HEDGING OF DEFAULTABLE GAME OPTIONS
IN A HAZARD PROCESS MODEL

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1 Valuation of Defaultable Game Options

We assume throughout that the evolution of the primary market can be modeled in terms of stochastic processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ where $\mathbb{P}$ is the statistical (objective) probability measure.

We are interested in studying a problem of time evolution of an arbitrage price of a game option. Therefore, we shall formulate the problem in a dynamic way by pricing the game option at any time $t \in [0, T]$, where T is the *maturity date*.

1.1 Primary Assets

Primary market is composed of the savings account B and of d risky assets, such that

- the *discount factor* process β, that is, the inverse of the savings account B, is a $\mathbb{G}$-adapted, finite variation, continuous, positive and bounded,
- the *risky assets* have an $\mathbb{R}^d$-valued, $\mathbb{G}$-semimartingale price process X,
- the risky assets pay dividends, whose cumulative value process D^X is an $\mathbb{R}^d$-valued, $\mathbb{G}$-adapted process of finite variation.

1.2 Cumulative Price

Given the price process X, we define the *cumulative price* $\widehat{X}$ of the asset as

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u^X .$$

The last term in the formula above represents the current value at time t of all dividend payments of the asset over the period $[0, t]$, under the assumption that dividends are reinvested in savings account.

1.3 Arbitrage-Free Property

We use the classic *fundamental theorem of asset pricing* due to Delbaen and Schachermayer.

We assume that the primary market model is free of arbitrage opportunities (though presumably incomplete) in the sense that there exists a *risk-neutral measure* $\mathbb{Q} \in \mathcal{M}$.

Note that $\mathcal{M}$ stands for the set of all probability measures $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ for which the discounted cumulative price $\beta \widehat{X}$ is a σ-martingale under $\mathbb{Q}$.

1.4 Default Time

Let τ_d be a $\mathbb{G}$-stopping time representing the *default time* of some reference entity.

We denote by $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ the *default indicator process*.

We denote by $\mathcal{G}_T^t$ the set of all $\mathbb{G}$-stopping times with values in $[t, T]$. Given a $\mathbb{G}$-stopping time $\bar{\tau} \in \mathcal{G}_T^0$, we write

$$\bar{\mathcal{G}}_T^t = \{\tau \in \mathcal{G}_T^t \,:\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}.$$

1.5 Defaultable Game Option

Definition 1.1 Let τ stand for $\tau_p \wedge \tau_c$. A *defaultable game option* is a game option with the cumulative discounted cash flows $\beta_t \pi(t; \tau_p, \tau_c)$, where the $\mathcal{G}_\tau$-measurable r.v. $\pi(t; \tau_p, \tau_c)$ is given by the formula

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_{(t,\tau]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \mathbb{1}_{\{\tau = T\}} \xi$$

for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, where

• the *dividend process* $D = (D_t)_{t \in [0,T]}$ equals

$$D_t = C_{t \wedge \tau_d} + \mathbb{1}_{\{\tau_d \le t\}} R_{\tau_d}$$

for some $\mathbb{G}$-adapted *coupon process* $C = (C_t)_{t \in [0,T]}$ and $\mathbb{G}$-predictable *recovery process* $R = (R_t)_{t \in [0,T]}$,

• the *put payment* process $L = (L_t)_{t \in [0,T]}$ is a $\mathbb{G}$-adapted process,

• the *call payment* $U = (U_t)_{t \in [0,T]}$ is a $\mathbb{G}$-adapted process such that $L_t \le U_t$ for $t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$,

• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable, such that $L_T \le \xi \le U_T$.

1.6 Main Pricing Result: Abstract Set-up

The following theorem characterizes the set of arbitrage prices of a game option in terms of values of related Dynkin games.

Theorem 1.1 *Assume that a process* Π *is a* $\mathbb{G}$*-semimartingale and if there exists* $\mathbb{Q} \in \mathcal{M}$ *such that* Π *is the value of the Dynkin game related to a game option, that is, for* $t \in [0, T]$

$$\text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t; \tau_p, \tau_c) \,\Big|\, \mathcal{G}_t\Big) = \Pi_t$$
$$= \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\pi(t; \tau_p, \tau_c) \,\Big|\, \mathcal{G}_t\Big).$$

Then Π is an arbitrage (ex-dividend) price of the game option. The converse holds true under the following integrability assumption:

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} \widehat{\mathcal{L}}_t \,\Big|\, \mathcal{G}_0\Big) < \infty \text{ a.s.}$$

where we write

$$\widehat{\mathcal{L}}_t = \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \Big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi \Big).$$

Proof. The proof relies on an extension of the theorem due to Kallsen and Kühn [11]. □

2 Valuation in the Hazard Process Set-Up

Our next objective is to derive convenient pricing formulae for an arbitrage price of a game option in the hazard process set-up.

Given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$ with $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ and a $\mathbb{G}$-stopping time τ_d, we assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ where the filtration $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ and where $\mathbb{F}$ is some *reference filtration*.

All the filtrations under consideration are assumed to satisfy the so-called *usual conditions*.

Assumption 2.1 (i) We assume that the process G given by

$$G_t = \mathbb{Q}(\tau_d > t \,|\, \mathcal{F}_t), \quad t \in [0, T],$$

is strictly positive and continuous so that the $\mathbb{F}$-*hazard process* $\Gamma_t = \ln G_t, t \in [0, T]$, is well defined and continuous.

(ii) In addition, we assume that Hypothesis (H) is satisfied, which means that all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales.

- Hypothesis (H) implies that the process Γ is non-decreasing.
- Under our assumptions we have that $\mathbb{Q}(\tau_d = \tau) = 0$ for any $\mathbb{F}$-stopping time τ.

For any $t \in [0, T]$, we denote by $\mathcal{F}_T^t$ the set of all $\mathbb{F}$-stopping times with values in $[t, T]$. Given $\bar{\tau} \in \mathcal{F}_T^0$, we denote

$$\bar{\mathcal{F}}_T^t = \{\tau \in \mathcal{F}_T^t \,:\, \tau \geq \bar{\tau}\}.$$

We find it convenient to make the following standing assumptions.

Assumption 2.2 (i) The discount factor process β is $\mathbb{F}$-adapted.
(ii) The coupon process C is $\mathbb{F}$-predictable.
(iii) The recovery process R is $\mathbb{F}$-predictable.
(iv) The payoff processes L, U are $\mathbb{F}$-predictable and the random variable ξ is $\mathcal{F}_T$-mesurable.
(v) The call protection $\bar{\tau}$ is an $\mathbb{F}$-stopping time.

2.1 First Lemma

Computation of the lower and upper value of the Dynkin game with respect to $\mathbb{G}$-stopping times can be reduced to the computation of values with respect to $\mathbb{F}$-stopping times.

Lemma 2.1 *We have*

$$\begin{aligned}
&\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\,\Big|\,\mathcal{G}_t\Big) = \\
&\qquad\operatorname{esssup}_{\tau_p\in\mathcal{F}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}^t_T}\mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\Big|\,\mathcal{G}_t\Big),\\
&\operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\,\Big|\,\mathcal{G}_t\Big) = \\
&\qquad\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}^t_T}\operatorname{esssup}_{\tau_p\in\mathcal{F}^t_T}\mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\Big|\,\mathcal{G}_t\Big).
\end{aligned}$$

2.2 Second Lemma

The computation of conditional expectations of cash flows with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of $\mathbb{F}$-*equivalent* cash flows with respect to $\mathcal{F}_t$.

Lemma 2.2 *Given stopping times $\tau_p \in \mathcal{F}_T^t$ and $\tau_c \in \bar{\mathcal{F}}_T^t$, we have that*

$$\mathbb{E}_{\mathbb{Q}}\Big(\pi(t;\tau_p,\tau_c)\,\Big|\,\mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\Big|\,\mathcal{F}_t\Big)$$

where $\widetilde{\pi}(t;\tau_p,\tau_c)$ is some $\mathcal{F}_\tau$-measurable random variable.

The random variable $\widetilde{\pi}(t;\tau_p,\tau_c)$ *is given by*

$$\beta_t\widetilde{\pi}(t;\tau_p,\tau_c) = \int_{(t,\tau]} \beta_u\, dD_u^t + \beta_\tau \mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p}^t + \beta_\tau \mathbb{1}_{\{\tau=\tau_c<\tau_p\}} U_{\tau_c}^t + \beta_\tau \mathbb{1}_{\{\tau=T\}} \xi^t$$

where $\tau = \tau_p \wedge \tau_c$ *and for any* $t \in [0,T]$ *and* $u \in [t,T]$

$$D_u^t = e^{\Gamma_t} \int_t^u e^{-\Gamma_s}\, dC_s + e^{\Gamma_t} \int_t^u R_s e^{-\Gamma_s}\, d\Gamma_s$$

and

$$L_u^t = e^{\Gamma_t-\Gamma_u} L_u, \quad U_u^t = e^{\Gamma_t-\Gamma_u} U_u, \quad \xi^t = e^{\Gamma_t-\Gamma_T}\xi.$$

2.3 Main Pricing Result: Hazard Process Set-up

Theorem 2.1 *Let* Π *be an arbitrage* $\mathbb{Q}$*-price for a game option. Then* $\Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ *for any* $t \in [0, T]$, *where*

$$\text{esssup}_{\tau_p \in \mathcal{F}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c) \,\Big|\, \mathcal{F}_t\Big) = \widetilde{\Pi}_t$$
$$= \text{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \text{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c)\Big| \mathcal{F}_t\Big).$$

Hence the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\pi}(t;\tau_p,\tau_c) \,\Big|\, \mathcal{F}_t\Big)$ *on* $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ *has the value* $\widetilde{\Pi}_t$, *which equals the pre-default value at time* t *of the game option under the pricing measure* $\mathbb{Q}$.

3 Valuation via BSDEs

Given a risk-neutral measure $\mathbb{Q} \in \mathcal{M}$, we shall now characterize an arbitrage $\mathbb{Q}$-price of a game option as a solution to a suitably chosen doubly reflected BSDE.

We work throughout under the standing assumption that this BSDE admits indeed a solution.

In [5], we show that in the case of convertible bonds in the framework of a jump-diffusion reduced-form credit model, the related BSDEs do have solutions.

3.1 Credit-Risk Adjusted Discount Factor

Let $\alpha_t = \beta_t \exp(-\Gamma_t)$ stand for the *credit-risk adjusted discount factor*.

Then $\widetilde{\pi}(t; \tau_p, \tau_c)$ satisfies

$$\alpha_t \widetilde{\pi}(t; \tau_p, \tau_c) = \int_t^{\tau} \alpha_u \, dD_u^* + \alpha_\tau \mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \alpha_\tau \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \alpha_\tau \mathbb{1}_{\{\tau = T\}} \xi$$

where we denote

$$D_t^* = C_t + \int_{[0,t]} R_u \, d\Gamma_u .$$

3.2 Doubly Reflected BSDE

Let M be an $\mathbb{R}^n$-valued, square-integrable $\mathbb{F}$-martingale.

We consider the following doubly reflected BSDE (1) with data $(F, \xi, L, U, \bar{\tau})$ and solution (Θ, Z, K^+, K^-)

$$\alpha_t \Theta_t = \alpha_T \xi + \alpha_T F_T - \alpha_t F_t + K_T - K_t - \int_t^T Z_u^\top \, dM_u$$

$$L_t \leq \Theta_t \leq \bar{U}_t, \quad t \in [0, T],$$

$$\int_0^T (\Theta_{u-} - L_{u-}) \, dK_u^+ = \int_0^T (\bar{U}_{u-} - \Theta_{u-}) \, dK_u^- = 0$$

where

- the process F is a given $\mathbb{F}$-semimartingale,

• $K = K^+ - K^-$ is the difference of two non-decreasing processes; in particular, $K_0^+ = K_0^- = 0$,

• the process $\bar{U} = (\bar{U}_t)_{t\in[0,T]}$ equals

$$\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t$$

with the convention that $0 \times \pm\infty = 0$.

We do not study here the problem of existence of a solution to the doubly reflected BSDE (1). In order to have a chance for the existence of a solution to (1) for any terminal condition ξ, it is typically assumed that the martingale M enjoys the predictable representation property.

3.3 Equivalent Formulation

The BSDE (1) can be rewritten as the following BSDE (2)

$$\alpha_t \widehat{\Theta}_t = \alpha_T \widehat{\xi} + K_T - K_t - \int_t^T Z_u^{\mathsf{T}} dM_u$$

$$\widehat{L}_t \leq \widehat{\Theta}_t \leq \widehat{U}_t, \quad t \in [0, T],$$

$$\int_0^T (\widehat{\Theta}_{u-} - \widehat{L}_{u-})\, dK_u^+ = \int_0^T (\widehat{U}_{u-} - \widehat{\Theta}_{u-})\, dK_u^- = 0$$

where $\widehat{\Theta}_t = \Theta_t + F_t$, so that $\widehat{\Theta}_T = \xi + F_T$, and

$$\widehat{L}_t = L_t + F_t, \quad \widehat{U}_t = \bar{U}_t + F_t.$$

This shows that the problem can be reduced to the case of $F = 0$ with suitably modified reflecting barriers.

3.4 Continuous Case

If the $\mathbb{F}$-martingale M, the $\mathbb{F}$-semimartingale F and the barriers L and U are continuous, it is natural to look for a solution of the BSDE (1) with continuous components Θ and (K^+, K^-) or equivalently, a solution of (2) with continuous components $\widehat{\Theta}$ and (K^+, K^-).

The following result establishes the link between a solution to a doubly reflected BSDE and an arbitrage $\mathbb{Q}$-price of the game option. Let us set

$$F_t = \alpha_t^{-1} \int_0^t \alpha_u \, dD_u^*.$$

3.5 Verification Principle for a Game Option

The following result links a solution to BSDE (1) with the price of a game option.

Theorem 3.1 *Assume that the BSDE* (1) *admits a solution* $(\widehat{\Theta}, Z, K^+, K^-)$ *and let, for* $t \in [0, T]$,

$$\widetilde{\Pi}_t = \widehat{\Theta}_t - F_t = \widehat{\Theta}_t - \alpha_t^{-1} \int_0^t \alpha_u \, dD_u^* .$$

Then the process $\Pi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Pi}_t$ *is an arbitrage* $\mathbb{Q}$*-price for the game option. Moreover, for any* $t \in [0, T]$ *and for any* $\varepsilon > 0$, *a pair of ε-optimal stopping times $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ for the related Dynkin game on $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$ is given by*

$$\tau_p^\varepsilon = \inf\Big\{ u \in [t,T] \,:\, \widetilde{\Pi}_u \le L_u + \varepsilon \Big\} \wedge T,$$
$$\tau_c^\varepsilon = \inf\Big\{ u \in [\bar{\tau} \vee t, T] \,:\, \widetilde{\Pi}_u \ge U_u - \varepsilon \Big\} \wedge T.$$

In the special case where K^+ and K^- are continuous, the pair of stopping times $(\tau_p^*, \tau_c^*) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ obtained by setting $\varepsilon = 0$ is a saddle-point of the game.

Proof. The present assumptions imply that $\widetilde{\Pi}_t$ is the value of the Dynkin game at time $t \in [0,T]$, with $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ as a pair of ε-optimal stopping times. $\square$

4 Hedging of Defaultable Game Options

We now examine the implications of existence of a solution to the doubly reflected BSDE on hedging strategies of a game option, under the assumption that the primary market is complete.

We shall work with a definition of hedging game options adapted from successive developments, starting with the definition of hedging for American options.

This definition will be shown to be consistent with the concept of arbitrage pricing of a game option.

4.1 The Model

We place ourselves within the hazard process set-up for some fixed $\mathbb{Q} \in \mathcal{M}$. Recall that we assumed that the hazard process Γ is continuous.

We define the *compensated jump martingale* M^d of the non-decreasing default indicator process $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ by setting

$$M_t^d = H_t - \Gamma_{t \wedge \tau_d}, \quad t \in [0, T].$$

4.2 Completeness of the Primary Market

For simplicity, we present here the special case of a complete market.

Assumption 4.1 There exists a $\mathbb{G}$-predictable, matrix-valued process Ξ_t, which is left-invertible for any $t \in [0, \tau_d]$, and such that

$$d(\beta_t \widehat{X}_t) = \beta_t \Xi_t \, d \begin{bmatrix} M_t \\ M_t^d \end{bmatrix}, \quad t \in [0, T].$$

4.3 Portfolio with Cost

We now recall the notion of a trading strategy with cost.

Definition 4.1 A *portfolio with cost* Q is a $\mathbb{G}$-predictable process (ζ^0, ζ) such that its wealth process V, given by

$$V_t = \zeta_t^0 B_t + \zeta_t^{\mathsf{T}} X_t, \quad t \in [0, T], \tag{1}$$

satisfies

$$d(\beta_t V_t) = \zeta_t^{\mathsf{T}} d(\beta_t \widehat{X}_t) - \beta_t \, dQ_t, \quad t \in [0, T]. \tag{2}$$

Remark 4.1 Due to Assumption 4.1, condition (2) is equivalent to

$$d(\beta_t V_t) = \beta_t \zeta_t^{\mathsf{T}} \Xi_t \, d \begin{bmatrix} M_t \\ M_t^d \end{bmatrix} - \beta_t \, dQ_t \, . \tag{3}$$

Given the cost Q, a $\mathbb{G}$-predictable process ζ and some initial value V_0, we can first define the process V by (2) and next find the $\mathbb{G}$-predictable process ζ^0 using (1).

Therefore, we shall focus only on the pair (V, ζ).

In the sequel, the process Q will be equal to the dividend process D of a game option.

4.4 Issuer Hedge

Definition 4.2 By an *issuer ε-hedge* for the game option we mean a triple (V, ζ, τ_c) such that:
(i) the portfolio (V, ζ) has the cost $Q = D$,
(ii) τ_c belongs to $\bar{\mathcal{G}}^0_T$,
(iii) for any $t \in [0, T]$ the process

$$V_{t\wedge\tau_c} - \mathbb{1}_{\{t\wedge\tau_c<\tau_d\}}\Big(\mathbb{1}_{\{t\le\tau_c<T\}}L_t + \mathbb{1}_{\{\tau_c<t\}}U_{\tau_c} + \mathbb{1}_{\{t=\tau_c=T\}}\xi\Big)$$

is greater or equal to $-\varepsilon$.

4.5 Holder Hedge

Definition 4.3 By a *holder ε-hedge* for the game option we mean a triple (V, ζ, τ_p) such that:

(i) the portfolio (V, ζ) has the cost $Q = -D$,

(ii) τ_p belongs to $\mathcal{G}_T^0$,

(iii) for any $t \in [\bar{\tau}, T]$ the process

$$V_{t\wedge\tau_p} + \mathbb{1}_{\{t\wedge\tau_p<\tau_d\}}\Big(\mathbb{1}_{\{\tau_p\leq t<T\}}L_{\tau_p} + \mathbb{1}_{\{t<\tau_p\}}U_t + \mathbb{1}_{\{\tau_p=t=T\}}\xi\Big)$$

is greater or equal to $-\varepsilon$.

In the case of $\varepsilon = 0$, we say that we deal with an *issuer hedge* and a *holder hedge* for the game option.

4.6 Main Hedging Result

Theorem 4.1 *Assume that the BSDE* (1) *admits a solution* $(\widehat{\Theta}, Z, \mathbf{K})$ *and define*

$$\widetilde{\Pi}_t = \widehat{\Theta}_t - F_t, \quad t \in [0, T],$$

so that the process $\Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ *is the arbitrage* $\mathbb{Q}$*-price for the game option.*

(i) Π_0 *is the least value* V_0 *such that for any* $\varepsilon > 0$, *there exists an issuer* ε*-hedge with initial value* V_0*. Starting from* $V_0 = \Pi_0$, *the hedge is given by* τ_c^ε *as in Theorem 3.1 (for* $t = 0$*) and*

$$\zeta_t^c = \mathbb{1}_{\{t \le \tau_d\}} (\Xi_t^{\mathsf{T}})^{-1} \begin{bmatrix} \alpha_t^{-1} Z_t \\ R_t - \Pi_{t-}^* \end{bmatrix}$$

where the $\mathbb{F}$*-adapted process* Π^* *equals* $\Pi_t^* = \widetilde{\Pi}_t + \alpha_t^{-1} K_t$ *for every* $t \in [0, T]$.

(ii) *For any* $\varepsilon > 0$, *an holder* ε*-hedge with initial value* $-\Pi_0$ *is furnished by* τ_p^ε *as in Theorem 3.1 (for* $t = 0$*) and* $\zeta^p = -\zeta^c$.

(iii) *If* K^+ *and* K^- *are continuous, we may set* ε *equal to 0 in (i)-(ii). Hence there exist the issuer hedge (resp. the holder hedge) of the game option with the initial value* Π_0 *(resp.* $-\Pi_0$*).*

4.7 Comments

It is easy to see that one could state analogous definitions and theorems regarding hedging a defaultable game option, starting from any $t \in [0, T]$. In other words, the fact that 0 is the inception date of the option is immaterial in Theorem 4.1.

A jump-diffusion model of credit risk, pricing and hedging problems can be addressed by solving suitable variational inequalities (see Bielecki et al. [5]).

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