# Finite Dimensional Realizations of Regime-Switching HJM models

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# Outline

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# The Model

Heath-Jarrow-Morton model with Musiela parametrization and Stratonovich dynamics:

$$\begin{cases} dr_t(x) & = \mu(r_t, y_t, x)\, dt + \sigma(r_t, y_t, x) \circ dW(t), \\ r_0(x) & = r_0^o(x). \end{cases}$$

where $\{y(t), t \geq 0\}$ is a continuous time Markov chain with finite state space $\{1, 2, \dots m\}$.
By the HJM drift condition:

$$\mu = \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, y_t, x) \int_0^x \sigma^\top(r_t, y_t, s)\, ds - \frac{1}{2}\sigma_r'(r_t, y_t, x)[\sigma(r_t, y_t, x)].$$

Infinite dimensional SDE on Hilbert space $\mathcal{H}$ of forward rate curves $x \mapsto r_t(x)$.

# Finite Dimensional Realizations (FDR)

## Definition

The SDE is said to have a FDR if there exist smooth $d$-dimensional vector fields $a$ and $b$, initial points $z_0 \in R^d$, $y_0 \in \{1, 2, \ldots, m\}$ and a mapping $G : R^d \to \mathcal{H}$ such that $r_t$ has a representation

$$\begin{aligned} r_t(x) &= G(z_t, x), \\ dz_t &= a(z_t, y_t)\, dt + b(z_t, y_t) \circ dW_t, \\ z_0 &= z^0, \\ y_0 &= y^0. \end{aligned}$$

# Main Questions

1. Necessary and sufficient conditions for existence of a FDR?
2. If existence is established, what does the realization look like?

# Previous Research

- **Framework:** Björk, Christensen (1999); Björk, Svensson (2001); Björk, Landén (2002).
- **Short rate models** with regime-switching: e.g. Hansen and Poulsen (2000); Landén (2000).
- **HJM models** with regime-switching: Valchev (2004).
- **HJM models** with a diffusion driving the volatility: Björk, Landén, and Svensson (2004).

# Answer to the first question

## Proposition

*The model admits a FDR*

*if and only if*

$$\dim \left\{ \mu_i, \sigma_i ; i \in \{1, 2, \ldots, m\} \right\}_{LA} < \infty$$

Here $\mu_i := \mu(r, i, x)$ and $\sigma_i := \sigma(r, i, x)$.

# Answer to the second question

## Proposition

*Assume that the Lie algebra* $\{\mu_1, \ldots, \mu_{n_1}, \ldots, \sigma_1, \ldots, \sigma_{n_2}\}_{LA}$ *is finite dimensional near* $r^0$ *and is spanned by an involutive system of independent vector fields* $\{f_1, \ldots, f_n\}$. *Then*

$$G(z_1, \ldots, z_n) = e^{z_n f_n} \cdots e^{z_1 f_1} r^0,$$

*where* $e^{z_i f_i}$ *denotes the flow mapping of the ODE* $\frac{dr_t}{dt} = f_i(r_t)$, $r_0 = r^0$.

# FDR with Deterministic Volatility

## Proposition

*If the volatility is deterministic then there exists a FDR if and only if the volatility is quasi-exponential in $x$.*

## Definition (Quasi-exponential function)

$$f(x) = \sum_i p_i(x)\, e^{\lambda_i x} + \sum_j e^{\alpha_j x}\left(p_j(x)\cos(\omega_j x) + q_j(x)\sin(\omega_j x)\right)$$

where $\lambda_i, \alpha_j, \omega_j \in \mathbb{R}$ and $p_i, p_j, q_j$ polynomials.

# Example: Constant Volatility
## Ho-Lee

**Ho-Lee**

$$\sigma(r, x, y) = \sigma(y)$$
$$\mu = \mathbb{F}r + \sigma^2(y)x,$$

**Lie Algebra**

$$\text{span}\{\mathbb{F}r, \mathbb{I}, x\}$$

**Forward curve manifold**

$$G(z_0, z_1, z_2) = r_0(x + z_0) + z_1 + z_2 x$$

**Realization**

$$dz_0 = dt$$
$$dz_1 = z_2\, dt + \sigma(y) dW_t$$
$$dz_2 = \sigma^2(y)\, dt$$

# FDR with Separable Volatility

Without regime switching: $\sigma(r,x) = \varphi(r)\lambda(x)$

(i) $\sigma(r,x) = \varphi(r,y)\lambda(x)$.

- Assume $\varphi(r,y) \neq 0 \; \forall r \in \mathcal{H}$ for at least 2 states of the Markov chain $y$.
- Then **FDR iff** $\lambda(x)$ **quasi exponential**. $\varphi(r,y)$ is allowed to be any smooth field.

(ii) $\sigma(r,x) = \varphi(r)\lambda(x,y)$.

- Set $\varphi^2 := \phi$. Assume $\phi''[\lambda_i, \lambda_j] \neq 0 \; \forall r \in \mathcal{H}$ and for at least 2 states of the Markov chain $y$.
- Then **FDR iff** $\lambda(x,y)$ **quasi-exponential function of** $x$ and choose $\varphi(r)$ freely.

(iii) $\sigma(r,x) = \varphi(r,y)\lambda(x,y)$.

- Same conclusion as in (ii)

# Example: Separable Volatility

## Cox Ingersoll Ross

**Short rate model**

$$dR = a(b - R)\,dt + \sigma\sqrt{R}\,dW_t$$

**Forward rate formulation**

$$\sigma(r, x) = \sqrt{r(0)}\lambda(x, \sigma, a)$$

where

$$\lambda(x, \sigma, a) = -\frac{\partial}{\partial x}\left(\frac{2\sigma(e^{\gamma x} - 1)}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}\right)$$

and where

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

# Summary

- The model admits a FDR iff a certain Lie algebra is finite dimensional.
- A realization is easily constructed from the vector fields spanning this Lie algebra.
- If $\sigma(x, y)$ is deterministic there exists a FDR iff $\sigma(x, y)$ is QE in $x$.
- If $\sigma(x, y) = \varphi(r)\lambda(x)$, modulo some condition on $\varphi(r)$, there exists a FDR iff $\lambda(x)$ is QE.

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