Continuous-Time Mean-Variance Portfolio Selection with Proportional Transaction Costs

ZUOQUAN XU
The Chinese University of Hong Kong

Joint work with:
Min Dai, National University of Singapore
Xun Yu Zhou, The Chinese University of Hong Kong

Quantitative Methods in Finance 2006, Sydney

Introduction

- Various continuous-time mean-variance models have been proposed and solved by Duffie, Jin, Li, Pliska, Richardson, Rorn, Zhou, etc.
- Davis and Norman considered a portfolio selection problem in a market with proportional transaction costs in *Portfolio Selection with Transaction Costs.*
- Dai and Yi studied finite horizon optimal investment problem in a market with transaction costs.

Market

A Black-Scholes Market:

Bank Account: $dR(t) = rR(t)dt$

Stock Account: $dS(t) = \alpha S(t)dt + \sigma S(t)dB(t)$

Assumption: The parameters r, α and σ are all constants, and $\alpha > r$.

Wealth Process

$M(t)$: the cumulative dollar value of stock bought up to time t

$N(t)$: the cumulative dollar value of stock sold up to time t

$X(t)$: the number of dollars in bank account at time t

$$dX(t) = rX(t)dt - (1+\lambda)dM(t) + (1-\mu)dN(t),$$

$Y(t)$: the number of dollars invested in the stock at time t

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dB(t) + dM(t) - dN(t),$$

Wealth Process

$W(t)$: the net wealth at time t

$$W(t) = X(t) + (1-\mu)Y(t)^{+} - (1+\lambda)Y(t)^{-}$$

$\mathcal{W}_s^{x,y}$: the attainable net wealth set

$$\mathcal{W}_s^{x,y} = \left\{ W \,\middle|\, \begin{array}{l} \text{There exits an admissible} \\ \text{process } (X,Y) \text{ such that} \\ X(s-) = x,\ Y(s-) = y \text{ and} \\ W(T) = W. \end{array} \right\}.$$

Mean-Variance Problem

MV problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{E}[W^2], \\ \text{subject to} & \mathbf{E}[W] = z, \\ & W \in \mathcal{W}_0^{x,y}. \end{array}$$

Assumption:

$$z > e^{rT}x + (1-\mu)e^{rT}y^+ - (1+\lambda)e^{rT}y^-.$$

Difficulties

- State constraint.
- The problem is a singular control problem in a finite time horizon.

Feasibility

Proposition 1: The MV problem is feasible if and only if

$$z \in \widetilde{\mathcal{D}},$$

where

$$\widetilde{\mathcal{D}} = \begin{cases} (e^{rT}x + (1-\mu)e^{rT}y^{+} - (1+\lambda)y^{-}, +\infty), & \text{if } T_0 > 0, \\ (e^{rT}x + (1-\mu)e^{rT}y, e^{rT}x + (1-\mu)e^{\alpha T}y], & \text{if } T_0 = 0,\ y > 0. \\ \emptyset, & \text{if } T_0 = 0,\ y \leqslant 0. \end{cases}$$

$$T_0 = \max\left\{T - \frac{1}{\alpha - r}\ln\frac{1+\lambda}{1-\mu}, 0\right\}.$$

Unconstrained Problem

U problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{E}[(W - \ell^*)^2], \\ \text{subject to} & W \in \mathcal{W}_0^{x,y}. \end{array}$$

Or, equivalently,

$$\begin{array}{ll} \text{minimize} & \mathbf{E}[W^2], \\ \text{subject to} & W \in \mathcal{W}_0^{x-\ell^* e^{-rT},y}. \end{array}$$

Value Function

$$V(t,x,y) = \inf_{W \in \mathcal{W}_t^{x,y}} \mathbf{E}[W^2].$$

HJB equation:

$$\begin{cases} \min\ \{\varphi_t + \mathcal{L}_0\varphi, (1-\mu)\varphi_x - \varphi_y, \varphi_y - (1+\lambda)\varphi_x\} = 0, \\ \varphi(T,x,y) = (x + (1-\mu)y^+ - (1+\lambda)y^-)^2. \end{cases}$$

$$\forall\ (t,x,y) \in [0,T) \times \mathscr{S}$$

$$\mathcal{L}_0\varphi = \frac{1}{2}\sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + rx\varphi_x.$$

$$\mathscr{S} = \left\{(x,y) \in \mathbb{R}^2 \middle|\, x + (1-\mu)y^+ - (1+\lambda)y^- < 0\right\}.$$

Equivalent Problem

$$\overline{V}(t,x) = \frac{1}{2}\ln V(t,x,1),$$

$$\begin{cases} w_t + \overline{\mathcal{L}}_0 w = 0, \text{ if } \frac{1}{x+1+\lambda} I(x) < w_x(t,x) < \frac{1}{x+1-\mu}, \\ w_t + \overline{\mathcal{L}}_0 w \geqslant 0, \text{ if } w_x(t,x) = \frac{1}{x+1+\lambda} I(x), \\ w_t + \overline{\mathcal{L}}_0 w \geqslant 0, \text{ if } w_x(t,x) = \frac{1}{x+1-\mu}, \\ w(T,x) = \ln(-x-(1-\mu)), \end{cases}$$

$$\forall\ (t,x) \in [0,T) \times \mathscr{X}.$$

$$\overline{\mathcal{L}}_0 w = \frac{1}{2}\sigma^2 x^2 (w_{xx} + 2w_x^2) - (\alpha - r + \sigma^2) x w_x + \alpha + \frac{1}{2}\sigma^2,$$

$$I(x) = \begin{cases} 1, & \text{if } x < -(1+\lambda), \\ -\infty, & \text{if } x \geqslant -(1+\lambda). \end{cases} \qquad \mathscr{X} = (-\infty, -(1-\mu)).$$

Double-Obstacle Problem

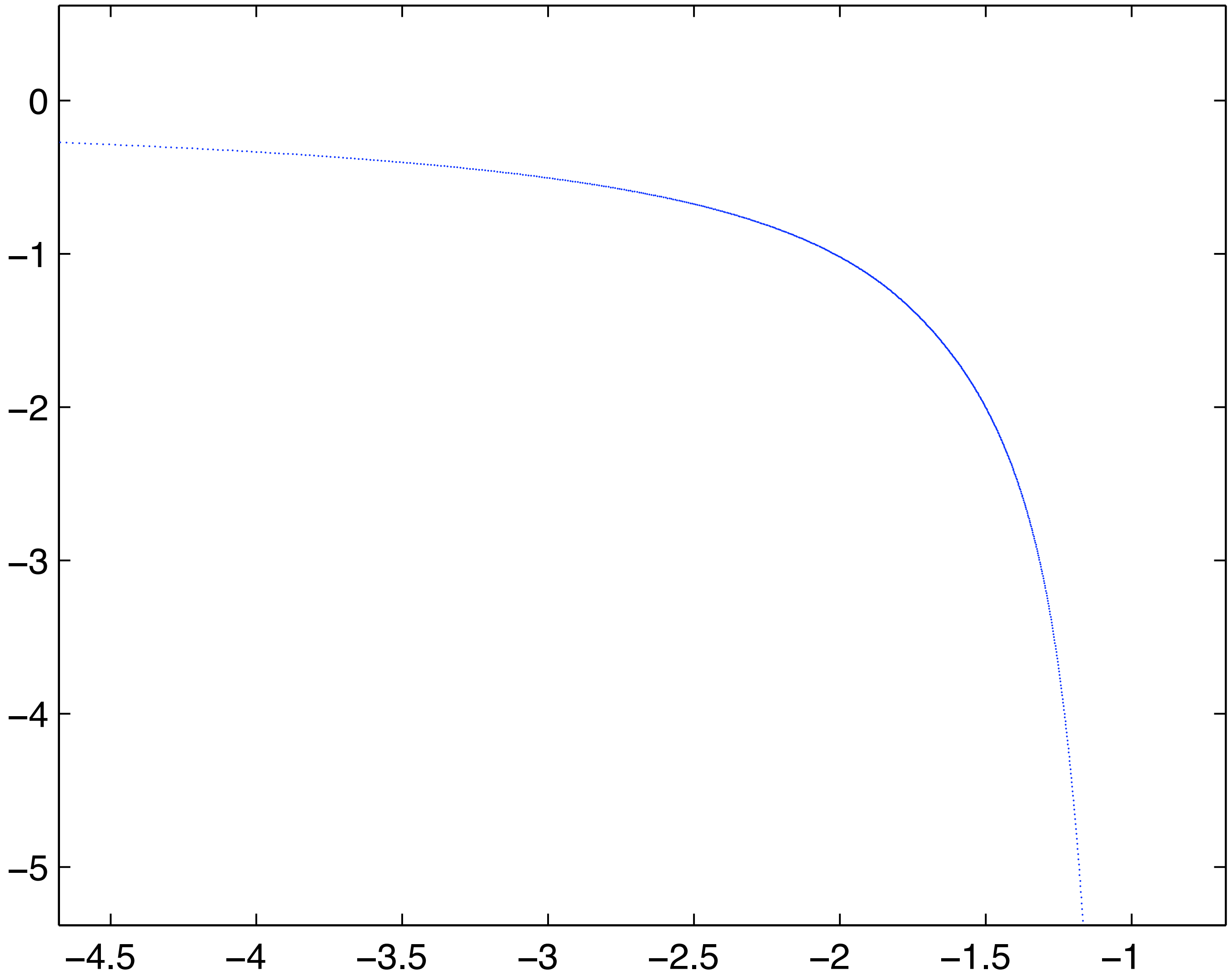
$$\widetilde{V}(t,x) = \frac{\partial}{\partial x}\overline{V}(t,x),$$

DO problem:

$$\begin{cases} v_t + \mathcal{L}v = 0, \text{ if } \frac{1}{x+1+\lambda}I(x) < v(t,x) < \frac{1}{x+1-\mu}, \\ v_t + \mathcal{L}v \leqslant 0, \text{ if } v(t,x) = \frac{1}{x+1+\lambda}I(x), \\ v_t + \mathcal{L}v \geqslant 0, \text{ if } v(t,x) = \frac{1}{x+1-\mu}, \\ v(T,x) = \frac{1}{x+1-\mu}. \end{cases}$$

$$\mathcal{L}v = \frac{1}{2}\sigma^2 x^2 v_{xx} - (\alpha - r)x v_x - (\alpha - r + \sigma^2)v + 2\sigma^2(x^2 v v_x + x v^2).$$

Solution to the Double-Obstacle Problem



$\alpha = 0.15 \quad r = 0.05 \quad \sigma = 0.2 \quad \lambda = 0.02 \quad \mu = 0.02 \quad T = 2$

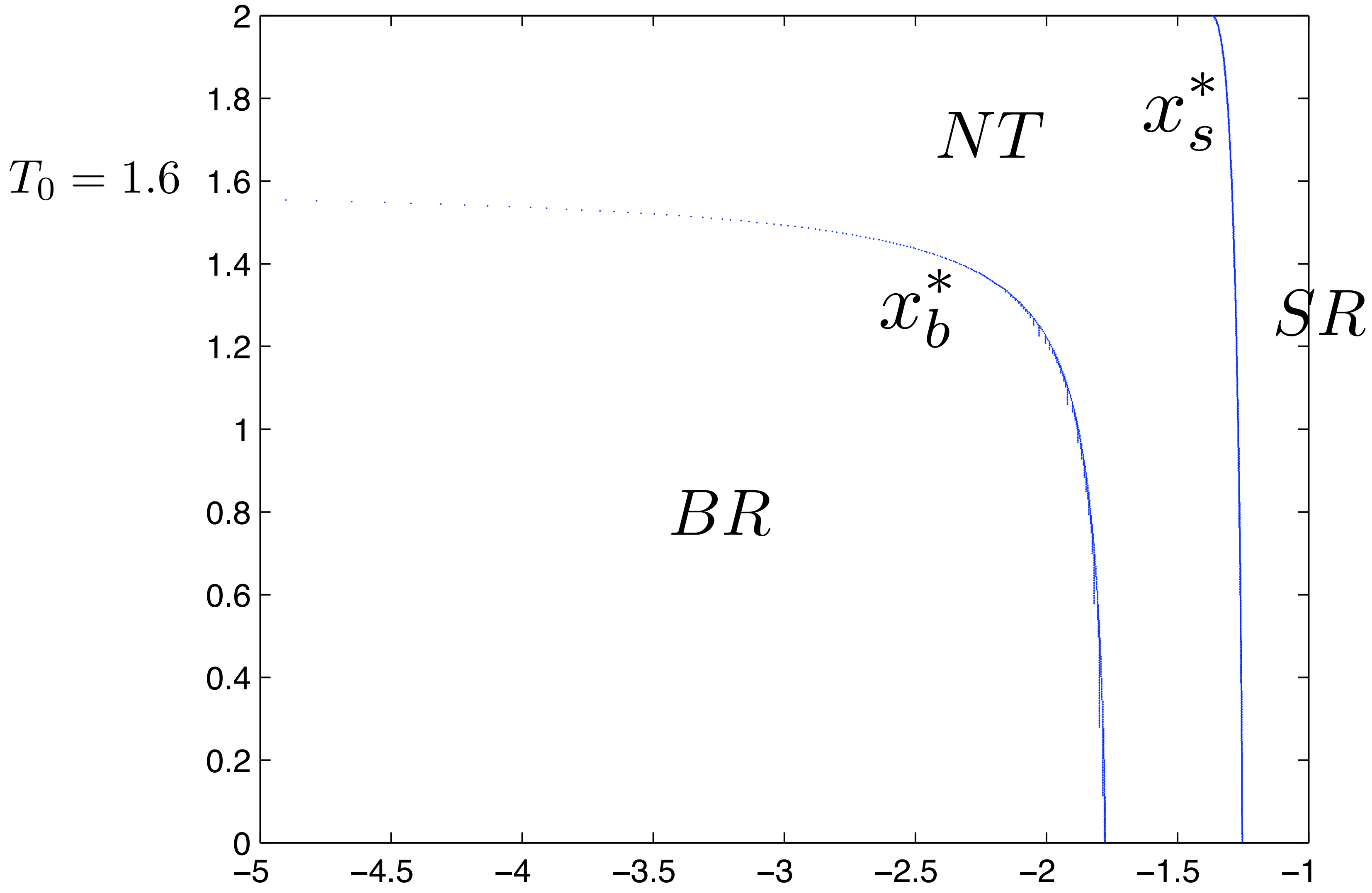
Three Regions

$$SR = \left\{ (t,x) \in [0,T) \times \mathscr{X} \;\middle|\; v(t,x) = \frac{1}{x+1-\mu} \right\},$$

$$BR = \left\{ (t,x) \in [0,T) \times \mathscr{X} \;\middle|\; v(t,x) = \frac{1}{x+1+\lambda} I(x) \right\},$$

$$NT = \left\{ (t,x) \in [0,T) \times \mathscr{X} \;\middle|\; \frac{1}{x+1+\lambda} I(x) < v(t,x) < \frac{1}{x+1-\mu} \right\}.$$

Properties of the Free Boundaries



$\alpha = 0.15 \quad r = 0.05 \quad \sigma = 0.2 \quad \lambda = 0.02 \quad \mu = 0.02 \quad T = 2$

Value Function

Theorem 3: The HJB equation has a unique solution:

$$V(t,x,y)=\begin{cases} y^2 e^{2w\left(t,\frac{x}{y}\right)}, & \text{if } y>0,\\ e^{2\mathfrak{B}(t)}(x+(1+\lambda)y)^2, & \text{if } y\leqslant 0, \end{cases}$$

where

$$\mathfrak{B}(t)=\int_t^T \frac{r x_b^{*2}(\tau)+(\alpha+r)(1+\lambda)x_b^*(\tau)+(\alpha+\frac{1}{2}\sigma^2)(1+\lambda)^2}{(x_b^*(\tau)+1+\lambda)^2}\,d\tau.$$

Lagrange Multiplier

Proposition 2: The Lagrange multiplier ℓ^* is determined by

$$V(0, x - \ell^* e^{-rT}, y) - (\ell^* - z)^2 = \sup_{\ell \in \mathbb{R}} (V(0, x - \ell e^{-rT}, y) - (\ell - z)^2),$$

or, equivalently

$$e^{-rT} V_x(0, x - \ell^* e^{-rT}, y) + 2\ell^* = 2z.$$

Lagrange Multiplier

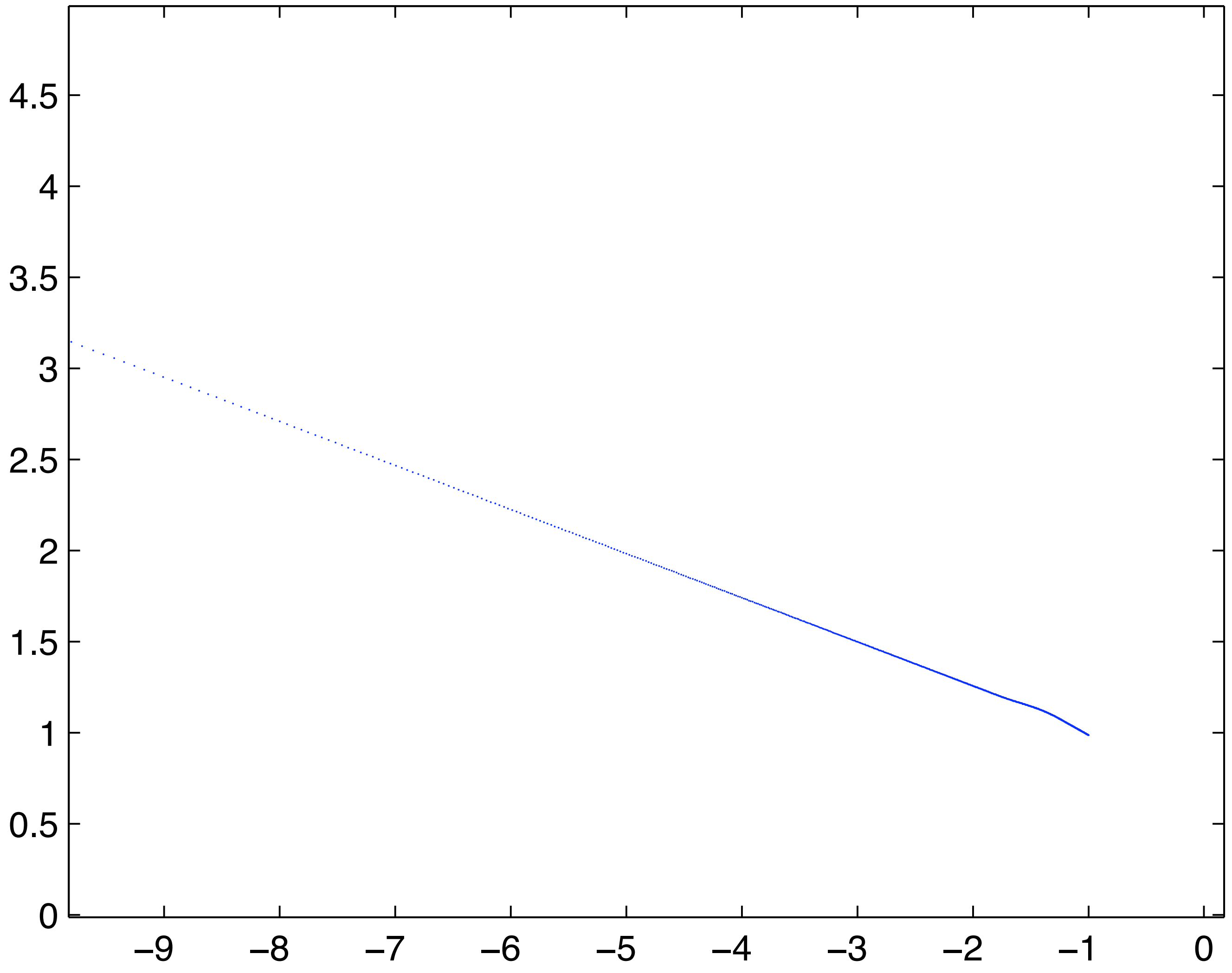
If $y \leqslant 0$, then

$$\ell^* = \frac{z - e^{2\mathfrak{B}(0) - rT}(x + (1+\lambda)y)}{1 - e^{2\mathfrak{B}(0) - 2rT}}.$$

If $y > 0$, set $\bar{\ell} = \frac{x - \ell^* e^{-rT}}{y} \leqslant \frac{x - z e^{-rT}}{y} < -(1-\mu)$, then

$$v\left(0, \bar{\ell}\right) e^{2w\left(0, \bar{\ell}\right) - 2rT} - \bar{\ell} + \frac{x - z e^{-rT}}{y} = 0.$$

Lagrange Multiplier



$\alpha = 0.15 \;\; r = 0.05 \;\; \sigma = 0.2 \;\; \lambda = 0.02 \;\; \mu = 0.02 \;\; T = 2$

Optimal Control

Market parameters:

$$\alpha = 0.15 \quad r = 0.05 \quad \sigma = 0.2 \quad \lambda = 0.02 \quad \mu = 0.02 \quad T = 2$$

Investor parameters:

$$x = -1 \qquad y = 1 \qquad z = 1.1$$

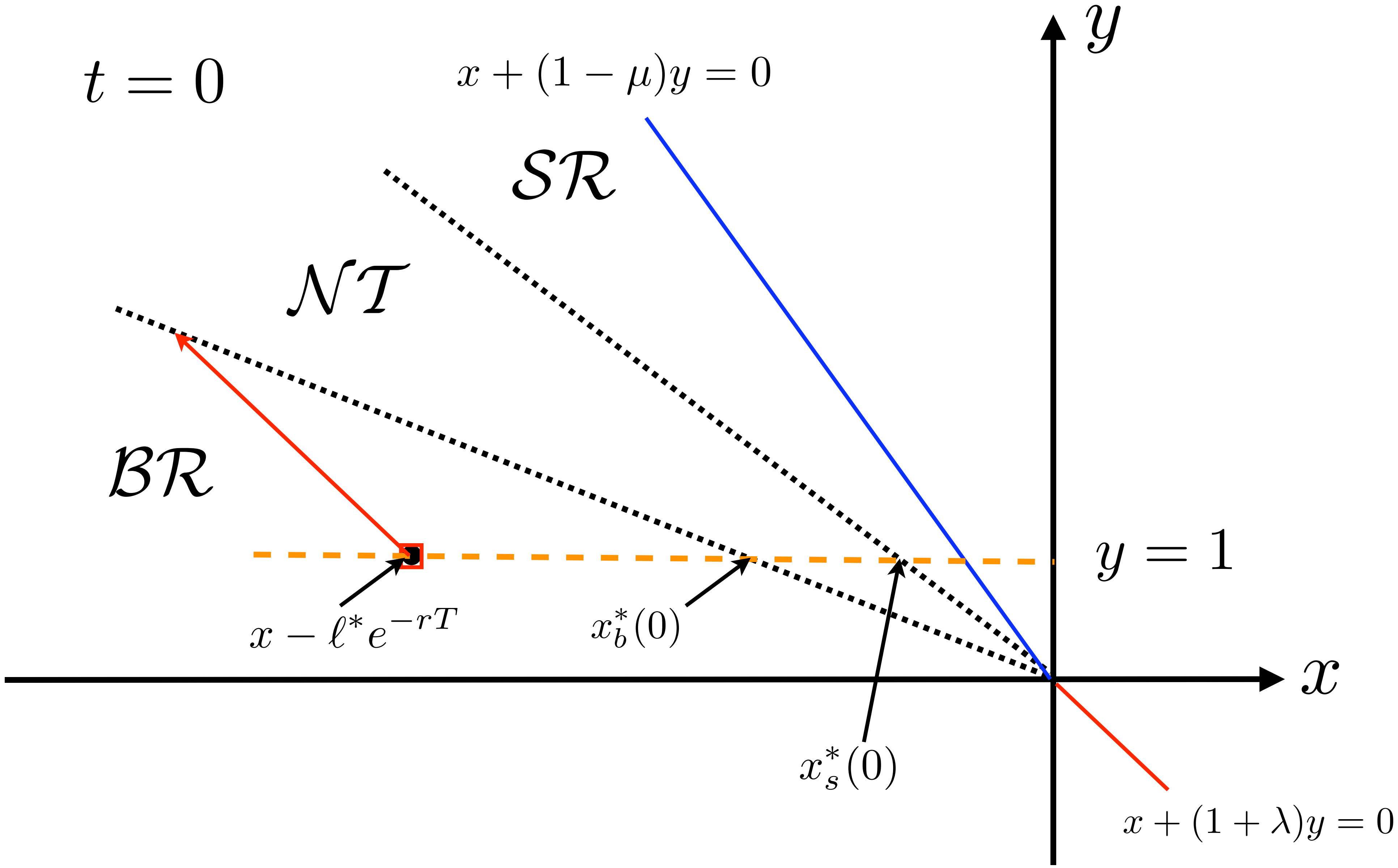
Lagrange multiplier:

$$\ell^* = 4.42$$

Initial position:

$$(x - \ell^* e^{-rT}, y) = (-5, 1)$$

Optimal Control



Thank you !

Q&A