Inflation Indexed Swaps & Swaptions

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QMF, Sydney, December 2006

What has been done?

Hughston (1998)

- General theory
- Foreign-currency analogy

Jarrow & Yildirim (2003)

- 3-factor HJM model
- TIPS
- Vanilla Inflation Option

Mercurio (2005)

- YYIIS, Caplets, Floorlets (ZCIIS)
- JY version of HJM
- 2 Market Models

What is new?

- HJM model with jumps
  - YYIIS
- Inflation Swap Market Models
  - ZCIISwaptions
  - YYIISwaptions
- HJM model
  - ZCIISwaptions
  - TIPStions
- No foreign-currency analogy is needed $\Rightarrow$
- We prove that the foreign-currency analogy holds for an arbitrary process

YYIIS= Year-on-Year Inflation Indexed Swaps

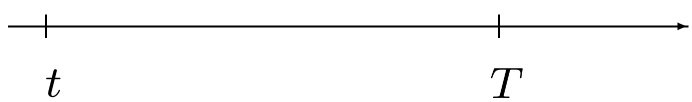
ZCIIS= Zero Coupon Inflation Indexed Swaps

Given

$\mathbf{I(t)}$: An arbitrary stochastic process

$\mathbf{p^n(t,T)}$: Price in dollar at t of a contract that pays out 1 dollar at T.

$\mathbf{p^{IP}(t,T)}$: Price in dollar at t of a contract that pays out $I(T)$ dollar at T.



Assume : There exist a market for $p^n(t,T)$ and $p^{IP}(t,T)$ for all T

Define : $p^r(t,T) = \frac{p^{IP}(t,T)}{I(t)}$

If $I(t)$ is the price of a hamburger

A nominal bond:

- Pays out 1 dollar at maturity.
- $p^n(t,T)$: the price of a nominal bond is in dollar

A hamburger-indexed bond:

- At maturity it pays out a dollar amount that is enough to buy 1 hamburger.
- $p^{IP}(t,T)$: the price of a hamburger-inflation protected bond is in dollar

A hamburger-real bond:

- Pays out 1 hamburger at maturity
- $p^r(t,T)$: the price of a real bond is in hamburgers

Note 1: CPI

Note 2: Temperature

Define

Forward rates:

$$f^i(t,T) = -\frac{\partial \text{ln } p^i(t,T)}{\partial T} \quad \text{for } i = r, n$$

Short rates:

$$r^i(t) = f^i(t,t) \qquad \text{for } i = r, n$$

Money Market Accounts:

$$B^i(t) = e^{\int_0^t r^i(s)ds} \quad \text{for } i = r, n$$

Assume

HJM model with Jumps:

Under the objective probability measure P:

$$\begin{aligned} df_t^r(T) &= \alpha_t^r(T)dt + \sigma_t^r(T)dW^P + \int_V \xi^r(t,v,T)\mu^P(dt,dv) \\ df_t^n(T) &= \alpha_t^n(T)dt + \sigma_t^n(T)dW^P + \int_V \xi^n(t,v,T)\mu^P(dt,dv) \\ dI_t &= I_t\mu_t dt + I_t\sigma_t^I dW^P + I_{t-}\int_V \gamma_t^I(v)\mu^P(dt,dv) \end{aligned}$$

Result

Under the nominal risk neutral measure:

$$
\begin{aligned}
\frac{dI_t}{I_{t-}} &= (r_t^n - r_t^r)dt + \sigma_t^I dW \int_V \gamma_t^I(v)\tilde{\mu}(dt, dv) \\
\frac{dp_t^n(T)}{p_{t-}^n(T)} &= r_t^n dt + \beta_t^n(T)dW + \int_V \delta_t^n(v, T)\tilde{\mu}(dt, dv) \\
\frac{dp_t^{IP}(T)}{dp_{t-}^{IP}(T)} &= r_t^n dt + \beta_t^{IP}(T)dW + \int_V \delta_t^{IP}(v, T)\tilde{\mu}(dt, dv) \\
\frac{dp_t^r(T)}{dp_{t-}^r(T)} &= a(t, T)dt + \beta_t^r(T)dW + \int_V \delta_t^r(v, T)\tilde{\mu}(dt, dv)
\end{aligned}
$$

Note: I has the same dynamics as an FX-rate!

Main idea of the Proof

1 Forward rates $\Rightarrow$ Bondprices (BKR)

2 Change measure from P to Q^n (Girsanov)

Now we have found the Q^n-drift of $p^n(t,T)$ and $p^{IP}(t,T)$ which we call $\mu_Q^n(t,T)$ and $\mu_Q^{IP}(t,T)$

3 By assumption

$$\frac{p^n(t,T)}{B^n(t)} \quad \frac{p^{IP}(t,T)}{B^n(t)} \quad \text{are } Q^n\text{-martingales}$$

i.e. $\mu_Q^n(t,T) = \mu_Q^{IP}(t,T) = r^n(t)$ for all maturities T.
$\Downarrow$
3 drift conditions
$\Downarrow$
One of the 3 conditions tells us that the Q^n-drift of the index I is equal to $r^n - r^r$

3 drift conditions

The drift conditions that has to be satisfied in order for the market to be free of arbitrage are:

$$\begin{aligned} \alpha^n(t,T) &= \sigma^n(t,T)\left(\int_t^T \sigma^r(t,s)ds - h(t)\right) \\ &+ \int_V \{\delta^n(t,v,T)+1\}\,\xi^n(t,v,T)\lambda_t(dv) \end{aligned}$$

$$\begin{aligned} \alpha^r(t,T) &= \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t) - h(t)\right) \\ &+ \int_V \left(1+\gamma^I(t,v)\right)\left(1+\delta^r(t,v,T)\right)\xi^r(t,v,T)\lambda_t(dv) \end{aligned}$$

$$\mu^I(t) = r^n(t) - r^r(t) - h(t)\sigma^I(t) - \int_V \gamma^I(t,v)\lambda_t(dv)$$

Note

We have derived the Q^n-dynamics of the process I_t by assuming that

$$\frac{p^n(t,T)}{B^n(t)} \quad \frac{p^{IP}(t,T)}{B^n(t)} \quad \text{are } Q^n\text{-martingales}$$

Comparing with for example Jarrow & Yildirim they derive the Q^n-dynamics of the process I_t using the additional assumption that $\frac{I(t)B^r(t)}{B^n(t)}$ is a Q^n-martingale. However this is not known apriori. We, instead show, by using Ito on I_t and $P^{IP}(t,T)$, that $\frac{I(t)B^r(t)}{B^n(t)}$ is indeed a Q^n-martingale.

Foreign Currency Analogy

Nominal vs Real

$p^n(t,T)$:	Price of nominal T-bond in dollar
$p^r(t,T)$:	Price of real T-bond in CPI units*
$I(t)$:	Price level (dollar per CPI-unit)
$I(t)p^r(t,T)$:	Price of a real T-bond in dollar denoted by $p^{IP}(t,T)$

Domestic vs Foreign

$p^n(t,T)$:	Price of domestic T-bond
$p^r(t,T)$:	Price of foreign T-bond
$I(t)$:	FX-rate (domestic per foreign unit)
$I(t)p^r(t,T)$:	Domestic price of foreign T-bond *

Assumptions

- Exist a market for nominal T-bonds and nominal indexed-bonds for all maturity dates

- The bond prices are differentiable wrt T

- Forward rate dynamics according to HJM with Jumps

- The market is free of arbitrage

- (All volatilities and the intensity are deterministic under the nominal risk neutral measure)

A Payer Swap

Definition:

- starts at time T_m
- At each payment date j where $j = m+1$, $m+2$, $\cdots$, T_M you pay

$$\alpha_j K$$

- and receive

$$\alpha_j \left[\frac{I(T_j)}{I(T_{j-1})} - 1 \right]$$

The price at time t is:

$$\sum_{j=m+1}^{M} \Pi \left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})} \right] - (K+1) \sum_{j=m+1}^{M} \alpha_j p(t, T_j)$$

A Payer Swap

The payoff function

$$\mathcal{X}_2 = \frac{I(T_2)}{I(T_1)}$$

The value at time t

$$\Pi\left[t, \mathcal{X}_2\right] = p^n(t, T_1) E_t^{T_1, n}\left[p^r(T_1, T_2)\right]$$

and

$$\begin{aligned} E_t^{T_1, n}\left[p^r(T_1, T_2)\right] &= E_t^{T_1, r} \frac{1}{L(t)} \left[\frac{p^r(T_1, T_2)}{p^r(T_1, T_1)} L(T_1)\right] \\ &= \frac{p^r(t, T_2) C(t, T_1, T_2)}{p^r(t, T_1)} \end{aligned}$$

where $C(t, T_1, T_2)$ is a convexity correction term.

The price

The price of the swap is at time t

$$\sum_{j=m+1}^{M} \alpha_j \frac{p^n(t,T_{j-1}) p^{IP}(t,T_j) C(t,T_{j-1},T_j)}{p^{IP}(t,T_{j-1})}$$

$$- \quad (K+1) \sum_{j=m+1}^{M} \alpha_j p^n(t,T_j)$$

Note

The price does only depend on bonds in the nominal market.

The Swap rate

The price is:

$$\begin{aligned} YYIIS_m^M(t,K) &= \sum_{j=m+1}^{M} \Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right] \\ &- (K+1) \sum_{j=m+1}^{M} \alpha_j p(t,T_j) \end{aligned}$$

The par swap rate is:

$$R_m^M(t) = \frac{\sum_{j=m+1}^{M} \Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right] - S_m^M(t)}{S_m^M(t)}$$

where

$$S_m^k(t) = \sum_{j=m+1}^{k} \alpha_j p_n(t,T_j)$$

The Swap rate

The par swap rate at time t

$$R_m^M(t) = \frac{\sum_{j=m+1}^{M} \frac{\alpha_j p^n(t,T_{j-1}) p^{IP}(t,T_j) C(t,T_{j-1},T_j)}{p^{IP}(t,T_{j-1})} - S_m^M(t)}{S_m^M(t)}$$

Where

$$S_m^k(t) = \sum_{j=m+1}^{k} \alpha_j p_n(t, T_j)$$

Note

Nasty distribution!

Summary

- We proved the FX-analogy   Today!

- HJM model with jumps
  - YYIIS   Today!

- Inflation Swap Market Models
  - YYIISwaptions
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- HJM model
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