. \tag{2.4}$$

For $m > n$, by the Jensen inequality we have $E(M^+_{\tau_m}|\mathscr{F}_{\tau_n}) \ge M^+_{\tau_n}$ a.s. Now, taking into the consideration $\lim\limits_{m\to\infty} E|M^+_\infty - M^+_{\tau_m}| = 0$, we claim that

$$E(M^+_\infty|\mathscr{F}_{\tau_n}) \ge M^+_{\tau_n} \tag{2.5}$$

Now, taking into account (1.1) and $e^{\varepsilon M^+_\infty} \le 1 + e^{\varepsilon M_\infty}$, we introduce the uniformly integrable martingale $E(e^{\lambda M^+_\infty}|\mathscr{F}_{\tau_n})$. By the Jensen inequality and (2.5),

$$E\Big(e^{\lambda M^+_\infty}|\mathscr{F}_{\tau_n}\Big) \ge e^{\lambda E(M^+_\infty|\mathscr{F}_{\tau_n})} \ge e^{\lambda M^+_{\tau_n}} \ge \mathfrak{z}_{\tau_n}(\lambda).$$

In other words, $(\mathfrak{z}_{\tau_n}(\lambda))_{n\geq 1}$ is majorized by the uniformly integrable martingale and so the family $(\mathfrak{z}_{\tau_n}(\lambda))_{n\geq 1} \in D$.

Thus, the desired statement holds by (2.4).

2) From (2.2), it is readily to derive that $\log \mathscr{E}_t(\lambda) \leq \frac{\lambda^2}{2}\langle M^c\rangle_t + G_t(\lambda)$, i.e. $\mathscr{E}_t(\lambda) \leq \frac{\lambda^2}{2}\langle M^c\rangle_\infty + G_\infty(\lambda) < \infty$ a.s. This fact, particularly, guarantees $\mathscr{E}_\infty(\lambda) < \infty$ a.s.

It is clear that

$$\begin{aligned}\big\{\mathscr{E}_\infty(\lambda)=0\big\} &\subseteq \Big\{\prod_{s>0}\big(1+\triangle G_s(\lambda)\big)e^{-\triangle G_s(\lambda)}=0\Big\}\\ &= \Big\{\sum_{s>0}\log\big(1+\triangle G_s(\lambda)\big)-\triangle G_s(\lambda)=-\infty\Big\}\\ &\subseteq \Big\{\sum_{s>0}\triangle G_s(\lambda)=\infty\Big\} \subseteq \Big\{G_\infty(\lambda)=\infty\Big\} \overset{a.s.}{=} \varnothing.\end{aligned}$$

□

3. The proof of Theorem 1.1

3.1. **The proof of (i) and (ii).**

Proof. 1) Let $(\tau_n)_{n\geq 1}$ be a localizing sequence for M, i.e. $(M_{\tau_n})_{n\geq 1} \in D$ and therefore $EM^-_{\tau_n} = EM^+_{\tau_n}, n \geq 1$. Due to the assumption $M^+ \in D$, we have $\lim_{n\to\infty} EM^+_{\tau_n} = EM^+_\infty < \infty$. Now, applying the Fatou theorem, we find that $EM^+_\infty \geq EM^-_\infty$ and therefore $EM^+_\infty \geq EM^+_\infty - EM^-_\infty = EM_\infty \geq 0$. □

2) Set $S_\lambda = \inf\{t : M^-_{S_\lambda} \geq \lambda\}$ and notice that

$$\{S_\lambda < \infty\} = \{\sup_{t\geq 0}(M^-_t) > \lambda\}.$$

Since $\triangle M_\infty = 0$ and $|\triangle M| \in D$, $(M_{t\wedge S_\lambda})_{t\geq 0}$ is the uniformly martingale and so $EM_{S_\lambda} = 0$.

Now, write

$$\begin{aligned}0 = EM_{S_\lambda} &= EM_\infty I_{\{S_\lambda=\infty\}} + EM_{S_\lambda} I_{\{S_\lambda<\infty\}}\\ &= EM_\infty I_{\{S_\lambda=\infty\}} + EM_{S_\lambda} I_{\{\sup_{t\geq 0}(-M_t)\geq\lambda\}}\\ &= EM_\infty I_{\{S_\lambda=\infty\}} + E(M_{S_\lambda}-\lambda) I_{\{S_\lambda<\infty\}}\\ &\quad + \lambda P(\sup_{t\geq 0}(M^-_t) > \lambda).\end{aligned}$$

Since $EM^+_\infty < \infty$ provides $\lim_{\lambda\to\infty} S_\lambda = \infty$ and $EM_\infty \geq 0$, we claim that the desired statement holds true, if $|M_{S_\lambda} - \lambda|, \lambda > 0$ is the uniformly integrable family. So, it remains to notice that $|M_{S_\lambda} - \lambda| \leq |\triangle M_{S_\lambda}| \leq K$. □

3.2. **The proof of (iii).** The next three lemmas play here a crucial role.

3.2.1. **Auxiliary results.**

Lemma 3.1.
$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big(1-e^{-\log \mathscr{E}_\infty(\lambda)}\Big)=EM_\infty.$$

Proof. Recall that $\lambda\le\varepsilon$ for ε involved in assumption (ii). By Proposition 2.1, $E\mathfrak{z}_\infty(\lambda)=1$. Hence,
$$\begin{aligned}E\frac{1}{\lambda}\Big(1-e^{-\log \mathscr{E}_\infty(\lambda)}\Big)&=E\frac{1}{\lambda}\Big(\mathfrak{z}_\infty(\lambda)-e^{-\log \mathscr{E}_\infty(\lambda)}\Big)\\&=E\frac{1}{\lambda}\Big(e^{\lambda M_\infty}-1\Big)e^{-\log \mathscr{E}_\infty(\lambda)}.\end{aligned}$$

The required statement follows from
$$\begin{aligned}&\lim_{\lambda\downarrow 0}\frac{1}{\lambda}e^{-\log \mathscr{E}_\infty(\lambda)}\Big(e^{\lambda M_\infty}-1\Big)=M_\infty,\\&\frac{1}{\lambda}e^{-\log \mathscr{E}_\infty(\lambda)}\big|e^{\lambda M_\infty}-1\big|\le e^{\varepsilon M_\infty}\end{aligned}$$
and the assumption $Ee^{\varepsilon M_\infty}<\infty$ (see (1.1)). □

Lemma 3.2.
$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big(1-e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big)=EM_\infty.$$

Proof. Due to Lemma 3.1, it suffices to show that
$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big|e^{-\log \mathscr{E}_\infty(\lambda)}-e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big|=0. \tag{3.1}$$

In order to verify (3.1), we estimate $\log\mathscr{E}_\infty(\lambda)$ from above and below via $\frac{\lambda^2}{2}\langle M\rangle_\infty$. Owing to $\log\mathscr{E}_\infty(\lambda)\le\frac{\lambda^2}{2}\langle M^c\rangle_\infty+G_\infty(\lambda)$, we have
$$\log\mathscr{E}_\infty(\lambda)\le\frac{\lambda^2}{2}\langle M\rangle_\infty\Big[1+\frac{\lambda}{3}Ke^{\lambda K}\Big] \tag{3.2}$$
Further, with
$$G^c_\infty(\lambda)=\int_0^\infty\int_{|z|\le K}\big(e^{\lambda z}-1-\lambda z\big)\nu^c(dt,dz)$$
where $\nu^c(dt,dz):=\nu(dt,dz)-\nu(\{t\},dz)$, we get
$$\begin{aligned}&\log\mathscr{E}_\infty(\lambda)\\&=\frac{\lambda^2}{2}\langle M^c\rangle_\infty+G^c_\infty(\lambda)+\sum_{t>0}\log\big(1+\triangle G_t(\lambda)\big)\\&\ge\frac{\lambda^2}{2}\langle M^c\rangle_\infty+\Big[1-\lambda Ke^{\lambda K}\Big]\int_0^\infty\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu^c(dt,dz)\\&+\sum_{t>0}\log\left(1+\Big[1-\lambda Ke^{\lambda K}\Big]\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\right).\end{aligned} \tag{3.3}$$
We choose λ so small to have $1-\lambda Ke^{\lambda K}>0$ and estimate from below the "$\sum_{t>0}\log$" in the last line from the above inequality by applying
$$\log(1+x)\ge x-0.5x^2,\ x\ge 0.$$

Write

$$\begin{aligned}\sum_{t>0}\log\left(1+\Big[1-\lambda Ke^{\lambda K}\Big]\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\right)\\ \ge\Big[1-\lambda Ke^{\lambda K}\Big]\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\\ -0.5\Big[1-\lambda Ke^{\lambda K}\Big]^2\left(\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\right)^2.\end{aligned}$$

Since $\nu(\{t\},|z|\le K)\le 1$, by the Cauchy-Schwarz inequality we find

$$\begin{aligned}&\left(\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\right)^2\\ &\qquad\le\frac{\lambda^4}{4}\int_{|z|\le K}z^4\nu(\{t\},dz)\le\frac{\lambda^4K^2}{4}\int_{|z|\le K}z^2\nu(\{t\},dz).\end{aligned}$$

So, finally we get

$$\begin{aligned}\sum_{t>0}\log\left(1+\Big[1-\lambda Ke^{\lambda K}\Big]\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\right)\\ \ge\left(\Big[1-\lambda Ke^{\lambda K}\Big]-\frac{\lambda^2}{8}K^2\Big[1-\lambda Ke^{\lambda K}\Big]^2\right)\int_{|z|\le K}\frac{\lambda^2}{2}z^2\nu(\{t\},dz)\end{aligned}\tag{3.4}$$

and now choose λ so small to have

$$\Big[1-\lambda Ke^{\lambda K}\Big]-\frac{\lambda^2}{8}K^2\Big[1-\lambda Ke^{\lambda K}\Big]^2\ge 1-\lambda C>0\tag{3.5}$$

for some constant $C>0$. Combining now (3.3), (3.4) and (3.5), we may choose a generic positive constant C and sufficiently small λ such that $\mathscr{E}_\infty(\lambda)\ge\big[1-C\lambda\big]\frac{\lambda^2}{2}\langle M\rangle_\infty$. Hence and with (3.2), for some generic positive constant $C>0$ and sufficiently small $\lambda>0$ we have

$$0<\big[1-C\lambda\big]\frac{\lambda^2}{2}\langle M\rangle_\infty\le\log\mathscr{E}_\infty(\lambda)\le\big[1+C\lambda\big]\frac{\lambda^2}{2}\langle M\rangle_\infty.$$

These inequalities provide

$$\frac{1}{\lambda}\Big|e^{-\log\mathscr{E}_\infty(\lambda)}-e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big|\le C\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\xrightarrow[\lambda\to 0]{}0.$$

Since $xe^{-x}\le e^{-1}$, the desired result holds by Lebesgue dominated theorem. □

Lemma 3.3.

$$\lim_{\lambda\to\infty}\lambda P\big(\langle M\rangle_\infty^{1/2}>\lambda\big)=c\Leftrightarrow\lim_{\lambda\to\infty}\lambda P\big([M,M]_\infty^{1/2}>\lambda\big)=c.$$

Proof. It suffices to establish

$$\begin{aligned}\overline{\lim_{\lambda\to 0}}\frac{P\big([M,M]_\infty^{1/2}>\lambda\big)}{P\big(\langle M\rangle_\infty^{1/2}>\lambda\big)}\le 1\\ \underline{\lim_{\lambda\to 0}}\frac{P\big([M,M]_\infty^{1/2}>\lambda\big)}{P\big(\langle M\rangle_\infty^{1/2}>\lambda\big)}\ge 1.\end{aligned}\tag{3.6}$$

Set $L = [M,M] - \langle M\rangle$. Since $[M,M]_\infty \le \langle M\rangle_\infty + \sup_{t\ge 0}|L_t|$, applying $(c+d)^{1/2} \le c^{1/2} + d^{1/2}$, $c,d \ge 0$, we find that

$$\begin{aligned} P\big([M,M]_\infty^{1/2} > \lambda\big) &\le P\big([\langle M\rangle_\infty + \sup_{t\ge 0}|L_t|]^{1/2} > \lambda\big) \\ &\le P\big(\langle M\rangle_\infty^{1/2} + \sup_{t\ge 0}|L_t|^{1/2} > \lambda\big) \\ &\le P\big(\langle M\rangle_\infty^{1/2} > (1-a)\lambda\big) + P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big),\ a\in(0,1). \end{aligned} \tag{3.7}$$

With $\lambda_a = (1-a)\lambda$, (3.7) is rewritten to

$$\lambda P\big([M,M]_\infty^{1/2} > \lambda\big) \le (1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\big) + \lambda P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big). \tag{3.8}$$

So, we will deal with the evaluation from above of $P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big)$. We show first that there exists an absolute positive constant C such that for any stopping time τ

$$E\sup_{t\le\tau}|L_t|^2 \le CK^2 E\langle M\rangle_\tau \tag{3.9}$$

where K is the bound for $|\triangle M|$. We use the fact that L is the purely discontinuous local martingale with

$$\begin{aligned} [L,L]_t &= \sum_{s\le t}(\triangle L_s)^2 = \sum_{s\le t}\big((\triangle M_s)^2 - \triangle\langle M\rangle_s\big)^2 \\ &= \sum_{s\le t}\left(\int_{|z|\le K} z^2(\mu(\{s\},dz) - \nu(\{s\},dz)\right)^2. \end{aligned}$$

The process $\langle L\rangle$, being the compensator of $[L,L]$, is defined as follows

$$\langle L\rangle_t = \sum_{s\le t}\left(\int_{|z|\le K} z^4(\nu(\{s\},dz) - \Big(\int_{|z|\le K} z^2\nu(\{s\},dz)\Big)^2\right);$$

the latter provides

$$\langle L\rangle_t \le \sum_{s\le t}\int_{|z|\le K} z^4(\nu(\{s\},dz) \le K^2\sum_{s\le t}\int_{|z|\le K} z^2(\nu(\{s\},dz) \le K^2\langle M\rangle_t$$

and, moreover, $K^2\langle M\rangle - \langle L\rangle$ is the increasing process.

We refer now to the Burkholder-Gundy inequality (see e.g. Theorem 1.9.7 in [9]): for any stopping time τ, $E\sup_{t\le\tau}|L_t|^2 \le CE[L,L]_\tau$. Owing to $E[L,L]_\tau = E\langle L\rangle_\tau$ and $K^2\langle M\rangle - \langle L\rangle$ is the increasing process, that is $E\langle L\rangle_\tau \le K^2 E\langle M\rangle_\tau$ too, we get (3.9).

Due to (3.9) and the fact that $\langle M\rangle$ is the predictable process, the Lenglart-Rebolledo inequality (see e.g. Theorem 1.9.3 in [9]) is applicable (notice that $\{\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\} \equiv \{\sup_{t\ge 0}|L_t| > a^4\lambda^4\}$).

$$\begin{aligned} P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big) &\le \frac{\lambda^{5/2}}{a^4\lambda^4} + P\big(CK^2\langle M\rangle_\infty > \lambda^{5/2}\big) \\ &= \frac{\lambda^{5/2}}{a^4\lambda^4} + P\big(\langle M\rangle_\infty^{1/2} > \lambda^{5/4}/(C^{1/2}K)\big), \end{aligned} \tag{3.10}$$

that is with $r = 1/(C^{1/2}K)$ and $\lambda_r = r^{-1}\lambda^{5/4}$

$$\lambda P\big(\sup_{t\le T_x}|L_t|^{1/2} > a\lambda\big) \le \frac{1}{a^4\lambda^{1/2}} + \frac{1}{r\lambda^{1/4}}\lambda_r P\big(\langle M\rangle_\infty^{1/2} > \lambda_r\big). \tag{3.11}$$

Now, (3.8) and (3.11) provide

$$\lambda P\big([M,M]_\infty^{1/2} > \lambda\big) \le (1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\big) + \frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}}\lambda_r P\big(\langle M\rangle_\infty^{1/2} > \lambda_r\big).$$

Assume that $c > 0$. Then, we get

$$\frac{P\big([M,M]_\infty^{1/2} > \lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \le \frac{(1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\big)}{\lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} + \frac{\frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}}\lambda_r P\big(\langle M\rangle_\infty^{1/2} > \lambda_r\big)}{\lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \xrightarrow[\lambda\to 0]{} \frac{1}{1-a} \xrightarrow[a\to 0]{} 1$$

and the first part from (3.6). Since the second part from (3.6) is established similarly, we give only a sketch of the proof. The use of

$$P\big(\langle M\rangle^{1/2} > \lambda\big) \le P\big([M,M]^{1/2} > (1-a)\lambda\big) + P\big(\sup_{t\ge 0}|L_t| > a\lambda\big),\ a\in(0,1),$$

provides $\frac{P\left([M,M])_\infty^{1/2} > (1-a)\lambda\right)}{P\left(\langle M\rangle_\infty^{1/2}>\lambda\right)} \ge 1 - \frac{P\left(\sup_{t\ge 0}|L_t|>a\lambda\right)}{P\left(\langle M\rangle_\infty^{1/2}>\lambda\right)}$ and the result.

If $c = 0$, we replace M by $M + M'$, where M' is independent of M^c local continuous martingale with $M'_0 = 0$ and $\langle M'\rangle_\infty < \infty$ a.s. and

$$\lim_{\lambda\to\infty}\lambda P\big(\langle M'\rangle_\infty^{1/2} > \lambda\big) = c' > 0.$$

Now, taking into account the obvious relations

$$[M+M', M+M'] = [M,M] + [M',M'] \text{ and } \langle M+M'\rangle = \langle M\rangle + \langle M'\rangle,$$

with $\delta \ne 0$ we find that $\lim_{\lambda\to\infty}\lambda P\big(\langle M+\delta M'\rangle_\infty^{1/2} > \lambda\big) = \delta^2 c' > 0$. So, by proved result we have $\lim_{\lambda\to\infty}\lambda P\big([M+\delta M', M+\varepsilon M']_\infty^{1/2} > \lambda\big) = \delta c'$ and so, by $P\big([M+\delta M', M+\delta M']_\infty^{1/2} > \lambda\big) \ge P\big([M,M]_\infty^{1/2} > \lambda\big)$,

$$\overline{\lim_{\lambda\to 0}}\,\lambda P\big([M,M]_\infty^{1/2} > \lambda\big) \le \delta c' \xrightarrow[\delta\to 0]{} 0.$$

□

3.2.2. **Final part of the proof for (iii).** We refer the Tauberian theorem.

Theorem.** ([4], XIII.5, Example (c)) *Let X be an $\mathbb{R}_+$-valued random variable such that $\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}X}\Big)$ exists in $\mathbb{R}$, then*

$$\sqrt{\frac{2}{\pi}}\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}X}\Big) = \lim_{\lambda\to\infty}\lambda P(X^{1/2} > \lambda).$$

Now, we are in the position to finish the proof of (ii). Letting $X = \langle M\rangle_\infty$, we find that

$$\sqrt{\frac{2}{\pi}}\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big) = \lim_{\lambda\to\infty}\lambda P(\langle M\rangle^{1/2} > \lambda).$$

At the same time, Lemmas 3.1 and 3.2 provide

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big) = \sqrt{\frac{2}{\pi}}EM_\infty$$

while by Lemma 3.3 $\lim\limits_{\lambda\to\infty}\lambda P\big([M,M]^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}}EM_\infty$. □

3.3. **Supplement.** As was mentioned in Introduction, the condition $|\triangle M| \le K$ might be too restrictive in some examples. We replace it by another one verifiable for some martingales (see examples for the discrete-time case in [10]).

Theorem 3.1. *Let $M \in \mathscr{M}^2_{\rm loc}$, $\langle M\rangle_\infty < \infty$ a.s., $M^+ \in D$ and* (1.1) *hold. Assume also there exist nonnegative integrable random variables ζ_1, ζ_2 such that for all sufficiently small $\lambda > 0$*

$$\frac{\lambda^2}{2}\langle M\rangle_\infty(1 - |\lambda|\zeta_1)^+ \le \log\mathscr{E}_\infty(\lambda) \le \frac{\lambda^2}{2}\langle M\rangle_\infty(1 + |\lambda|\zeta_2). \tag{3.12}$$

Then

$$\lim_{\lambda\to\infty}\lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}}EM_\infty.$$

Proof. Notice that only (3.1) has to be verified under (3.12).

If (3.12) holds, we have

$$\begin{aligned}\frac{1}{\lambda}\Big|e^{-\log\mathscr{E}_\infty(\lambda)} - e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big| &\le \Big(\zeta_2 \vee \frac{|1-(1-\zeta_1\lambda)^+|}{\lambda}\Big)\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\\ &\le \big(\zeta_2\vee\zeta_1\big)\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}.\end{aligned}$$

The right hand side of this inequality converges to zero, as $\lambda \to 0$ and is bounded by $e^{-1}(\zeta_2 \vee \zeta_1)$.

Then, (3.1) holds by Lebesgue dominated theorem. □

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