

Smooth Convergence in the Binomial Model

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Black-Scholes price

Call option: $K =$ strike price, $T =$ time to maturity

$S_0 =$ the initial stock price,

$r =$ the risk-free rate,

$\sigma =$ volatility,

The Black-Scholes call option price at $t = 0$ is

$$C_{BS} = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

Binomial price

Binomial model:

$$\begin{array}{c} Su \\ S \\ Sd \end{array}$$

n = the number of periods, $\Delta t = \frac{T}{n}$

Cox, Ross, and Rubinstein (CRR) model:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

$C(n)$ = the n -period binomial model call option price

Central Limit Theorem implies $C(n) \rightarrow C_{BS}$
as $n \rightarrow \infty$

Example

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$

(i) $K = 19$ (in the money)

Pictures

(ii) $K = 20$ (at the money)

Pictures

Why do we get the oscillation in the in the money case but not in the at the money case?

Tian's Flexible Binomial Model

Tian, Y.S., A flexible binomial option pricing model, *The Journal of Futures Markets* **19**, 817-843 (1999).

Tian adjusts u and d so that K falls on a terminal stock price.

$$u = e^{\lambda\Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{\lambda\Delta t - \sigma\sqrt{\Delta t}}.$$

Here

$$\lambda = \frac{2(\gamma - j_0)}{\sigma\sqrt{Tn}},$$

where

$$S_0 u_0^\gamma d_0^{n-\gamma} = K$$

with

$$u_0 = e^{\sigma\sqrt{\Delta t}}, \quad d_0 = e^{-\sigma\sqrt{\Delta t}}$$

and j_0 is the integer nearest γ .

λ is selected so that $S_0 u^{j_0} d^{n-j_0} = K$.

Example

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$

$$K = 19 \text{ (in the money)}$$

Use Tian's parameters

Get monotonic convergence.

Why?

Smooth Convergence

Definition: *If*

$$e(n) = C(n) - C_{BS} = \frac{f(n)}{n^m} + o\left(\frac{1}{n^m}\right)$$

as $n \rightarrow \infty$, where $f(n)$ is bounded and does not tend to zero, we say the rate of convergence is of order $1/n^m$.

If $f(n)$ can be taken as constant, we say the convergence is smooth.

smooth convergence implies monotonic convergence

Acceleration of Convergence

When the convergence is smooth of order $1/n^m$, we find that

$$r(2n) = \frac{e(n)}{e(2n)} \rightarrow \rho = 2^m \quad \text{as } n \rightarrow \infty.$$

If we define the sequence

$$\hat{C}(2n) = \frac{\rho C(2n) - C(n)}{\rho - 1},$$

we find that

$$\hat{C}(2n) - C_{BS} = e(2n) \frac{\rho - r(2n)}{\rho - 1}$$

so that $\hat{C}(2n)$ converges faster to C_{BS} than the original sequence.

Theorem: For the n -period binomial model, where

$$u = e^{\sigma\sqrt{\Delta t} + \lambda\sigma^2\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t} + \lambda\sigma^2\Delta t},$$

with λ an arbitrary bounded function of n , the price of a European call option with strike price K satisfies

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n - 12\sigma^2 T(\Delta_n^2 - 1)}{n} + o\left(\frac{1}{n}\right)$$

where $\Delta_n = 1 - 2 \frac{\log(S_0/K) + n \log d}{\log(u/d)}$ and

$$A_n = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)(r - \lambda\sigma^2) - 12T^2(r - \lambda\sigma^2)^2.$$

CRR Model ($\lambda = 0$)

For the n -period CRR model, the price of a European call option with strike price K satisfies

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A - 12\sigma^2 T(\Delta_n^2 - 1)}{n} + o\left(\frac{1}{n}\right),$$

where $\Delta_n = 1 - 2 \operatorname{frac} \left[\frac{\log(S_0/K) + n \log d}{\log(u/d)} \right]$ and

$$A = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)r - 12T^2 r^2.$$

Diener, F., Diener, M., Asymptotics of the price oscillations of a vanilla option in a tree model, *Mathematical Finance* **14**, 271-293 (2004)

Walsh (2003): $\lambda = r/\sigma^2$

Walsh, J.B.: The rate of convergence of the binomial tree scheme. Finance and Stochastics **7**, 337-361 (2003)

Explanation of Oscillations in CRR model

If j_0 is the integer such that

$$S_{j_0-1} = S_0 u^{j_0-1} d^{n-j_0+1} < K \leq S_0 u^{j_0} d^{n-j_0} = S_{j_0},$$

then

$$\frac{1 - \Delta_n}{2} = \frac{\log(S_{j_0}/K)}{\log(S_{j_0}/S_{j_0-1})}.$$

$$\Delta_n = \begin{cases} 1 & \text{if } K = S_{j_0} \\ 0 & \text{if } K = \sqrt{S_{j_0} S_{j_0-1}} \end{cases}$$

and approaches -1 as K approaches S_{j_0-1} .

CRR model: position of K oscillates between the two adjacent stock prices so that Δ_n oscillates between 1 and -1 .

Example: out of the money

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$

$$K = 19$$

$$C(n) = C_{BS} - \frac{1.91[0.12 + 0.24(\Delta_n^2 - 1)]}{n} + o\left(\frac{1}{n}\right)$$

$$\Delta_n = \pm 1:$$

$$C(n) = C_{BS} - \frac{0.24}{n} + o\left(\frac{1}{n}\right)$$

$$\Delta_n = 0:$$

$$C(n) = C_{BS} + \frac{0.22}{n} + o\left(\frac{1}{n}\right)$$

Example: at the money

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$
$$K = 20$$

$$C(n) = C_{BS} - \frac{2.26[0.13 + 0.24(\Delta_n^2 - 1)]}{n} + o\left(\frac{1}{n}\right)$$

if n even, $\Delta_n = 1$:

$$C(n) = C_{BS} - \frac{0.28}{n} + o\left(\frac{1}{n}\right)$$

if n is odd, $\Delta_n = 0$:

$$C(n) = C_{BS} + \frac{0.26}{n} + o\left(\frac{1}{n}\right)$$

A Few Words about the Proof

Our fundamental tool is a result in

Uspensky, J. V., *Introduction to mathematical probability*, McGraw-Hill, 1937

on approximating the binomial distribution by the normal distribution.

Smooth Convergence in the Flexible Binomial Model

$$\Delta_n = 1, \quad \lambda = \lambda_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Theorem. *For the flexible binomial model, (a) the price of a digital call option with strike price K satisfies*

$$C_d(n) = e^{-rT} N(d_2) + \frac{e^{-rT} e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi n}} + O\left(\frac{1}{n}\right);$$

(b) the price of a European call option with strike price K satisfies

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A}{n} + o\left(\frac{1}{n}\right),$$

$$A = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)r - 12T^2 r^2.$$

Example: flexible binomial model

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$
$$K = 19$$

digital call option:

$$C_d(n) = e^{-rT} N(d_2) + \frac{0.34}{\sqrt{n}} + O\left(\frac{1}{n}\right);$$

call option:

$$C(n) = C_{BS} - \frac{0.24}{n} + o\left(\frac{1}{n}\right)$$

Centre Binomial Model

j_0 is chosen so that

$$S_{j_0-1} = S_0 u_0^{j_0-1} d_0^{n-j_0+1} < K \leq S_0 u_0^{j_0} d_0^{n-j_0} = S_{j_0},$$

where

$$u_0 = e^{\sigma\sqrt{\Delta t}}, \quad d_0 = e^{-\sigma\sqrt{\Delta t}}$$

as in the CRR model.

In the centre model, $\lambda = \lambda_n$ is defined so that

$$K^2 = S_{j_0-1} S_{j_0},$$

that is, K is the geometric average of S_{j_0-1} and S_{j_0} .

$$\Delta_n = 0, \quad \lambda = \lambda_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

Theorem. For the centre binomial model, (a) the price of a digital call option with strike price K satisfies

$$C_d(n) = e^{-rT} N(d_2) + \frac{e^{-rT} e^{-d_2^2/2} A}{\sqrt{2\pi} n} + o\left(\frac{1}{n}\right)$$

$$A = \frac{1}{24\sigma^2} \left[\sigma^2 (d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1) + 4\sigma\sqrt{T} (2 - d_1 d_2 - d_1^2) r + 12T d_1 r^2 \right];$$

(b) the price of a European call option with strike price K satisfies

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}} A}{24\sigma\sqrt{2\pi T} n} + o\left(\frac{1}{n}\right),$$

$$A = -\sigma^2 T (-6 + d_1^2 + d_2^2) + 4T (d_1^2 - d_2^2) r - 12T^2 r^2$$

Example: centre binomial model

$$S_0 = 20, T = 0.5, r = 0.06, \sigma = 0.2$$

$$K = 19$$

digital call option:

$$C_d(n) = e^{-rT} N(d_2) - \frac{0.26}{n} + o\left(\frac{1}{n}\right)$$

call option:

$$C(n) = C_{BS} + \frac{0.22}{n} + o\left(\frac{1}{n}\right)$$