A minimal dissipation type-based classification in irreversible thermodynamics and microeconomics

A. M. Tsirlin, Vladimir Kazakov[+] and N. A. Kolinko
Program System Institute, Russian Academy of Science,
set. Botic, Perejaslavl-Zalesky, Russia 152020
tsirlin@sarc.botik.ru
[+]School of Finance and Economics, University of Technology,
PO Box 123, Broadway NSW 2007 Australia,
Vladimir.Kazakov@uts.edu.au

June 16, 2003

Annotation

We formulate the problem of finding classes of kinetic dependencies in irreversible thermodynamic and microeconomic systems for which minimal dissipation processes belong to the same type. We show that this problem is an inverse optimal control problem and solve it. The commonality of this problem in irreversible thermodynamics and microeconomics is emphasized.

1 Introduction and problem formulation

Analogy between reversible processes in thermodynamic and microeconomic systems has been long known [1], [2], [3],[5], [11], et al. This analogy is based on the fact that both systems include large numbers of elementary subsystems (molecules, economic agents), which are not directly controllable. Control here can only be applied on the macro level. We shall call them macro controllable systems [4].

The Finite Time Thermodynamics (FTT) ([6],[7], [8], [9] et al) studies limiting possibilities of thermodynamic systems subject to condition that the average rates of some flows are given. These problems can be roughly divided into three classes:
(A) the problems of limiting rate of objective flux (limiting power of heat engine, limiting productivity of thermal processes of gas and liquid separation, etc.),
(B) the problem of minimal energy use for given rate of objective flow (limiting efficiency of heat engine with given power, minimal heat used for separation with given rate, etc.),
(C) the problems of constructing realizability areas in a state space where coordinates are the average rates of flows in the system.

Solutions of FTT problems (B), (C) are the *minimal dissipation processes* [14], defined as processes with minimal entropy production subject to given rates of flows. The conditions of minimal dissipation jointly with thermodynamic balances on mass, energy and entropy, determine these processes.

Similar problems arise in microeconomics -
(A) the problem of limiting rate of capital extraction from the system, subject to some constraints of its structure,
(B) the problem of maximal norm of profit subject to given rate of capital extraction,
(C) the problem of construction of realizability areas in the space of flows of capital and goods.
The role of entropy here is played by the prosperity function. One of the proofs of its existence is based on Ville's axiom [10]. These problems were considered in [4], where the notions of system's prosperity and capital dissipation were introduced. The former is defined as the amount of capital that can be potentially extracted from the system, and the latter as the rate of reduction in profitability. It was also shown there that microeconomic balances on assets and capital and on the prosperity jointly with the conditions of minimal dissipation determine the boundary of the realizability area of microeconomic systems.

In their turn, the conditions of minimal dissipation are determined by the system's kinetics, that is, by the dependence of the flows of mass, energy and assets between its subsystems on the driving forces (differences in temperatures, concentrations, asset price estimates, etc). Different kinetics can correspond to the same characteristic solution of minimal dissipation problem. For example, the condition of constant temperature ratio of contacting subsystems in minimal dissipation process holds for a number of different laws of heat transfer. This leads to the problem of finding classes of kinetic dependencies, all members of which have minimal dissipation processes that belong to the same class. We shall call it classification problem with respect to conditions of minimal dissipation.

2 Dissipation in thermodynamic and microeconomic systems

A thermodynamic system and a microeconomic system both are described by variables that can be conveniently divided into two groups - extensive and intensive. If two systems are combined into a new system then the extensive variables of combined system are the sums of the extensive variables of the initial systems. For instance, when two identical systems are combined the volume of the resulting system is twice the initial system's volume. Capital and asset inventories in microeconomic systems are summed similarly. Intensive variables do not change when similar systems are combined. For instance, is two systems with identical temperatures are combined then the combined system's temperature is the same. The asset price estimates in microeconomics behave similarly.

Microeconomic subsystem (agent) with asset inventory N estimates asset price p as the derivative of its prosperity function S on N ([2], [4])

$$p = \frac{\partial S}{\partial N}.$$

If contact between two microeconomic agents with different values of intensive variables is established then they trade and exchange flow arise. Asset price estimate p plays here the same role as the temperature in thermodynamics. The flow of asset depends on the difference between the trading price and the asset price estimate in the same way the heat flow between two bodies depends on their temperatures' difference.

Thus, non-zero difference between agent's asset price estimate and trading price is necessary to have non-zero rates of asset flows. In thermodynamics dissipation is minimal in reversible processes with infinitely small rates of exchange flows. Similarly, trading costs in microeconomic systems are minimal where exchange is reversible and exchange rates are infinitely small. If exchange rates' are finite then trading is conducted irreversibly and trading costs exceed the minimal-possible ones. This additional cost is called capital dissipation. They are similar to entropy production in thermodynamics. These and other analogies between irreversible processes in thermodynamics and microeconomics are described in details in ([2]).

If contact is established between two subsystems with different values of intensive variables then exchange flows occur. One can control these flows by controlling the values of subsystems' intensive variables. It is useful to single out the class of subsystems whose intensive variables are controllable. Heat engine's working body with controllable volume (and therefore controllable temperature) belongs to this class. Economic intermediaries, who set optimal (from their viewpoint) prices, also belong to this class. We shall call such systems active. Passive systems differ from active ones because their intensive variables change only as a result of changes to their extensive variables.

3 Problem formulation

We denote all extensive variables and the intensive variables of the passive systems as x and the intensive variables of the active systems as u. Variables u are the problem's control variables. The dependence of rate of flows on the driving forces is determined by kinetics of the process and is described by the function $n(x, u)$. This function determines conditions of minimal dissipation [14]

$$F\left(n(x,u), x, u, n_x, n_u\right) = const, \tag{1}$$

where n_x, n_u denote partial derivatives of the flow n on the corresponding variables.

These conditions are obtained by solving optimal control problem where dissipation is minimized subject to given average rates of exchange processes in the system. Any minimal dissipation process must obey these conditions.

Additional constrains can be imposed on the function n. For example, the following condition can be used

$$n(x,u) = 0 \qquad \text{for} \quad x = u. \tag{2}$$

If the difference $x - u$ changes sign then the flow n changes sign also.

A simple example of obtaining these conditions of optimality is given in Section 4.1.

Suppose that the conditions of minimal dissipation (1) can be written in the following form

$$\varphi(x^*, u^*) = const, \tag{3}$$

where φ is some predefined function. In its turn, this function can depend on $n(x,u)$. We denote optimal (minimal dissipation) processes and their parameters with superscript $*$.

The aim of this paper is to find the conditions that must be imposed on the kinetic function $n(x,u)$ to guarantee that its conditions of minimal dissipation (1) have the form (3) for a given function φ.

First we will describe the general schema of solution of the described problem. It is based on the Statement 1: *The solution of minimal dissipation problem obeys condition (3) if and only if function $n(x,u)$ is a solution of the equation*

$$\frac{F_x}{F_u} = \frac{\varphi_x}{\varphi_u}. \tag{4}$$

Indeed, from the condition (3) follows that

$$\varphi_x dx = -\varphi_u du,$$

and from (3) it follows that

$$F_x dx = -F_u du,$$

therefore (4) holds. It is clear that the inverse is also true - if (1) holds and (4) does not, then the condition (3) is also violated.

The left hand side of the equality (4) depends on the form of the function n and its partial derivatives and can be used to derive partial differential equation for the function n. Its general solution gives the class of dependencies. We will consider a number of examples of solution of this classification problem for particular systems, which demonstrate the class of problems considered is general enough.

4 Thermodynamic systems

It turned out that for many, but not all, minimal dissipation processes in FTT the entropy production is constant (time and space independent). We will derive conditions when minimal dissipation thermodynamic process corresponds to constant entropy production. We consider heat exchange first and then generalize the obtained results for a wider class of thermodynamic processes.

4.1 Irreversible heat exchange

Minimal dissipation heat exchange process is defined as a heat exchange during which given amount of heat Q is removed from a body with the temperature $T(t)$ and finite heat capacity in given time τ in such a fashion that the resulting increase of system's entropy S is minimal. Temperature of the coolant $T_0(t)$ is the control variable. The dependence of the heat flow $n(T_0, T)$ between the body that is cooled and the coolant on their temperatures T and T_0 is called the law of heat transfer.

Formally the problem is stated as

$$\Delta S = \int_0^\tau n(T_0, T)(1/T_0 - 1/T)dt \to \min$$

subject to constraints

$$\int_0^\tau n(T_0, T)dt = Q,$$

$$C\frac{dT}{dt} = -n(T_0, T), \quad T(0) = T_0.$$

where C is the heat capacity of body that is being cooled.

Because T is a monotonic function and we can replace t as independent variable with T. We get $dt = -\frac{C}{n(T_0,T)}dT$ and the minimal dissipation problem takes the form

$$\int_{T_0}^{T(\tau)} (\frac{1}{T} - \frac{1}{T_0})dT \to \min$$

subject to given duration of the process τ

$$-\int_{T_0}^{T(\tau)} \frac{C}{n(T_0, T)}dT = \tau.$$

The Lagrange function of the transformed problem has the form

$$L = \int_{T_0}^{T(\tau)} (\frac{1}{T} - \frac{1}{T_0} - \lambda \frac{C}{n(T_0,T)}) dT.$$

λ here is a Lagrange multiplier, that is independent on T.

The conditions of optimality for the transformed problem are reduced to the condition that the derivative of the integrand of this Lagrange function on T is equal zero on the optimal solution for all T. That is, the conditions (1) here take the following form [12]

$$\frac{T^2}{n^2(T_0,T)} \frac{\partial n}{\partial T_0} = const, \quad n(T_0,T_0) = 0. \tag{5}$$

Condition of constant temperature difference. What is required from the law of heat transfer $n(T_0,T)$, to guarantee that the condition (3) with the function $\varphi(T_0,T) = T_0 - T$ holds? That is, for which laws of heat transfer the condition of minimal dissipation corresponds to constancy of the temperature differences? The answer to this question is given by the Statement 2: *The condition of minimal dissipation corresponds to constant temperature difference for such and only such laws of heat transfer that can be represented in the following form*

$$n(T_0,T) = \frac{M(T_0 - T)T^2}{1 + R(T)M(T_0 - T)}. \tag{6}$$

The proof of this and other statements are given in Appendix.
The following law of heat transfer gives an example of heat transfer law with temperature dependent heat transfer coefficient that obeys (6)

$$n(T_0,T) = \alpha T^2(T_0 - T).$$

4.2 When minimal dissipation thermodynamic process corresponds to constant entropy production?

The minimal dissipation problem for thermodynamic process with the scalar variable x has the following form:

$$\overline{\sigma} = \frac{1}{\tau} \int_0^{\tau} n(x,u)R(x,u)dt \to \min_u \tag{7}$$

subject to constraints

$$\dot{x} = f(x,u), \quad x(0) = x_0, \tag{8}$$

$$\int_0^\tau n(x,u)dt = \Delta N. \tag{9}$$

Here $\overline{\sigma}$ is the entropy production. The condition (8) characterizes the rate of change of the intensive variable of the system (temperature, pressure, chemical potential), $R(x,u)$ is the driving force of the process and $n(x,u)$ is the flow. The condition (9) determines the average rate of the flow. The derivation of the necessary condition of optimality for the problem (7)–(9) is similar to the derivations in Section 3.1. The conditions of optimality here have the following form [14]

$$F = \varphi(x,u) = \frac{n^2(x,u)}{n_u} R_u = Const. \tag{10}$$

It is required to determine for which dependencies $n(x,u)$ the entropy production is constant on the optimal solution, that is, where

$$\varphi(x,u) = n(x,u)R(x,u) = Const,$$

Thus, for this function φ we seek conditions on the kinetic function $n(x,u)$ that guarantee that on minimal dissipation processes (3) with this φ holds. After taking into account (4) we obtain

$$\frac{2\frac{n_x}{n} + \frac{R_{ux}}{R_u} - \frac{n_{ux}}{n_u}}{2\frac{n_u}{n} + \frac{R_{uu}}{R_u} - \frac{n_{uu}}{n_u}} = \frac{\varphi_x}{\varphi_u} = \frac{Rn_x + nR_x}{Rn_u + nR_u}. \tag{11}$$

The condition (11) singles out the class of minimal dissipation processes for which entropy production is independent on time and length. It is clear that this condition holds for both Newton and Fourier heat transfers.

It is been suggested that entropy production in minimal dissipation processes is always constant. From the condition (11) it is clear that for heat transfer this assumption is true for many but not all laws of heat transfer.

5 Microeconomic systems

Consider a market when one asset is traded and where only one intermediary (monopolist) buys and resells this commodity. This intermediary chooses the price $c(t)$ to minimize the price it pays (or to maximize the price is received) to acquire the inventory ΔN in the given period of time τ. The dependence $n(x, u)$ describes the demand/supply (for buying/selling) function. The asset price estimate p is the minimal price for which the flow of purchasing is zero. The condition of minimal capital dissipation (3) can be interpreted as the condition of minimal trading costs. It holds if the trading costs are minimal subject to the given average rates of trading.

5.1 Conditions of constancy of optimal premium

Formally the problem of optimal trading has the form

$$I = \int_0^\tau n(c,p)c dt \to \min_c$$

subject to constraints

$$\int_0^\tau n(c,p)dt = \Delta N, \qquad n > 0,$$

$$\dot{N} = -n(c,p), \qquad N(0) = N_0.$$

Here $c(t)$ is the trading price and $p(N)$ is a given function. The conditions of optimality of this problem has the following form [13]

$$F = \frac{1}{n^2(p,c)} \frac{\partial n}{\partial c} = const, \quad n(c,p) = 0 \quad \text{for} \quad c = p. \tag{12}$$

It's derivation is very similar to condition of optimality for heat transfer in section 4.1 It is clear that if $n(p, c)$ leads to the condition of optimality

$$\varphi(c,p) = c - p = const. \tag{13}$$

then it is guaranteed that the optimal premium (the difference between offered price and price estimate $\delta = c(t) - p$) is constant and time-independent.

The condition (12) is identical to (21) and the solution of the problem of finding the optimal constant premium price is identical to the solution of the problem of optimal irreversible heat exchange up to multiplier $1/T^2$. Therefore, the class of function $n(c,p)$ that guarantees the fulfilment of (13) has the form

$$n(c,p) = \frac{M(c-p)}{1 + R(p)M(c-p)}, \tag{14}$$

The condition (14) singles out the dependencies of flows on trading prices and asset price estimates for which the optimal premium is constant.

The expression (14) can be rewritten as

$$n(c,p) = \frac{\mu(\delta)}{1 + R(p)\mu(\delta)}.$$

Since

$$\int_0^\tau n(c,p)dt = \Delta N,$$

we get

$$\mu(\delta) = \frac{\Delta N}{\int_0^\tau \frac{dt}{1 + R(p(t))\mu(\delta)}}. \tag{15}$$

This condition determines the premium δ.

The process's irreversibility is characterized by the integral

$$\Delta S = \int_0^\tau \delta n(c,p)dt = \delta \Delta N.$$

The average dissipation (trading costs) is

$$\overline{\sigma} = \frac{\Delta S}{\tau} = \frac{\delta \Delta N}{\tau}. \tag{16}$$

The equalities (15), (16) determine the irreversibility of the process for any function $n(c,p)$, which has the form (14).

5.2 Conditions of constancy of the optimal flow of goods

The condition of optimality for trading (12) leads to the condition of constancy of the flow $n(p, c)$ on the optimal solution $c^*(p)$ if the left hand side of the equality (12) depends on n only

$$F(p, c) = \varphi(n(p, c)),$$

where φ is an arbitrary function or, which is the same, when n_c is a function of n

$$n_c(p, c^*) = \zeta(n(p, c^*)) \quad \forall p. \tag{17}$$

The Statement 3 holds that: *The minimal cost of trading corresponds to a constant time-independent flow of commodity if and only if the demand function can be represented as*

$$n(c, p) = (c - p)M(c - p). \tag{18}$$

Here M is an arbitrary nonnegative function of price difference.
The optimal dependence $c^*(p)$ is determined by the condition

$$(c^* - p)M(c^* - p) = n^* = \frac{\Delta N}{\tau}. \tag{19}$$

Example.
Suppose the dependence $n(c, p)$ is defined as

$$n(c, p) = \alpha \cdot arctg(c - p), \qquad \text{for} \quad c > p.$$

Because this expression obeys the condition (18), we obtain the following dependence of the optimal price $c^*(t)$ on time

$$c^*(t) = p(N^*) + tg\frac{\Delta N}{\tau\alpha},$$

here

$$N^*(t) = N_0 - \frac{\Delta N}{\tau}t.$$

The optimal flow of commodity is constant and equal to $\frac{\Delta N}{\tau}$.

Conclusion

In this paper the problem of thermodynamic systems' classification on the basis of the type of their minimal dissipation processes is formulated and solved. The minimal dissipation processes correspond to minimal-possible energy consumption and single out the boundary of thermodynamically feasible processes - realizability area. Minimal dissipation processes are obtained by solving optimal control problem for given kinetics. In this paper the inverse problem of finding kinetics using given conditions of optimality is solved.
The class of processes where minimal dissipation corresponds to a constant rate of entropy production is constructed.
Similar problems are solved for microeconomic systems. Capital dissipation here describes dissipation and can be interpreted as trade cost resulted from finite rate of trade. The results obtained in this paper allow us to find the class of demand functions for which optimal trading obeys some a priory given condition (e.g. the condition of constant optimal premium, that is, constant difference between equilibrium price and trading price).
The obtained results allow us to divide thermodynamic and microeconomic processes into classes of equivalent processes that have common-type minimal dissipation processes.

6 Acknowledgement

This work is supported by RFFI (grant 01-01-00020 and 02-06-80445), the School of Finance and Economics, UTS, AC3 and the Capital Markets CRC.

7 Appendix.

7.1 Proof of the Statement 2.

Suppose that the function $m(T_0, T)$ is defined as

$$m(T_0, T) = \frac{n(T_0, T)}{T^2}, \tag{20}$$

Suppose m and $n(T_0, T)$ are continuously differentiable. Substitution into (5) yields

$$F = \frac{1}{m^2(T_0, T)} \frac{\partial m}{\partial T_0} = const, \quad m(T_0, T_0) = 0 \tag{21}$$

and

$$\frac{F_T}{F_{T_0}} = \frac{\varphi_T}{\varphi_{T_0}} = -1, \tag{22}$$

or

$$F_T + F_{T_0} = 0. \tag{23}$$

Let us obtain F_T and F_{T_0}

$$\begin{aligned} F_T &= \tfrac{1}{m^2(T_0,T)} \tfrac{\partial^2 m}{\partial T_0 \partial T} - \tfrac{2}{m^3(T_0,T)} \tfrac{\partial m}{\partial T} \tfrac{\partial m}{\partial T_0}, \\ F_{T_0} &= \tfrac{1}{m^2(T_0,T)} \tfrac{\partial^2 m}{\partial T_0 \partial T_0} - \tfrac{2}{m^3(T_0,T)} \tfrac{\partial m}{\partial T_0} \tfrac{\partial m}{\partial T_0}. \end{aligned} \tag{24}$$

After substitution (24) into (23) we obtain

$$m_{T_0 T} + m_{T_0 T_0} = \frac{2}{m} m_{T_0}(m_{T_0} + m_T),$$

or

$$\frac{\partial}{\partial T_0}(m_T + m_{T_0}) = \frac{2}{m} m_{T_0}(m_T + m_{T_0}). \tag{25}$$

Formula (25) can be rewritten as

$$\frac{\frac{\partial}{\partial T_0}(m_{T_0} + m_T)}{m_{T_0} + m_T} = 2\frac{m_{T_0}}{m},$$

or

$$\frac{\partial}{\partial T_0} \ln|m_{T_0} + m_T| = 2\frac{\partial}{\partial T_0} \ln|m|.$$

Thus

$$\frac{\partial}{\partial T_0} \ln|\frac{m_T + m_{T_0}}{m^2}| = 0. \tag{26}$$

From (26) it follows that the expression under the derivative is an arbitrary continuous function of T

$$\ln|\frac{m_T + m_{T_0}}{m^2}| = \xi(T).$$

or

$$\frac{m_T + m_{T_0}}{m^2} = -f(T). \tag{27}$$

We will solve the equation (27) using the method of characteristics

$$\dot{T}_0 = 1, \quad \dot{T} = 1, \quad \dot{m} = -f(T)m^2.$$

The solutions of these equations are

$$T_0(t) = t + r_1, \quad T(t) = t + r_2,$$

$$\dot{m} = -f(t + r_2)m^2 \Rightarrow \frac{1}{dt}\left(\frac{1}{m}\right) = f(t + r_2),$$

$$\Rightarrow m(t) = \frac{1}{\int f(t + r_2)dt + c}, \tag{28}$$

where c is a constant, $f(t)$ is a continuous function. After taking into account (28), eliminating t and replacing dt with dT, we obtain its common solution in the following form

$$m(T_0, T) = \frac{1}{\int f(T)dT + \mu(T_0 - T)}, \tag{29}$$

where f and μ are arbitrary functions. We took into account here that because of (28) the difference $(T_0 - T)$ and any function of it are constant.

Suppose the function $\mu(T_0 - T)$ has the form

$$\mu(T_0 - T) = \frac{1}{M(T_0 - T)}.$$

Since functions f and μ in (28) are arbitrary functions, this solution can be rewritten in equivalent form

$$m(T_0, T) = \frac{M(T_0 - T)}{1 + R(T)M(T_0 - T)}, \tag{30}$$

where $R(T) = \int f(T)dT$ is differentiable on all of its arguments.

In order to take into account the condition $m(T_0, T) = 0$ for $T_0 = T$, we impose additional condition $M(0) = 0$ on the function $M(T_0 - T)$. After taking into account (20) and (30) we obtain the dependence $n(T_0, T)$ of the general form (6).

7.2 Proof of the Statement 3.

After taking into account (4) from the condition (17) we obtain

$$\frac{n_{cc}}{n_{cp}} = \frac{n_c}{n_p} \Rightarrow \frac{\partial}{\partial c} \ln \left| \frac{n_c}{n_p} \right| = 0 \Rightarrow \frac{n_c}{n_p} = r(p),$$

where r is an arbitrary function. This yields the following equation, which determines the form of the function n,

$$n_c - r(p)n_p = 0, \quad n(p,c) = 0 \quad \text{for} \quad p = c. \tag{31}$$

The equation of characteristic is

$$\dot{c} = 1, \quad \dot{p} = -r(p).$$

Thus

$$c(t) = c_0 + t, \quad \mu(p) = t - t_0, \tag{32}$$

where $\mu(p)$ is an arbitrary diffeerentiable function such that

$$\frac{d\mu}{dp} = -\frac{1}{r(p)}.$$

Elimination of t from (32) yields the first integral of the equation (31)

$$\mu(p) - c = t_0 - c_0 = const,$$

therefore the general solution is

$$n(c,p) = M[\mu(p) - c].$$

After taking into account that $n(p,c) = 0$ for $c = p$ we obtain a class of demand function (18), for which the flow asset is constant at the optimal solution.

References

[1] Lichnerowicz M., Annales de l'Institut Henri Poincaré, (1970).

[2] Martinás K., *Irreversible microeconomics*, In K. Martinás, M. Moreau (eds.) Complex Systems in Natural and Economic Sciences (Mátrafüred, 1995).

[3] Rozonoer L.I., Automation and remote control, (1973), 5, I, 115–132; 6, II, 65–79; 8, III, 82–103.

[4] Tsirlin A.M., *Optimization methods in irreversible thermodynamics and microeconomics* (Moscow, Science, 2002).

[5] Samuelsen P.A., THESIS, Winter 1993, **1**, 1, 184–202.

[6] Andresen B., *Finite-time thermodynamics* (Copenhagen, 1983).

[7] Berry R.S., Kasakov V.A., Sieniutycz S., Szwast Z. and Tsirlin A.M., *Thermodynamic Optimization of Finite Time Processes*, (Wiley,Chichester, 1999).

[8] Salamon P., Nitzan A., Andresen B. and Berry R.S., Phys. Rev. **A**, (1980) 21, 2115-2129.

[9] Sieniutycz S. and Salamon P., *Finite-Time Thermodynamics and Thermoeconomics*, (Taylor & Francis, 1990).

[10] *Ville J.*, Rev. Economics Studies, **19**, (1951), 123-128.

[11] Von Neumann J. A, Review of Economic Studies, **3**, (1945), 1-9.

[12] Kuznecov A.G., Rudenko A.V., Tsirlin A.M., Automation and Remote Control, (1985), 6, 56-62.

[13] Kolinko N. A., Tsirlin A.M., Proceeding of the international conference "Intellectual technologies in control problems", (1999), 172-177.

[14] Tsirlin A.M., Mironova V.A., Kazakov V., Amelkin S.A., Phys. Rev. E, (1998), **58**, 1.