Modeling the Volatility and Expected Value of a Diversified World Index

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Abstract. This paper considers a diversified world stock index in a continuous financial market with the growth optimal portfolio (GOP) as the reference unit or benchmark. Diversified broadly based portfolios, which include major world stock market indices, are shown to approximate the GOP. It is demonstrated that a key financial quantity is the drift of the discounted GOP, which can be expressed explicitly using a certain quadratic variation term. Using real market approximations for the discounted GOP it is shown that its drift does not vary greatly in the long term. For a diversified world index this leads to a natural model where the discounted index is a time transformed squared Bessel process of dimension four. The inverse of the squared GOP volatility then follows a square root process of dimension four.

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1 Introduction

Despite the rich literature that exists on continuous time financial market modeling there is still no single model that has been widely accepted. For some recent accounts one can refer to Merton (1992), Duffie (1996), Rebonato (1998), Musiela & Rutkowski (1997) and Björk (1998). Further, a substantial body of empirical evidence has been accumulated on the dynamics of market indices, see, for instance, Cont & da Fonseca (2002). These empirical findings suggest that at least a two-factor model is required to characterize an index.

In this paper we consider a diversified world stock portfolio, which covers the entire world stock market and address the key issue of how volatility should be modeled. By application of a limit theorem on diversified portfolios, see Platen (2003), the world stock portfolio will be shown to approximate the *growth optimal portfolio* (GOP), see Kelly (1956) and Long (1990). The *total market price for risk* determines the volatility of the GOP. It is the most important factor that drives the world stock index. A natural choice for the second factor is the short rate.

By using the *discounted GOP drift* we will show that the discounted GOP always displays the dynamics of a time transformed squared Bessel process of dimension four, see Platen (2002). In this framework it turns out that the corresponding time transformation does not fluctuate greatly. In addition, the inverse of the squared volatility for the discounted GOP is shown to be a time transformed square root process of dimension four. It will be demonstrated that the resulting model, where the discounted GOP drift increases on average according to the net growth rate of the market, captures some of the key empirical features observed for the world stock index. This occurs under the simple assumption of constant parameters.

Section 2 describes the benchmark approach and shows that the diversified world stock portfolio approximates the GOP. In Section 3 the world stock index is modeled and its volatility is analyzed in Section 4.

2 Benchmark Model

2.1 Primary Securities

Let us consider a continuous *financial market model* with $d+1$ *primary security account processes* $S^{(0)}$, $S^{(1)}$, $\ldots$, $S^{(d)}$, $d \in \{1, 2, \ldots\}$. These are the value processes of share accounts and savings accounts. It is assumed that for a primary security account the accrued income is always reinvested. The value of the jth primary security account at time t is denoted by $S^{(j)}(t)$ for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. We also assume that $S^{(j)}(t)$ is the unique strong solution of the stochastic differential equation (SDE)

$$dS^{(j)}(t) = S^{(j)}(t)\left\{a^j(t)\,dt + \sum_{k=1}^{d} b^{j,k}(t)\,dW^k(t)\right\} \tag{2.1}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$ with $S^{(j)}(0) > 0$ and $S^{(0)}(0) = 1$. Here the uncertainty is modeled by d independent standard Wiener processes $W^k = \{W^k(t),\, t \in [0,T]\}$, $k \in \{1,2,\dots,d\}$, which are defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0,\infty)$, fulfilling the usual conditions, see Protter (1990). The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ models the evolution of market information over time. Here $\mathcal{A}_t$ denotes the information available at time $t \in [0,T]$.

The jth *appreciation rate* $a^j = \{a^j(t),\, t \in [0,T]\}$ and (j,k)th *volatility* $b^{j,k} = \{b^{j,k}(t),\, t \in [0,T]\}$ are predictable stochastic processes such that

$$\int_0^T \left(|a^j(s)| + (b^{j,k}(s))^2\right) ds < \infty \tag{2.2}$$

for $j,k \in \{1,2,\dots,d\}$, see Protter (1990). For convenience, we set

$$b^{0,k}(t) = 0 \tag{2.3}$$

for $k \in \{1,2,\dots,d\}$ such that $S^{(0)}$ denotes the savings account with *short rate*

$$r(t) = a^0(t) \tag{2.4}$$

at time t. Furthermore, the *volatility matrix* $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ is for Lebesgue-almost-every $t \in [0,T]$ assumed to be *invertible*. This ensures that the uncertainty that is modeled by the Wiener processes $W^1,\dots,W^d$ uniquely determines security prices and vice versa.

Let $S = \{S(t) = (S^{(0)}(t),\dots,S^{(d)}(t))^\top,\, t \in [0,T]\}$ denote the vector of primary security accounts. Here $A^\top$ is the transpose of a vector or matrix A. By using the appreciation rate vector $a(t) = (a^1(t), a^2(t),\dots,a^d(t))^\top$ and the unit vector $\mathbf{1} = (1,1,\dots,1)^\top$, we introduce the *market price for risk* vector

$$\begin{aligned}\theta(t) &= (\theta^1(t), \theta^2(t),\dots,\theta^d(t))^\top \\ &= b^{-1}(t)\,[a(t) - r(t)\,\mathbf{1}]\end{aligned} \tag{2.5}$$

for $t \in [0,T]$. The market price for risk (2.5) allows us to rewrite the SDE (2.1) in the form

$$dS^{(j)}(t) = S^{(j)}(t)\left\{r(t)\,dt + \sum_{k=1}^{d} b^{j,k}(t)\,[\theta^k(t)\,dt + dW^k(t)]\right\} \tag{2.6}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$.

2.2 Strategies and Portfolios

We say that a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top,\ t \in [0,T]\}$ is a *strategy* if δ is S-integrable, see Protter (1990). The jth component $\delta^{(j)}(t) \in (-\infty, \infty)$ of the strategy δ denotes the number of units of the jth primary security account that are held at time $t \in [0,T]$ in the corresponding portfolio, $j \in \{0, 1, \ldots, d\}$. For a given strategy δ let $S^{(\delta)}(t)$ be the value of the corresponding portfolio at time t. This means that

$$S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, S^{(j)}(t) \tag{2.7}$$

for $t \in [0,T]$. A portfolio process $S^{(\delta)}$ and the corresponding strategy δ are called *self-financing* if

$$dS^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, dS^{(j)}(t) \tag{2.8}$$

for all $t \in [0,T]$. Thus, all changes in the value of a self-financing portfolio are due to corresponding gains from trade in the primary security accounts. Since we will consider in the following only self-financing strategies and corresponding self-financing portfolios, we omit from now on the word "self-financing".

For a given strategy δ we introduce the jth proportion $\pi_\delta^{(j)}(t)$ of the value of the jth primary security account held at time t in a strictly positive portfolio $S^{(\delta)}(t)$, given by

$$\pi_\delta^{(j)}(t) = \delta^j(t)\, \frac{S^{(j)}(t)}{S^{(\delta)}(t)} \tag{2.9}$$

for $t \in [0,T]$ and $j \in \{0, 1, \ldots, d\}$. Of course, the sum of the proportions always equals one, that is

$$\sum_{j=0}^{d} \pi_\delta^{(j)}(t) = 1 \tag{2.10}$$

for all $t \in [0,T]$. A strictly positive portfolio value $S^{(\delta)}(t)$ satisfies, according to (2.8) and (2.6), the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t)\, r(t)\, dt + \sum_{k=1}^{d} \beta_\delta^k(t)\, (\theta^k(t)\, dt + dW^k(t)) \tag{2.11}$$

with kth *portfolio volatility*

$$\beta_\delta^k(t) = \sum_{j=0}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t) \tag{2.12}$$

for $t \in [0, T]$. By application of the Itô formula, it follows from (2.11) that the logarithm of a strictly positive portfolio $S^{(\delta)}(t)$ satisfies the SDE

$$d \log \left(S^{\delta}(t) \right) = (r(t) + g_{\delta}(t))\, dt + \sum_{k=1}^{d} \beta_{\delta}^{k}(t)\, dW^{k}(t) \tag{2.13}$$

with *portfolio net growth rate*

$$g_{\delta}(t) = \sum_{k=1}^{d} \beta_{\delta}^{k}(t) \left(\theta^{k}(t) - \frac{1}{2}\, \beta_{\delta}^{k}(t) \right) \tag{2.14}$$

for $t \in [0, T]$. Here the portfolio volatility $\beta_{\delta}^{k}(t)$, given in (2.12), depends on the proportions.

2.3 Growth Optimal Portfolio

In a financial market model one has the freedom to choose a numeraire or benchmark. We choose as our benchmark the *growth optimal portfolio* (GOP), see Kelly (1956), Long (1990), Karatzas & Shreve (1998) or Platen (2002), and denote its value at time $t \in [0, T]$ by $S^{(\delta_*)}(t)$. The GOP is the portfolio that maximizes the portfolio net growth rate $g_{\delta}(t)$, see (2.14). The optimal proportions $\pi^{(\delta_*)} = \{\pi^{(\delta_*)}(t) = (\pi_{\delta_*}^{(1)}(t), \ldots, \pi_{\delta_*}^{(\delta)}(t))^{\top},\, t \in [0, T]\}$ follow directly by solving the first order equations for the corresponding quadratic maximization problem for the portfolio net growth rate $g_{\delta}(t)$ and are given by

$$\pi^{(\delta_*)}(t) = \left(b^{-1}(t)\right)^{\top} \theta(t) \tag{2.15}$$

for $t \in [0, T]$, see (2.5). The GOP then satisfies by (2.11) and (2.15) the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t) \left[r(t)\, dt + \sum_{k=1}^{d} \theta^{k}(t) \left(\theta^{k}(t)\, dt + dW^{k}(t) \right) \right] \tag{2.16}$$

for $t \in [0, T]$, see also Karatzas & Shreve (1998), where we assume that $S^{(\delta_*)}(0) > 0$.

From (2.16) and (2.5) it follows that the GOP volatilities $\theta^{k}(t)$, $k \in \{1, 2, \ldots, d\}$, are the corresponding market prices for risk. Note that the *risk premium* of the GOP appearing in (2.16) equals the square $|\theta(t)|^2$ of the *total market price for risk*

$$|\theta(t)| = \sqrt{\sum_{k=1}^{d} (\theta^{k}(t))^2} \tag{2.17}$$

for $t \in [0, T]$. The risk premium of the portfolio in (2.11) is given by the correlation of the returns of the portfolio with the returns of the GOP. This is consistent with the capital asset pricing model, see Merton (1973), if one interprets the market portfolio as GOP or as a proxy for the GOP. We call the above model a *benchmark model*. For practical applications of a benchmark model it is important to find a good approximation for the GOP. We address this problem in the following section which presents a limit theorem for diversified portfolios.

2.4 Approximate Growth Optimal Portfolios

In order to approximate the GOP using diversified portfolios, we formulate the following definitions.

Definition 2.1 *A strictly positive portfolio process $S^{(\delta)}$ is called a* diversified portfolio *(DP) if finite constants $K_1 > 0$, $K_2 > 0$ and $K_3 \in \{1, 2, \ldots\}$ exist, independent of d, such that*

$$|\pi_\delta^{(j)}(t)| \leq \frac{K_1}{d^{\frac{1}{2}+K_2}} \tag{2.18}$$

almost surely for all $j \in \{0, 1, \ldots, d\}$, $d \in \{K_3, K_3 + 1, \ldots\}$ and $t \in [0, T]$.

This means that the proportion $\pi_\delta^{(j)}(t)$ of the value of a DP, which is invested in the jth primary security account, need to decrease slightly faster than $d^{-\frac{1}{2}}$ as $d \to \infty$. This is, for instance, the case if equal proportions are used.

When we express a given portfolio $S^{(\delta)}(t)$ in units of the GOP, then we call the ratio

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} \tag{2.19}$$

the corresponding *benchmarked portfolio*. By application of the Itô formula and using (2.19), (2.11), (2.12) and (2.16) it follows that the benchmarked portfolio $\hat{S}^{(\delta)}(t)$ satisfies the SDE

$$d\hat{S}^{(\delta)}(t) = -\hat{S}^{(\delta)}(t) \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_\delta^{(j)}(t)\, \sigma^{j,k}(t)\, dW^k(t) \tag{2.20}$$

with jth *specific volatility*

$$\sigma^{j,k}(t) = b^{j,k}(t) - \theta^k(t) \tag{2.21}$$

for $t \in [0, T]$, $j \in \{0, 1, \ldots, d\}$ and $k \in \{1, 2, \ldots, d\}$. This allows us to introduce the kth *total specific volatility*

$$\hat{\sigma}^k(t) = \sum_{j=0}^{d} |\sigma^{j,k}(t)| \tag{2.22}$$

for $t \in [0, T]$ and $k \in \{1, 2, \ldots, d\}$.

Definition 2.2 *A benchmark model is called* regular *if there exist finite constants K_3 and K_4, independent of d, such that*

$$E\left(\left(\hat{\sigma}^k(t)\right)^2\right) \leq K_4 \tag{2.23}$$

for all $t \in [0,T]$, $k \in \{1,2,\ldots,d\}$ and $d \in \{K_3, K_3+1, \ldots\}$.

This is a property that arguably can be assumed for the world stock market consisting of all stocks traded on the existing exchanges. By (2.13) and (2.14) the difference between the logarithms of the GOP $S^{(\delta_*)}(t)$ and a given strictly positive portfolio $S^{(\delta)}(t)$ satisfies the SDE

$$d(\log(S^{(\delta_*)}(t)) - \log(S^{(\delta)}(t))) = \frac{1}{2}\, R^d_\delta(t)\, dt - \sum_{k=1}^{d}\sum_{j=0}^{d} \pi^{(j)}_\delta(t)\, \sigma^{j,k}(t)\, dW^k(t) \tag{2.24}$$

with *tracking rate*

$$R^d_\delta(t) = \sum_{k=1}^{d}\left(\sum_{j=0}^{d} \pi^{(j)}_\delta(t)\, \sigma^{j,k}(t)\right)^2 \tag{2.25}$$

for $t \in [0,T]$. Note that the tracking rate equals the squared diffusion coefficient of the SDE (2.24). It can be interpreted as a measure of the distance between a given portfolio $S^{(\delta)}(t)$ and the GOP $S^{(\delta_*)}(t)$ at time $t \in [0,T]$.

Definition 2.3 *For an increasing number d of risky primary security accounts we call a strictly positive portfolio $S^{(\delta)}$ an* approximate GOP *if the corresponding sequence of tracking rates $(R^d_\delta(t))_{d\in\{1,2,\ldots\}}$ vanishes in probability, that is for each $\varepsilon > 0$ we have*

$$\lim_{d\to\infty} P\left(R^d_\delta(t) > \varepsilon\right) = 0 \tag{2.26}$$

for all $t \in [0,T]$.

Under the above assumptions the following limit theorem is proved in Appendix A, see also Platen (2003).

Proposition 2.4 *For a regular benchmark model a diversified portfolio is an approximate GOP.*

This result allows us to conclude that a world stock portfolio that is a DP approximates the GOP. An obvious approximate GOP is the market capitalization weighted MSCI Growth World Stock Index (MSCI). In Figure 1 we plot for the period from 1970 until 2003 the MSCI when discounted by a US Dollar savings account. We use the daily MSCI data and short rates as provided by Thomson Financial. Because of Proposition 2.4 we interpret the MSCI as the GOP and use in the following both names interchangeably.

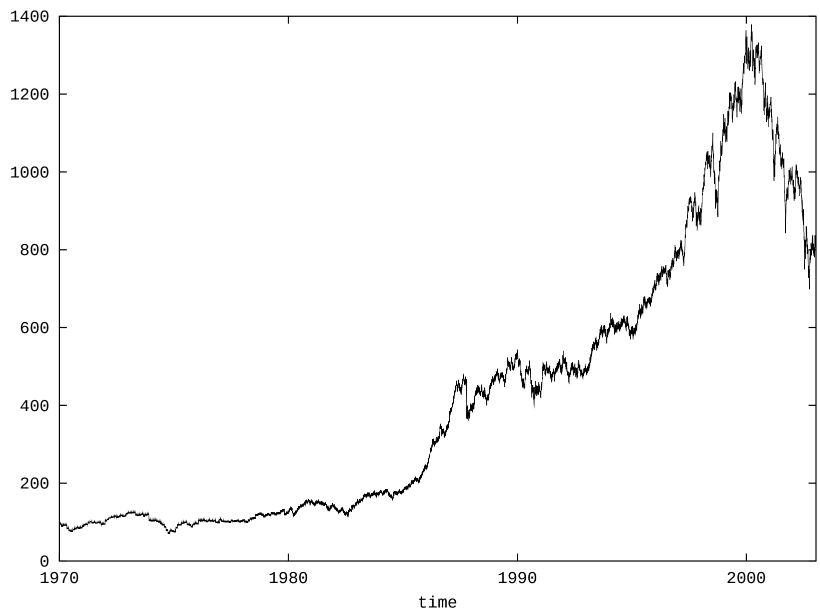

Figure 1: Discounted MSCI.

3 Modeling a World Stock Index

3.1 Discounted GOP

Let us discount the GOP $S^{(\delta_*)}(t)$ given in (2.16) by the savings account $S^{(0)}(t)$, see (2.1) - (2.4). The *discounted* GOP

$$\bar{S}^{(\delta_*)}(t) = \frac{S^{(\delta_*)}(t)}{S^{(0)}(t)} \tag{3.1}$$

satisfies by application of the Itô formula using (2.16) and (2.1) - (2.4) the SDE

$$d\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(t)\,|\theta(t)|\,(|\theta(t)|\,dt + d\hat{W}(t)) \tag{3.2}$$

for $t \in [0,T]$. The standard Wiener process $\hat{W} = \{\hat{W}(t),\, t \in [0,T]\}$ in (3.2) is characterized by the stochastic differential

$$d\hat{W}(t) = \frac{1}{|\theta(t)|}\sum_{k=1}^{d} \theta^k(t)\,dW^k(t), \tag{3.3}$$

for $t \in [0,T]$. By discounting the GOP we separate the impact of the short rate from that of the GOP volatility, see (2.16). Recall that Figure 1 displayed the discounted MSCI, which was interpreted as the discounted GOP for the world stock market.

Noting the form of (3.2) we introduce as a parameter process $\alpha = \{\alpha(t),\, t \in [0,T]\}$, which is set equal to the drift of the discounted GOP. We refer to it as the *discounted GOP drift*, where

$$\alpha(t) = \bar{S}^{(\delta_*)}(t)\,|\theta(t)|^2 \tag{3.4}$$

for $t \in [0, T]$. This parametrization leads to a GOP volatility of the form

$$|\theta(t)| = \sqrt{\frac{\alpha(t)}{\bar{S}^{(\delta_*)}(t)}}. \tag{3.5}$$

Thus, by (3.2), (3.4) and (3.5) we obtain for the discounted GOP the SDE

$$d\bar{S}^{(\delta_*)}(t) = \alpha(t)\,dt + \sqrt{\alpha(t)\,\bar{S}^{(\delta_*)}(t)}\,d\hat{W}(t) \tag{3.6}$$

for $t \in [0, T]$. Since $\alpha(t)$ appears in the drift and $\sqrt{\alpha(t)}$ in the diffusion coefficient of the SDE (3.6) it is natural to introduce the *GOP time* $\varphi = \{\varphi(t),\, t \in [0, T]\}$ with

$$\varphi(t) = \frac{1}{4}\int_0^t \alpha(s)\,ds \tag{3.7}$$

for $t \in [0, T]$. Note that the GOP time is, in general, random. For the discounted GOP process $X = \{X(\varphi),\, \varphi \in [0, \varphi(T)]\}$ considered in GOP time with

$$X(\varphi(t)) = \bar{S}^{(\delta_*)}(t) \tag{3.8}$$

we obtain by (3.6) the SDE

$$dX(\varphi) = 4\,d\varphi + \sqrt{4\,X(\varphi)}\,d\hat{W}_\varphi \tag{3.9}$$

for $\varphi \in [0, \varphi(T)]$ with $X(0) = \bar{S}^{(\delta_*)}(0)$, where

$$d\hat{W}_{\varphi(t)} = \sqrt{\frac{\alpha(t)}{4}}\,d\hat{W}(t) \tag{3.10}$$

for $t \in [0, T]$. It follows from (3.9) that X is in GOP time a *squared Bessel process of dimension four*, see Revuz & Yor (1999). Therefore, the process $\bar{S}^{(\delta_*)} = \{\bar{S}^{(\delta_*)}(t),\, t \in [0, T]\}$ in (3.6) is a time transformed squared Bessel process of dimension four using the above GOP time as the natural choice for an intrinsic time transformation.

3.2 Expected Discounted GOP

By application of the Itô formula it follows from (3.9) that the square root $\sqrt{X(\varphi)}$ of the discounted GOP, when expressed in GOP time φ, satisfies the SDE

$$d\sqrt{X(\varphi)} = \frac{3}{2\sqrt{X(\varphi)}}\,d\varphi + d\hat{W}_\varphi \tag{3.11}$$

for $\varphi \in [0, \varphi(T)]$. Note that the diffusion term in (3.11) is simply a standard Wiener process in GOP time.

For a given sequence of observation times $t_0 < t_1 < \ldots$ with maximum time step size $h \geq t_i - t_{i-1} > 0$, $i \in \{1, 2, \ldots\}$, let us introduce for $t \in [0, T]$ the integer

$$i_t = \max\{\ell \in \{0, 1, \ldots\} : t_\ell \leq t\} \tag{3.12}$$

as the largest index ℓ of observation times t_ℓ not greater than t.

The *quadratic variation* $\langle L \rangle = \{\langle L \rangle_t,\, t \in [0, T]\}$ of a continuous stochastic process $L = \{L(t),\, t \in [0, T]\}$ is for each $t \in [0, T]$ defined as the limit in probability

$$\langle L \rangle_t \overset{\mathrm{P}}{=} \lim_{h \to 0} \langle L \rangle_{h,t}, \tag{3.13}$$

where $\langle L \rangle_{h,t}$ is the *approximate quadratic variation*

$$\langle L \rangle_{h,t} = \sum_{i=1}^{i_t} (L(t_i) - L(t_{i-1}))^2. \tag{3.14}$$

The quadratic variation of a solution of an SDE, with a standard Wiener process as the diffusion term as in (3.11), is time itself. It therefore follows from (3.7), (3.8) and (3.11) that the quadratic variation of $\sqrt{\bar{S}^{(\delta_*)}}$ is the GOP time, that is

$$\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t = \left\langle \sqrt{X} \right\rangle_{\varphi(t)} = \varphi(t) \tag{3.15}$$

for all $t \in [0, T]$. Figure 2 shows the approximate quadratic variation of the square root of the discounted MSCI for the period from 1970 until 2003. This is a proxy for the GOP time $\varphi(t)$ by (3.15). One notes that the GOP time is monotonically increasing and does not appear to fluctuate greatly. By (3.7) and (3.15) it follows that the discounted GOP drift, see (3.5), is proportional to the slope of the GOP time. This allows us to formulate the following result.

Corollary 3.1 *The discounted GOP drift is equal to four times the slope of the quadratic variation of the square root of the discounted GOP, that is*

$$\alpha(t) = 4 \, \frac{d}{dt} \left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t \tag{3.16}$$

for $t \in [0, T]$.

This is a fundamental relationship that links the market trend with the overall market fluctuations. Note that we have not made any major assumptions on the dynamics of the market. We have merely parameterized the GOP dynamics by using the discounted GOP drift. As a consequence of Corollary 3.1 and (3.6) it follows that the increase in the expected discounted GOP must equal four times that of the expected increase of the GOP time. More precisely, we obtain the following result, where we refer to (3.7) and (3.15) with respect to the GOP time $\varphi(t)$.

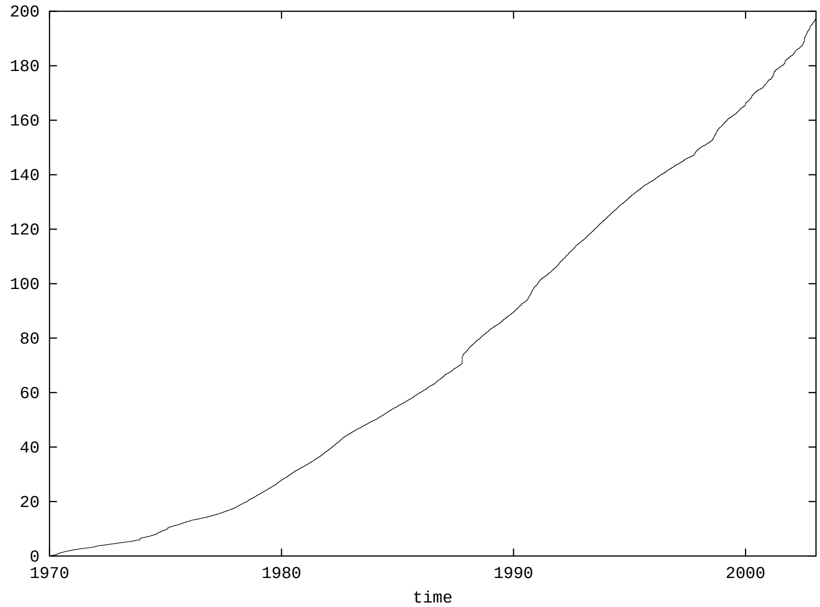

Figure 2: Approximate quadratic variation $\langle\sqrt{\bar{S}^{(\delta_*)}}\rangle_{h,t}$.

Corollary 3.2 *Under the assumption that the driftless process* $M = \{M(t) = \bar{S}^{(\delta_*)}(t) - 4\,\varphi(t),\ t \in [0,T]\}$ *is an* $(\underline{\mathcal{A}}, P)$*-martingale we have the relation*

$$E\left(\bar{S}^{(\delta_*)}(s)\,\middle|\,\mathcal{A}_t\right) = \bar{S}^{(\delta_*)}(t) + 4\,\left\{E(\varphi(s)\,\middle|\,\mathcal{A}_t) - \varphi(t)\right\} \tag{3.17}$$

for $t \in [0,T]$.

The martingale assumption on M is a rather modest requirement and is likely to be satisfied for real markets. Corollary 3.2 can be interpreted as a law for the conditional *expected future value* of the discounted world stock index. To illustrate this we plot in Figure 3 the logarithm of the discounted MSCI, that is $\log(\bar{S}^{(\delta_*)}(t))$, together with the logarithm of the sum of the initial discounted MSCI plus four times the observed GOP time, that is $\log(\bar{S}^{(\delta_*)}(0) + 4\,\langle\sqrt{\bar{S}^{(\delta_*)}}\rangle_t)$. Based on this figure and analysis the market was probably undervalued from about 1974 until 1986. Market prices seem to have been almost right during the following nine years but were overvalued from 1995 until about 2000 when the new technology bubble burst and global recessionary forces emerged. It is interesting to note that the overall market now seems to have become slightly undervalued in the first part of 2003.

Corollary 3.2 provides a fundamental relationship that permits us to quantify the expected evolution of the world stock index if one assumes that the GOP time would behave in a similar manner to what it did in the past. One must emphasize that this general relationship is simply a consequence of the structure of the discounted GOP and does not require any further assumptions or conditions.

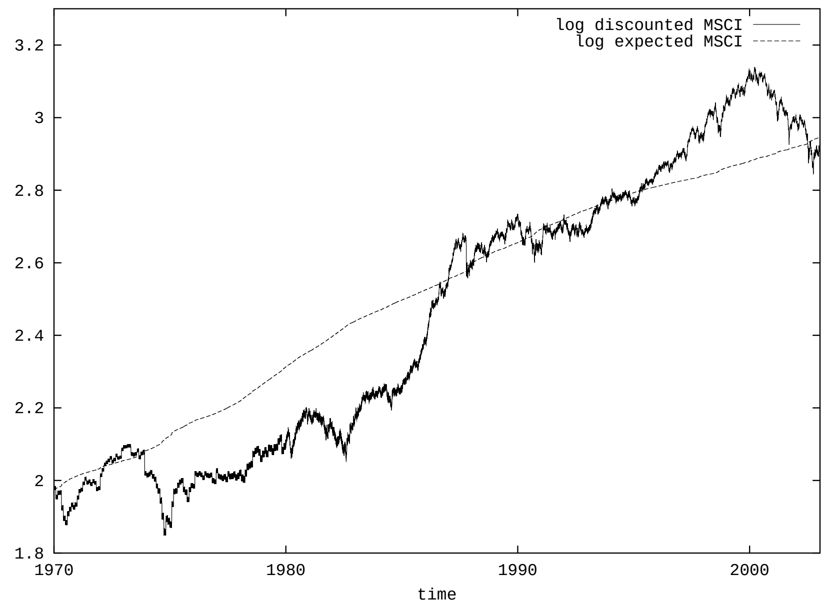

Figure 3: Logarithm of discounted and expected MSCI.

3.3 Normalized GOP

As shown in Figure 2 the GOP time $\varphi(t)$ appears to be fairly smooth and does not fluctuate greatly. For this reason we further parameterize the discounted GOP drift $\alpha(t)$ in the form

$$\alpha(t) = \alpha_0 \exp\left\{\eta \int_0^t m(s)\, ds\right\} m(t) \tag{3.18}$$

for $t \in [0, T]$, $\alpha_0 > 0$. Here η is the *net growth rate* of the market. For simplicity, we assume that it is a constant. The net growth rate captures the average growth of the discounted GOP drift. In addition, we call the factor $m(t)$ appearing in (3.18) the *market activity* at time t, which can be modeled by an adapted stochastic process that fluctuates around the value one.

Furthermore, we introduce the *normalized GOP* $Y(t)$ at time t, which is set equal to the discounted GOP expressed in units of the discounted GOP drift, but without the market activity $m(t)$. That is we define

$$Y(t) = \frac{\bar{S}^{(\delta_*)}(t)}{\alpha_0} \exp\left(-\eta \int_0^t m(s)\, ds\right) \tag{3.19}$$

for $t \in [0, T]$. It is straightforward to see by application of the Itô formula and using (3.19) and (3.6) that $Y(t)$ satisfies the SDE

$$dY(t) = \eta \left(\frac{1}{\eta} - Y(t)\right) m(t)\, dt + \sqrt{Y(t)\, m(t)}\, d\hat{W}(t) \tag{3.20}$$

for $t \in [0, T]$ with initial value

$$Y(0) = \frac{\bar{S}^{(\delta_*)}(0)}{\alpha_0}. \tag{3.21}$$

The normalized GOP is a square root process, which inherits the dimension four from the squared Bessel process $\bar{S}^{(\delta_*)}$. The process Y is determined by its initial value $Y(0)$, the net growth rate η and the market activity $m(t)$ for $t \in [0, T]$. The square root of this process satisfies by application of the Itô formula and (3.20) the SDE

$$d\left(\sqrt{Y(t)}\right) = \left(\frac{3}{8\sqrt{Y(t)}} - \frac{\eta}{2}\sqrt{Y(t)}\right) m(t)\, dt + \frac{\sqrt{m(t)}}{2}\, d\hat{W}(t) \tag{3.22}$$

for $t \in [0, T]$. Since the diffusion coefficient in (3.22) equals $\frac{1}{2}\sqrt{m(t)}$, we obtain the following representation for the market activity

$$m(t) = 4\,\frac{d}{dt}\left\langle \sqrt{Y} \right\rangle_t \tag{3.23}$$

for $t \in [0, T]$. Equation (3.23) is evidently a powerful relationship that allows us to observe and calibrate the market activity through the quadratic variation of $\sqrt{Y(t)}$.

4 Volatility of the World Stock Index

In the following let us check whether the MSCI can be reasonably calibrated by using the previously introduced parametrization. For simplicity, we assume constant market activity $m(t) = 1$. From equation (3.16) it follows by (3.18) that

$$\alpha_0 = 4\,\frac{d}{dt}\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_0. \tag{4.1}$$

The estimated value of $\alpha_0 = 10.5$ is obtained from the slope of the curve shown in Figure 2. Let us also estimate the value of the net growth rate η. By application of the Itô formula we obtain from (3.6) and (3.5) for the logarithm of the discounted GOP the SDE

$$d\,\log(\bar{S}^{(\delta_*)}(t)) = \frac{1}{2}\,|\theta(t)|^2\, dt + |\theta(t)|\, d\hat{W}(t) \tag{4.2}$$

for $t \in [0, T]$. Thus, for the period $[t, s]$, we get from (4.2) and (3.5) for the discounted GOP the growth rate

$$\bar{g}^{\delta_*}_{t,s} = \frac{1}{s-t}\, E\left(\log\left(\frac{\bar{S}^{(\delta_*)}(s)}{\bar{S}^{(\delta_*)}(t)}\right) \,\Big|\, \mathcal{A}_t\right) = \frac{1}{2\,(s-t)} \int_t^s E\left(|\theta(z)|^2 \,\Big|\, \mathcal{A}_t\right) dz \tag{4.3}$$

for $t \in [0, T]$ and $s \in [t, T]$. The explicitly known transition density for the squared Bessel process X of dimension four can now be employed to compute the expectation of the squared GOP volatility $|\theta(t)|^2 = \frac{\alpha(t)}{X(\varphi(t))}$, see (3.5). We then obtain from (4.3), as shown in Appendix B, the limit

$$\lim_{\varepsilon\to\infty} \bar{g}^{\delta_*}_{t,t+\varepsilon} = \eta. \tag{4.4}$$

Thus, it follows from (4.4) for $m(t) = 1$ that the net growth rate η equals the long-term average growth rate of the discounted GOP.

The long-term history from 1900 until 2000 of an inflation adjusted market capitalization weighted world stock index was reconstructed in Dimson, Marsh & Staunton (2002) and provided a net growth rate estimate of about $\eta = 0.048$. With this parameter estimate and the previously obtained parameter $\alpha_0 = 10.5$ we can now calculate the corresponding trajectory of the normalized MSCI $Y(t)$, see (3.19), which is shown in Figure 4.

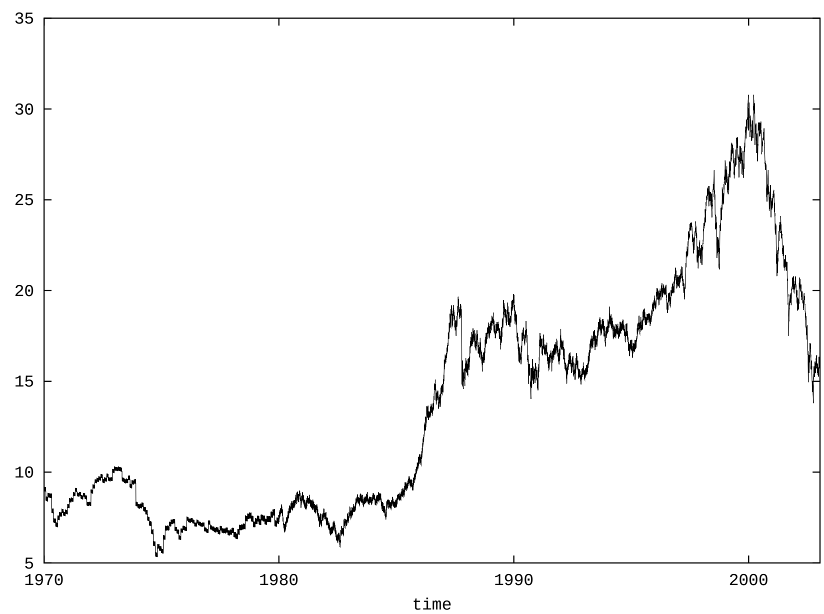

Figure 4: The normalized MSCI $Y(t)$.

One notes that the normalized MSCI behaves in a similar manner to the discounted MSCI plotted in Figure 1, but the long-term exponential growth is no longer present. From the estimated value of the net growth rate and the SDE (3.20) it is clear that the long-term average value of the normalized GOP $Y(t)$ is about $\frac{1}{\eta} = 20.8$. It can be seen that this level was approximately attained for the years 1996 and 2001. Note that the speed of adjustment parameter $\eta = 0.048$ is rather low. Consequently, the expected half-life of a shock on the normalized MSCI Y is about $\frac{\log(2)}{\eta} = 14.4$ years. This finding is supported by the trajectory displayed in Figure 4. For the constant parameter settings considered here, the volatility $|\theta(t)|$ of the discounted GOP equals, according to relations (3.5) and

(3.19), the inverse of the square root of the normalized GOP, that is

$$|\theta(t)| = \sqrt{\frac{1}{Y(t)}}. \tag{4.5}$$

Consequently, according to this analysis, the volatility is a very slowly mean-reverting process. For the period 1970 until 2003 the volatility of the MSCI is shown in Figure 5. Inspection of this plot indicates that the volatilities are contained within the interval $[0.15, 0.45]$.

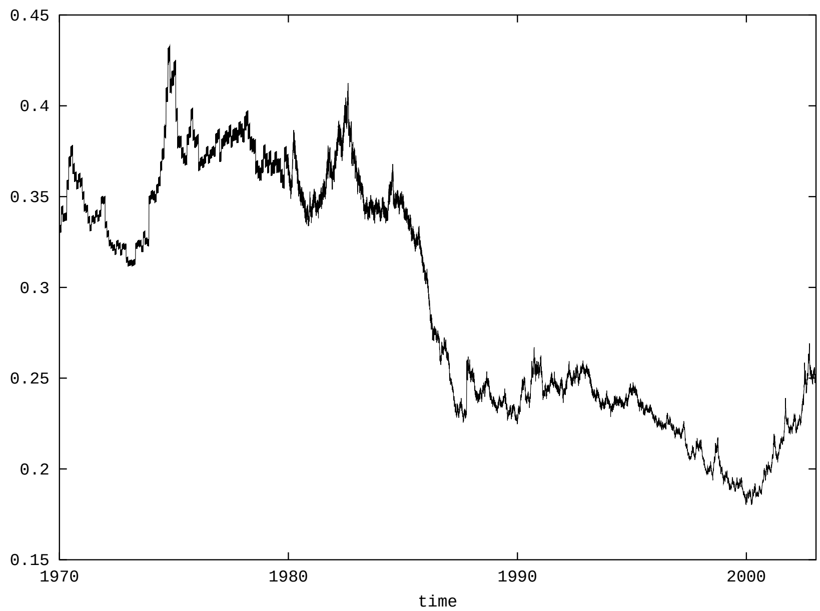

Figure 5: Volatility $|\theta(t)|$ of MSCI.

A comparison of the volatility shown in Figure 5 with that of the discounted GOP shown in Figure 1 reveals a strong negative correlation, which is due to the fact that the squared volatility is proportional to the inverse of the normalized MSCI. This feature reflects the well-known *leverage effect*, see Black (1976), that is characteristic of stock market indices.

The approximate quadratic variation of the square root of the normalized MSCI $\sqrt{Y(t)}$ is displayed in Figure 6. The quadratic variation shown is quite linear with an approximate slope of $\frac{1}{4}$. This corresponds to the average slope suggested in (3.7). One could now model the market activity process $m = \{m(t), t \in [0, T]\}$ to obtain greater modeling precision. The exponential of an Ornstein-Uhlenbeck process with strong mean reversion and some superimposed seasonal pattern is a good candidate for a model of the market activity. However, it is apparent that even for constant parameters the above described benchmark model reflects quite well the overall market behaviour.

The approximate linearity of $\langle\sqrt{Y}\rangle_t$ with slope $\frac{1}{4}$ indicates by application of Lévy's theorem, see Karatzas & Shreve (1991), that the diffusion term of the square root of the normalized GOP is approximately one-half of a standard Wiener process. This is an important empirical fact. It suggests that the volatility of the GOP is in reality close to the inverse of the square root of a four dimensional square root process, see (4.5) and (3.20). This kind of stochastic volatility is significantly different from a deterministic function of time, as used by the standard Black-Scholes model.

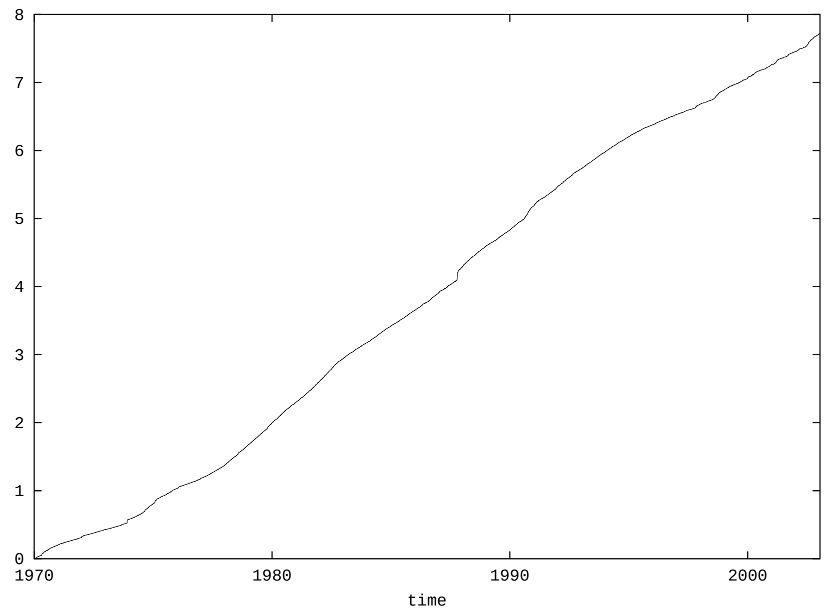

Figure 6: Quadratic variation $\langle\sqrt{Y}\rangle_{h,t}$ of square root of normalized MSCI for constant parameters.

An important fact to consider here is that we have modeled and calibrated the evolution of the discounted GOP, see (3.1), in units of a US Dollar savings account. This provides us with information about the evolution of the value of the US Dollar savings account relative to the world stock index. Of course, it is possible to apply the above analysis for any other primary security account and portfolio. This results in a characterization of the evolution of the value of securities with respect to the world stock index, which will be described in a forthcoming paper. In particular, the seasonal and stochastic components in the market activity will be identified and modeled using intraday data.

Conclusion

The paper derives a general relationship between the expected value of a discounted world index and the quadratic variation of its square root. The volatility of the world index is modeled by the inverse of a square root process of dimension four. The approach can be generalized by using a stochastic market activity. This analysis applies to all denominations of a diversified world index. Future research will focus on modeling stochastic market activity and the incorporation of jumps in primary security accounts.

A Appendix

Proof of Proposition 2.4:

We estimate by using (2.25), (2.18) and (2.23) for a DP $S^{(\delta)}$ in a regular benchmark model its expected tracking rate. That is

$$\begin{aligned} e^d_\delta(t) = E\left(R^d_\delta(t)\right) &\leq \sum_{k=1}^d E\left(\left(\sum_{j=0}^d |\pi^{(j)}_\delta(t)|\,|\sigma^{j,k}(t)|\right)^2\right) \\ &\leq \sum_{k=1}^d \left(\frac{(K_1)^2}{d^{1+2K_2}}\,K_4\right) \\ &\leq (K_1)^2\,K_4\,d^{-2K_2} \end{aligned}$$

for $t \in [0,T]$, where $d \in \{K_3, K_3+1, \ldots\}$. Consequently, since $K_2 > 0$ it follows by the Markov inequality for any given $\varepsilon > 0$ that

$$\lim_{d\to\infty} P\left(R^d_\delta(t) > \varepsilon\right) \leq \lim_{d\to\infty} \frac{1}{\varepsilon}\, e^d_\delta(t) = 0$$

for all $t \in [0,T]$. This proves by Definition 2.3 the Proposition 2.4. □

B Appendix

Proof of relation (4.4):

By application of the transition density for a squared Bessel process, see Revuz & Yor (1999), we obtain

$$E(X(\varphi)^{-1}) = X(0)^{-1}\left(1 - \exp\left(\frac{-X(0)}{2\,\varphi}\right)\right).$$

Thus with $\varphi(t) = \frac{1}{4}\int_0^t \alpha_0\, \exp(-\eta\, s)ds$ and $Y(t) = \frac{X(\varphi(t))}{\alpha_0\,\exp(\eta t)}$ we get

$$E\left((Y(t))^{-1}\right) = (Y(0))^{-1}\,\exp(-\eta\, t)\left(1 - \exp\left(-\frac{Y(0)\,\alpha_0}{2\,\varphi(t)}\right)\right).$$

Therefore

$$\lim_{t\to\infty} E\left(|\theta(t)|^2\right) = \lim_{t\to\infty} E\left((Y(t))^{-1}\right) = 2\,\eta$$

proves the limit condition (4.4). □

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