Optimal liquidation against a Markovian limit order book

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Modelling objectives

- Framework to investigate optimal multiperiod liquidation strategies against a limit order book

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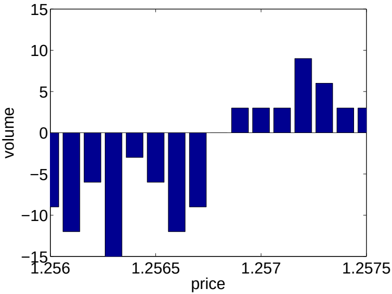
- Opportunity costs of delayed trading

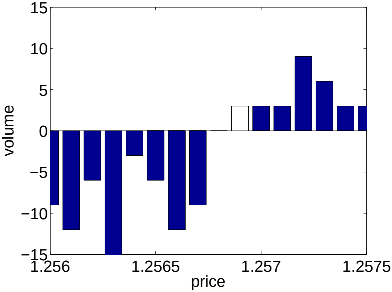
Order book model

Following [SFGK03], we specify a Markov process S_t on the large state space of order books S. Need transition rates for...



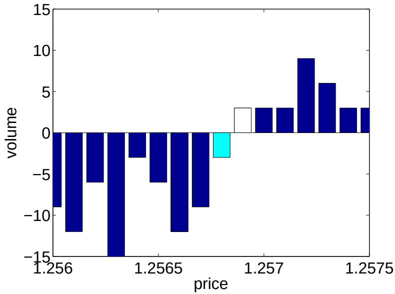
Order book model

Market order arrival: e.g. fixed Poisson rate μ for buys and sells



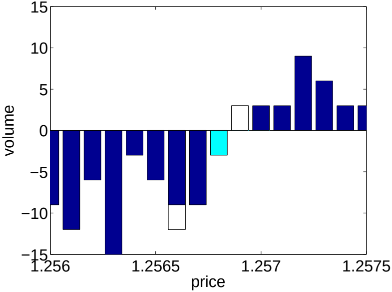
Order book model

Limit order arrival: e.g. fixed Poisson rate λ per tick below best ask for buy orders, v.v. for sell orders



Order book model

Cancellation of existing limit orders: e.g. outstanding limit order die at a rate χ



Set-up of optimisation

- Objective (cf. [AC00]) is to sell V_0 units of an asset to maximise risk-adjusted average sale price, using market orders only

$$\sup_{\tau_1 \leq \ldots \leq \tau_{V_0} < T} \mathbb{E}(U(\sum_{i=1}^{V_0} p^{bid}(S_{\tau_i}))) \quad (1)$$

where S_t follows the above dynamics, except at times $\tau_1, \ldots, \tau_{V_0}$, when a market buy order is removed; τ_i adapted to full observation filtration generated by S_t.

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- Solution will map $(s, v, c, T - t) \in \mathcal{S} \times \{1, \ldots V_0\} \times \mathbb{Z} \times \mathbb{R} \mapsto \{\text{sell 1 unit,..., sell v units, wait}\}$

Bellman equation

- Defining the value function ϕ for the problem in the usual way, we have the following dynamic programming equation

$$\phi(s, v, c, T-t) = max \begin{cases} \phi(TMS(s), v-1, c+p^{\mathrm{bid}}(s), T-t) \\ \mathbb{E}(\phi(S_{t+\delta t}, v, c, T-(t+\delta t))|S_t = s) \end{cases} \quad (2)$$

$$\phi(s, v, c, 0) = \begin{cases} U(c), \ v = 0 \\ -\infty, \ o/w \end{cases} \quad (3)$$

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- Direct solution difficult due to curse of dimensionality

Reduction of the problem

- Put $U(c) = -e^{-\gamma c}$. Then the solution can be written in terms of the order book shape and not its location via $\phi(s, v, c, T-t) = -e^{-\gamma c + vp^{\mathrm{bid}}(s)}\overline{\phi}(\overline{s}, v, T-t)$.

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- Let $T \to \infty$, and look for a t-independent solution.
- Note that $\phi(\cdot, v, \cdot, \cdot)$ depends explicitly on $\phi(\cdot, v-1, \cdot, \cdot)$, so we can solve sequentially for increasing v.
- For each v, it remains to find optimal strategy mapping $\overline{s} \in \overline{\mathcal{S}} \mapsto \{\mathrm{sell}, \mathrm{wait}\}$

Numerical schemes

- Design low-dimensional approximation architecture for $\overline{\phi}$

$$\overline{\phi}(\overline{s}) \approxeq \psi(\beta, \overline{s}) \tag{4}$$

using e.g. state aggregation, radial basis functions etc. Involves defining a distance on $\overline{\mathcal{S}}$ which captures relevant similarity of order book shapes.

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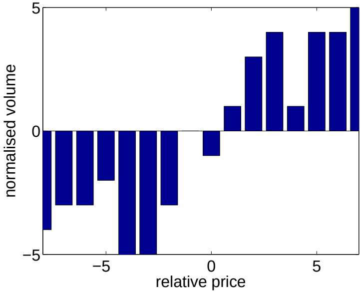
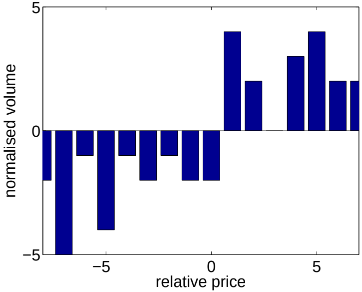
- Tune β so that ψ approximately satifies Bellman equation over $\overline{\mathcal{S}}$, using e.g. approximate value iteration, Q-learning ([TR96], [Ber05])

Example results - preliminary



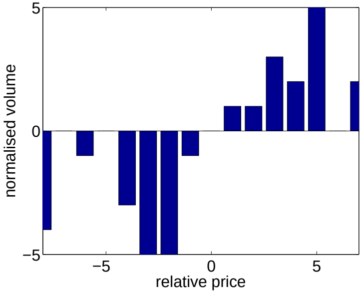
In sell region when $v = 1$ but not when $v = 2$

Example results - preliminary



In sell region when $v = 2$ (in fact sell 2 units) but not when $v = 1$

Example results - preliminary



Not in sell region for $v = 1$ or $v = 2$

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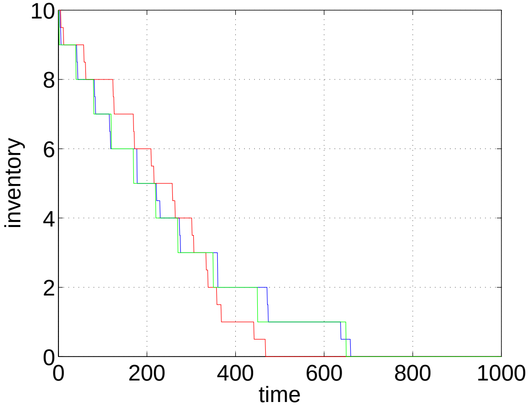

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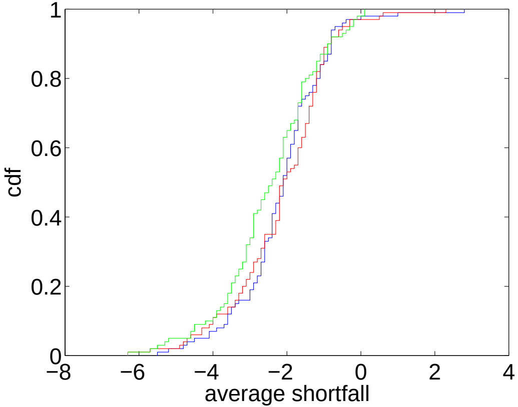

Outlook

- Improvement and analysis of numerical schemes
- Upper bounds for value function using e.g. duality
- Allow our agent to use limit sell orders
- Expand state space to capture salient non-Markovian features of limit order book process
- Partial observation of limit order book state, with application to FX market

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