Credit derivatives with recovery of market value for multiple firms

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http://www.comp.metro-u.ac.jp/~tanakake/paper/CDwithRMV.pdf

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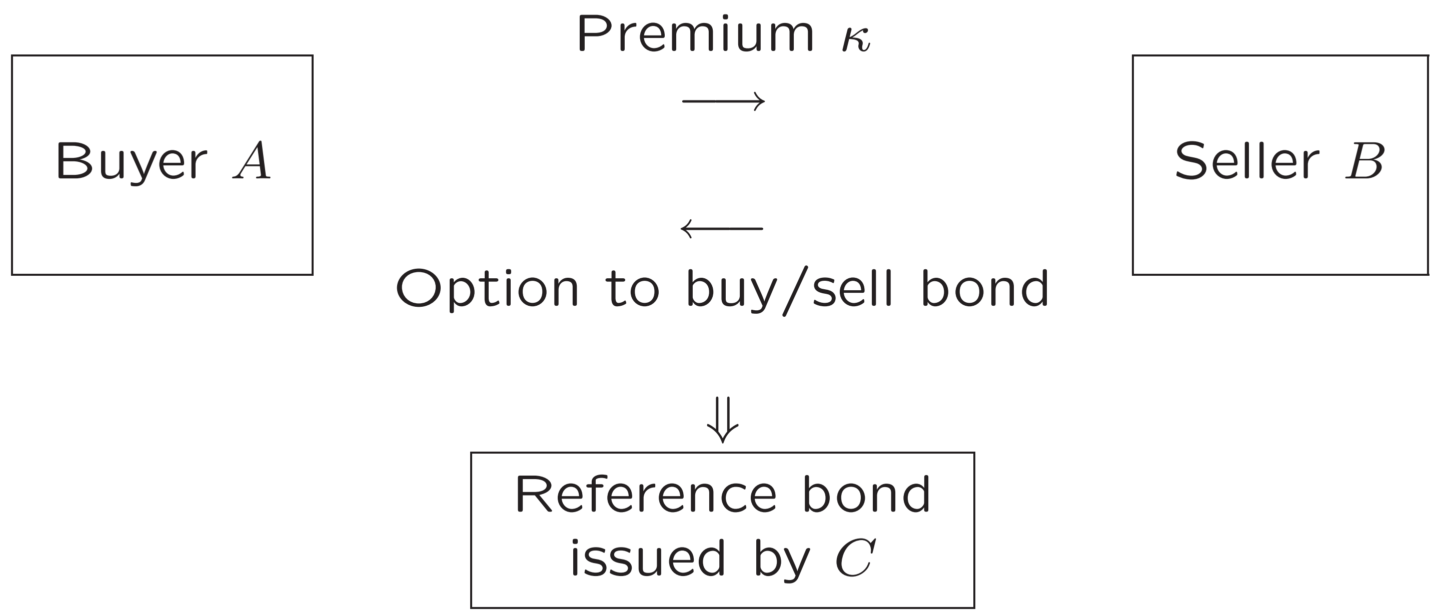
Outline

1. Credit derivatives
2. Valuation with RMV for multiple reference firms
3. Survival contingent forward measure
4. Application : Vulnerable option on defaultable bond and Vulnerable option on CDS
5. Approximation by Gram–Charlier Expansion
6. Survival contingent measure : Case of 2-period model

1. Credit derivatives

- Transaction between two parties to hedge default risk of 3rd party
- Counterparty in OTC is another source of default risk
- Our goal : Valuation of credit derivatives with multiple reference entities (A,B,C) subject to recovery of market value (RMV)
- Recovery of market value : protection buyer gets paid a fractional amount of the market value upon the default
- Example : Vulnerable option on defaultable bond, Option on CDS

Vulnerable Bond Option



(1) Recovery of Market Value (RMV)

Risk adjusted short rate = Default-free rate + loss rate × intensity

(1a) One reference firm : Zero-cpn bond (Duffie and Singleton, 1999)

$$\widetilde{P}^C(t,T) = E\Big[e^{-\int_t^T R_s ds} \mid \mathcal{F}_t\Big], \quad R_t = r_t + \delta^C h_t^C$$

(1b) Two reference firms (Asymmetric RMV) :
Swap contract (Duffie and Huang, 1996)

$$\begin{aligned} V_t &= E\Big[e^{-\int_t^T R(V_s,s)ds} Y_T \mid \mathcal{F}_t\Big] \\ R(v,t) &= r_t + \delta^A h_t^A \mathbf{1}_{\{v<0\}} + \delta^B h_t^B \mathbf{1}_{\{v\geq 0\}} \end{aligned}$$

(1c) This paper : Three reference firms
We will show that cash flows in credit derivatives with three ref. firms should be discounted by an adjusted short rate

$$R_t = r_t + \delta^A h_t^A + \delta^B h_t^B + \delta^C h_t^C$$

(2) Change of Measure

Change of measure $Q \to Q^A$ / Change of numéraire $B(t) \to A(t)$

$$\frac{dQ^A}{dQ}\Big|_{\mathcal{F}_t} = L_t^A = \frac{A(t)}{B(t)}\frac{B(0)}{A(0)}$$

(2a) Forward measure (Equivalent measure) $A(t) = P(t,T)$

$$C_t = B(t)E^Q\Big[\frac{Y_T}{B(T)} \mid \mathcal{F}_t\Big] = P(t,T)E^{Q^{fwd}}\Big[Y_T \mid \mathcal{F}_t\Big]$$

(2b) Survival measures and Survival contingent forward measure

	Measure	Equiv.	Numéraire $A(t)$
Schönbucher (2000)	survival "forward" measure Q^{sv}	abs. cont.	defaultable bond with zero recovery
CDGH	survival "spot" measure	abs. cont.	defaultable money market a/c with zero recovery
This paper	survival contingent forward measure Q^{svc}	equivalent	pre-default price with RMV

CDGH : Collin-Dufresne, Goldstein and Hugonnier (2004)

2. Model setup

$(\Omega, \mathcal{F}, Q)$	on $[0, T^*]$
Q	a martingale measure with a default-free rate r
A, B, C	three firms subject to default risk
τ^i	default time of firm i, $\quad \tau^{ABC} = \min(\tau^A, \tau^B, \tau^C)$

Assumption 1. (1) h^i : intensity process of τ^i $\quad (i = A, B, C)$

(2) filtrations $\mathbb{G} = \{\mathcal{G}_t : t \in [0, T^*]\}$ and $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$

$$\mathcal{F}_t = \mathcal{G}_t \vee \sigma(\mathbf{1}_{\{\tau^A \le s\}}, \mathbf{1}_{\{\tau^B \le s\}}, \mathbf{1}_{\{\tau^C \le s\}} : 0 \le s \le t)$$

(3) Any $(Q, \mathbb{G})$-martingale is also a $(Q, \mathbb{F})$-martingale

(4) No simultaneous defaults by any two (hence more) firms

$$R_t \equiv r_t + \delta^A h_t^A + \delta^B h_t^B + \delta^C h_t^C$$

3. Defaultable security

A defaultable security with the maturity T
- pays the cumulative dividened D until T and Y at T
- RMV recovery for defaults of A, B, C

in case of default of firm i ($i = A, B, C$) prior to the maturity T, the security holder receives $1 - \delta^i$ times the pre-default price and the contract is terminated without further payments

Example including

A: buyer of an option, B: seller, C: reference bond issuer

Recovery depends on first-to-default

the first defaulting firm and the fractional recovery ratio of the firm

Recursive equation associated with RMV for security price S

$$S_t = E\Big[\int_t^T \Big(-\big(R_s - h_s^A - h_s^B - h_s^C\big)S_s ds + \mathbf{1}_{\{\tau^{ABC}>s\}} dD_s\Big) + \mathbf{1}_{\{\tau^{ABC}>T\}} Y \mid \mathcal{F}_t\Big]$$

Define pre-default price V

$$V_t = E\Big[\int_t^T \exp\Big(-\int_t^s R_u du\Big) dD_s + \exp\Big(-\int_t^T R_u du\Big) Y \mid \mathcal{G}_t\Big]$$

with a spirit that cash flows in the defaultable security should be discounted by an adjusted short rate $R_t = r_t + \delta^A h_t^A + \delta^B h_t^B + \delta^C h_t^C$

Theorem 1. *Let D be a $\mathbb{G}$-predictable bounded process of FV and Y be a $\mathcal{G}_T$-measurable, integrable random variable. Suppose $\Delta V_{\tau^{ABC}} = 0$. Then under Assumption 1 the price S_t of the claim at time t is given by*

$$S_t = \mathbf{1}_{\{\tau^{ABC}>t\}} V_t.$$

Assumption 2. (1) W : an n-dimensional standard Brownian motion

$$\mathcal{G}_t = \sigma(W_s : 0 \le s \le t) \vee \mathcal{N}$$

(2) A diffusion X (state vector) taking values in $D \subset \mathbb{R}^n$

drives r and h^i : $\quad r_t = r(X_t), \quad h_t^i = h^i(X_t)$

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

$$\tau^i = \inf\{t \ge 0 : \int_0^t h^i(X_s)ds \ge \eta^i\}$$

Corollary 1. *Under Assumptions 1 and 2, S_t with $D \equiv 0$ is given by*

$$S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} E\Big[\exp\Big(-\int_t^T R(X_u)du\Big) Y \mid \mathcal{G}_t\Big].$$

Typical case : $\quad Y = \max(\sum_i a_i \widetilde{P}(T, T_i), 0)$

The pre-default price V_t can be further simplified under a survival contingent forward measure for RMV

4. Survival contingent forward measure for RMV

(Step 1) Pre-default price of fictitious & defaultable zero-cpn bond subject to RMV on A, B, C

$$F(t,T) = E\left[\exp\left(-\int_t^T R(X_s)ds\right) \mid \mathcal{G}_t\right]$$

$$dF(t,T) = F(t,T)R(X_t)dt + F(t,T)\sigma^F(t,T)^\top dW_t$$

(Step 2) For fixed T we define a martingale L^T by

$$\Lambda_t = \exp\left(-\int_0^t \sum_{i=A,B,C} \delta^i h^i(X_s)ds\right)$$

$$L_t^T = \frac{F(t\wedge T, T)\Lambda_{t\wedge T}}{\exp\left(\int_0^{t\wedge T} r(X_s)ds\right)F(0,T)} = \frac{1}{F(0,T)}E\left[\exp\left(-\int_0^T R(X_s)ds\right) \mid \mathcal{G}_{t\wedge T}\right]$$

Λ is chosen so that L^T becomes a positive martingale

(Step 3) An equivalent measure $Q^{svc} = Q^T \sim Q$ can be defined by

$$\frac{dQ^T}{dQ}\bigg|_{\mathcal{F}_t} = L_t^T$$

and is called survival contingent forward measure for RMV.

(i) $Q^T \sim Q$

(ii) h^i remains the intensity of H^i under Q^T

(iii) $W_t^T = W_t - \int_0^t \sigma^F(s,T)ds$ is a Brownian motion under Q^T

Under Q^T the pre-default price of a contingent claim with RMV is expressed as if $F(t,T)$ were a bond price

Theorem 2. *Under Assumptions 1 and 2, the price S is given by*

$$S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} F(t,T) E^T \Big[Y \mid \mathcal{G}_t \Big], \quad t < T,$$

where E^T is the expectation with respect to Q^T

Note

$$S_t = \mathbf{1}_{\{\tau^{ABC} > t\}} V_t, \quad S : \text{security price}, \quad V : \text{pre-default price}$$

(1) The relative price of $F(\cdot, T)$ w.r.t. money market account may not be a martingale because of (i) pre-default price and (ii) RMV.

(2) However, by construction, the relative price of $F(\cdot, T)\Lambda$ w.r.t. money market account is a martingale.

(3) By definition, $F(\cdot, T)\Lambda$ is a numéraire for any asset price S under Q^T

: $(S + D + \text{Recovery payoff})/(F(\cdot, T)\Lambda)$ is a $(Q^T, \mathbb{F})$-martingale

(4) By Th.2, $F(\cdot, T)$ is a numéraire for pre-default price V of any defaultable security subject to the same RMV under Q^T

: $V/F(\cdot, T)$ is a $(Q^T, \mathbb{G})$-martingale (hence $(Q^T, \mathbb{F})$-martingale)

(5) Survival contingent measure vs Survival measure

Survival cont. fwd measure Q^T	Survival fwd measure $\overline{Q}^T$
numéraire $F(t,T)\Lambda_t$	numéraire $\mathbf{1}_{\{\tau^{ABC}>t\}}F(t,T)\overline{\Lambda}_t$
$L_t^T = \frac{F(t,T)\Lambda_t}{\exp\left(\int_0^t r(X_s)ds\right)F(0,T)}$ $\Lambda_t = \exp\left(-\int_0^t \sum_i \delta^i h^i(X_s)ds\right)$	$\overline{L}_t^T = \frac{\mathbf{1}_{\{\tau^{ABC}>t\}}F(t,T)\overline{\Lambda}_t}{\exp\left(\int_0^t r(X_s)ds\right)F(0,T)}$ $\overline{\Lambda}_t = \Lambda_t \exp\left(\int_0^t \sum_i h^i(X_s)ds\right)$
$Q^T \sim Q$	$\overline{Q}^T \ll Q$

$$Q^T(G) = \overline{Q}^T(G) \quad \text{for } G \in \mathcal{G}_t$$

5. Application : Approximation of Option Price

Pre-default price of Option

$$= F(t,T)E_t^T\Big[\max\Big(\sum a_i \widetilde{P}(T,T_{i_1})\cdots\widetilde{P}(T,T_{i_n}),0\Big)\Big]$$

a. Survival contingent forward measure

b. Payoff on exercise : **polynomial of defaultable bond prices**

(or exponentially affince fct. of state vector)

c. Approximate density fct. of payoff by Gram–Charlier expansion

Vulnerable option on a defaultable bond

$$V_t = F(t,T_0)E^{T_0}\Big[\max\Big(\sum_{j=1}^{N} a_j \widetilde{P}^C(T_0,T_j) - K, 0\Big) \mid X_t\Big]$$

Vulnerable option to enter into a CDS

$$V_t = F(t,T_0)E^{T_0}\Big[\max\Big(G,0\Big) \mid X_t\Big]$$

The payoff upon the exercise of the vulnerable option to enter into a CDS is given by

$$E\Big[\sum_{j=1}^{N} e^{-\int_{T_0}^{T_j} r(X_u)du}\Big(-\kappa \mathbf{1}_{\{\tau^C > T_j\}} + \mathbf{1}_{\{T_{j-1} < \tau^C \le T_j\}}\Big(a + bP(T_j, U)\Big)\Big) \mid \mathcal{F}_{T_0}\Big],$$

which becomes $\mathbf{1}_{\{\tau^{ABC} > T_0\}} G$ where

$$\begin{aligned} G &= \sum_{j=1}^{N}\Big(-\kappa G(T_0, T_j, T_j) + a\Big(G(T_0, T_j, T_{j-1}) - G(T_0, T_j, T_j)\Big) \\ &\qquad + b\Big(G(T_0, U, T_{j-1}) - G(T_0, U, T_j)\Big)\Big), \quad \text{and} \\ G(t, S, T) &= E\Big[\exp\Big(-\int_t^S r(X_u)du - \int_t^T h^C(X_u)du\Big) \mid X_t\Big] \end{aligned}$$

which is an exponentially affine function of the state vector X_t if X is affine and r and h^i are linear functions of X.

6. Gram–Charlier Expansion

Orthogonal expansion of density function

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big)$$

1. ϕ : density function of normal distribution $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)$
2. H_n : Hermite polynomial $H_n(x)\phi(x) = (-1)^n \frac{d^n}{dx^n}\phi(x)$
3. q_n : coefficient dependent of cumulants c_i $(i = 1, 2, \ldots, n)$

$$q_0 = 1, \quad q_1 = q_2 = 0, \quad q_3 = \frac{c_3}{3! c_2^{3/2}}, \quad q_4 = \frac{c_4}{4! c_2^2}, \quad q_5 = \frac{c_5}{5! c_2^{5/2}}, \ \ldots$$

$$E\left[\max(Y, 0)\right] = \int_0^{\infty} x f(x) dx = \int_0^{\infty} x \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) dx$$

Approximation of Option Price

1. Bond moments under forward measure

$$\mu^T(t, T_0, \{U_1, \cdots, U_m\}) \equiv E^T\Big[\prod_{i=1}^{m} \widetilde{P}^C(T_0, U_i) \mid \mathcal{F}_t\Big]$$

2. Swap moments are obtained by bond moments and cash flows

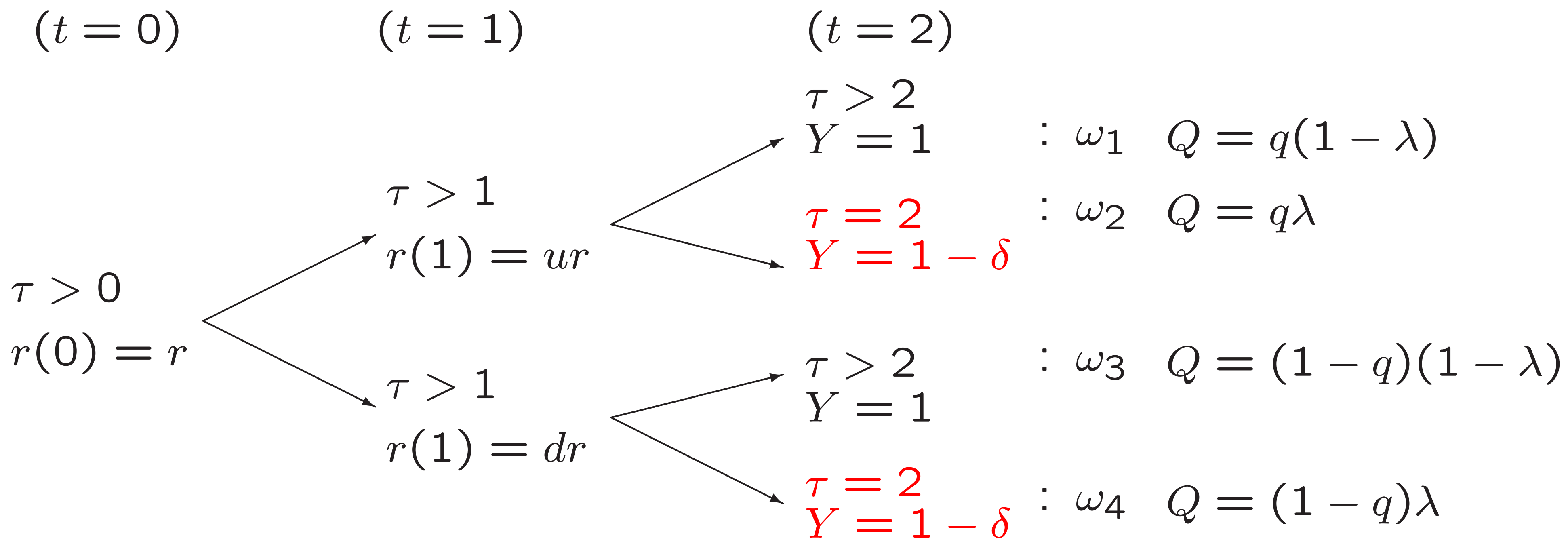
$$\begin{aligned} M_m(t) &= E^T\Big[SV(T)^m \mid \mathcal{F}_t\Big] = E^T\Big[\Big(\sum_{i=0}^{N} a_i \widetilde{P}^C(T, T_i)\Big)^m \mid \mathcal{F}_t\Big] \\ &= \sum_{0 \le i_1, \ldots, i_m \le N} a_{i_1} \cdots a_{i_m} \mu^T(t, T, \{T_{i_1}, \cdots, T_{i_m}\}) \end{aligned}$$

3. Approximated option price $(C_k = c_k F(t,T)^k)$

$$\begin{aligned} V_t \approx\; & C_1 N\Big(\frac{C_1}{\sqrt{C_2}}\Big) + \sqrt{C_2}\phi\Big(\frac{C_1}{\sqrt{C_2}}\Big) \\ & + \sqrt{C_2}\phi\Big(\frac{C_1}{\sqrt{C_2}}\Big) \sum_{k=3}^{L} (-1)^k q_k H_{k-2}\Big(\frac{C_1}{\sqrt{C_2}}\Big) \end{aligned}$$

7. Survival contingent measure : Case of 2-period model

$$Q\{r(1) = ur\} = q, \qquad Q\{r(1) = dr\} = 1 - q,$$
$$Q\{\tau > 1\} = 1, \qquad Q\{\tau = 2\} = \lambda$$



Recall that

1. Pre-default price V is obtained in risk-neutral world Q.

2. F/B is not a Q-martingale, but $F\Lambda/B$ is a Q-martinagle.

3. Λ is constructed so that $F\Lambda/B$ is a Q-martinagle.

4. A survival contingent forward measure Q^{svc} is defined as the relative price of any asset with respect to $F\Lambda$ is a martingale.

5. From the definition we can obtain the probabilities of rate movement and default under Q^{svc}.

Q^{svc} coincides with Q^{fwd} if the intensity is deterministic

$Q^{fwd} \longrightarrow Q^{sv}$: same q, but λ is changed

$Q^{fwd} \longrightarrow Q^{svc}$: same q and same λ

ω	Q	Q^{fwd}	Q^{sv}	Q^{svc}
ω_1	$q(1-\lambda)$	$q^{fwd}(1-\lambda)$	$Q^{fwd}\{\omega_1\}\dfrac{1}{1-\delta\lambda}$	$Q^{fwd}\{\omega_1\}$
ω_2	$q\lambda$	$q^{fwd}\lambda$	$Q^{fwd}\{\omega_2\}\dfrac{1-\delta}{1-\delta\lambda}$	$Q^{fwd}\{\omega_2\}$
ω_3	$(1-q)(1-\lambda)$	$(1-q^{fwd})(1-\lambda)$	$Q^{fwd}\{\omega_3\}\dfrac{1}{1-\delta\lambda}$	$Q^{fwd}\{\omega_3\}$
ω_4	$(1-q)\lambda$	$(1-q^{fwd})\lambda$	$Q^{fwd}\{\omega_4\}\dfrac{1-\delta}{1-\delta\lambda}$	$Q^{fwd}\{\omega_4\}$

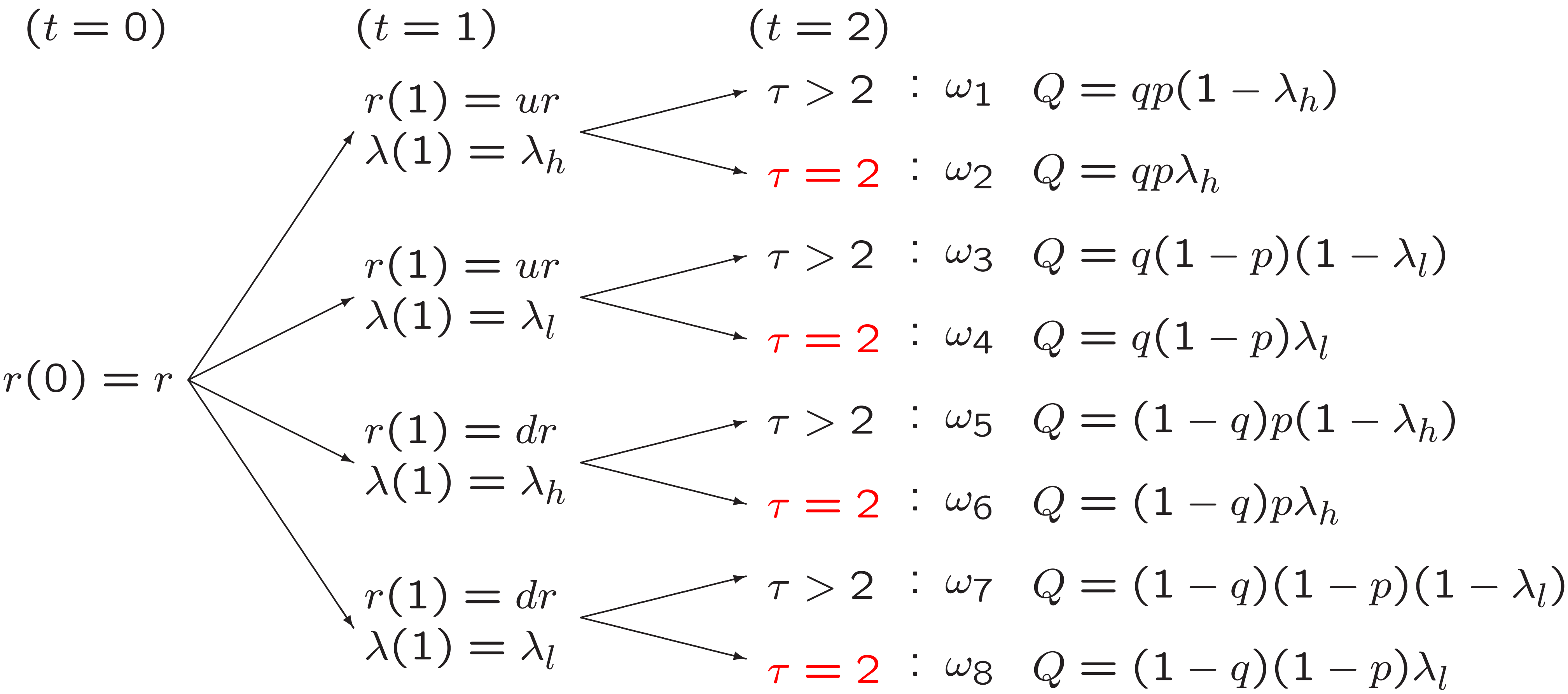
$$q^{fwd} = q^{sv} = q^{svc} = \frac{q(1+ur)^{-1}}{q(1+ur)^{-1} + (1-q)(1+dr)^{-1}}$$

$$\lambda^{fwd} = \lambda^{svc} = \lambda, \qquad \lambda^{sv} = \frac{\lambda(1-\delta)}{1-\delta\lambda}$$

$$Q\{r(1) = ur\} = q, \qquad Q\{r(1) = dr\} = 1 - q,$$

$$Q\{\lambda(1) = \lambda_h\} = p, \qquad Q\{\lambda(1) = \lambda_l\} = 1 - p,$$

$$Q\{\tau > 1\} = 1, \qquad Q\{\tau = 2 \mid t = 1\} = \lambda(1)$$



Q^{svc} differs from Q^{fwd} if the intensity is stochastic

$Q^{fwd} \to Q^{sv}$: same q and same p, but λ is changed

$Q^{fwd} \to Q^{svc}$: same q and same λ, but p is changed

	Q	Q^{fwd}	Q^{sv}	Q^{svc}
Prob. of higher rate	q	q^{fwd}	q^{fwd}	q^{fwd}
Prob. of higher intensity	p	p	p	$p(1-\delta\lambda_h)$
Higher value of intensity	λ_h	λ_h	λ_h^{sv}	λ_h
Lower value of intensity	λ_l	λ_l	λ_l^{sv}	λ_l

$$q^{fwd} = q^{sv} = q^{svc} = \frac{q(1+ur)^{-1}}{q(1+ur)^{-1} + (1-q)(1+dr)^{-1}}$$

$$p^{fwd} = p^{sv} = p, \qquad p^{svc} = p(1-\delta\lambda_h)$$

$$\lambda^{fwd}(1) = \lambda^{svc}(1) = \lambda(1), \qquad \lambda^{sv}(1) = \frac{\lambda(1)(1-\delta)}{1-\delta\lambda(1)}$$

(Step 3) An equivalent measure $Q^{svc} = Q^T \sim Q$ can be defined by

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and is called survival contingent forward measure for RMV.

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Under Q^T the pre-default price of a contingent claim with RMV is expressed as if $F(t,T)$ were a bond price

Theorem 3. *Under Assumptions 1 and 2, the price S is given by*

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where E^T is the expectation with respect to Q^T

7. Summary

1. Recovery of market value for multiple fimrs
2. Survival contingent forward measure
3. Vulnerable option on defaultable bond, Option on CDS