Well-posedness of utility-maximization

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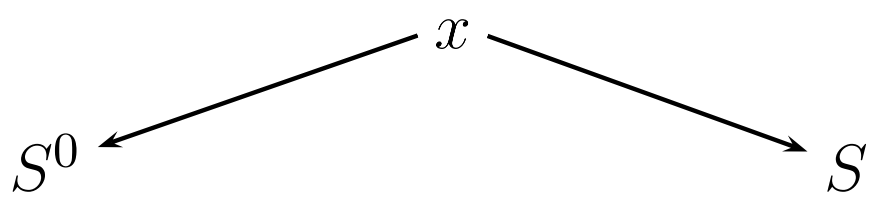
Agenda

- The optimal portfolio choice
  - Existence
  - Uniqueness
  - **Stability**
- Modeling financial markets
  - Complete
  - **Incomplete**
- Results about continuity of optimizers
  - Market specification (**drift** parameter)
  - Preference specification (**utility** function)

Optimal Portfolio Choice

- How does an investor optimally choose between a risky security (S) and a risk-free security (S^0)?



- The investor is described by
  - Preferences; given by a utility function, U
  - Beliefs; given by a set of probabilities, $\mathbb{P}$
- The investor is constrained in this choice (no arb)
  - The financial market allows a state-price density, Y

Solution - Abstract

- The investor seeks a strategy θ to maximize utility:

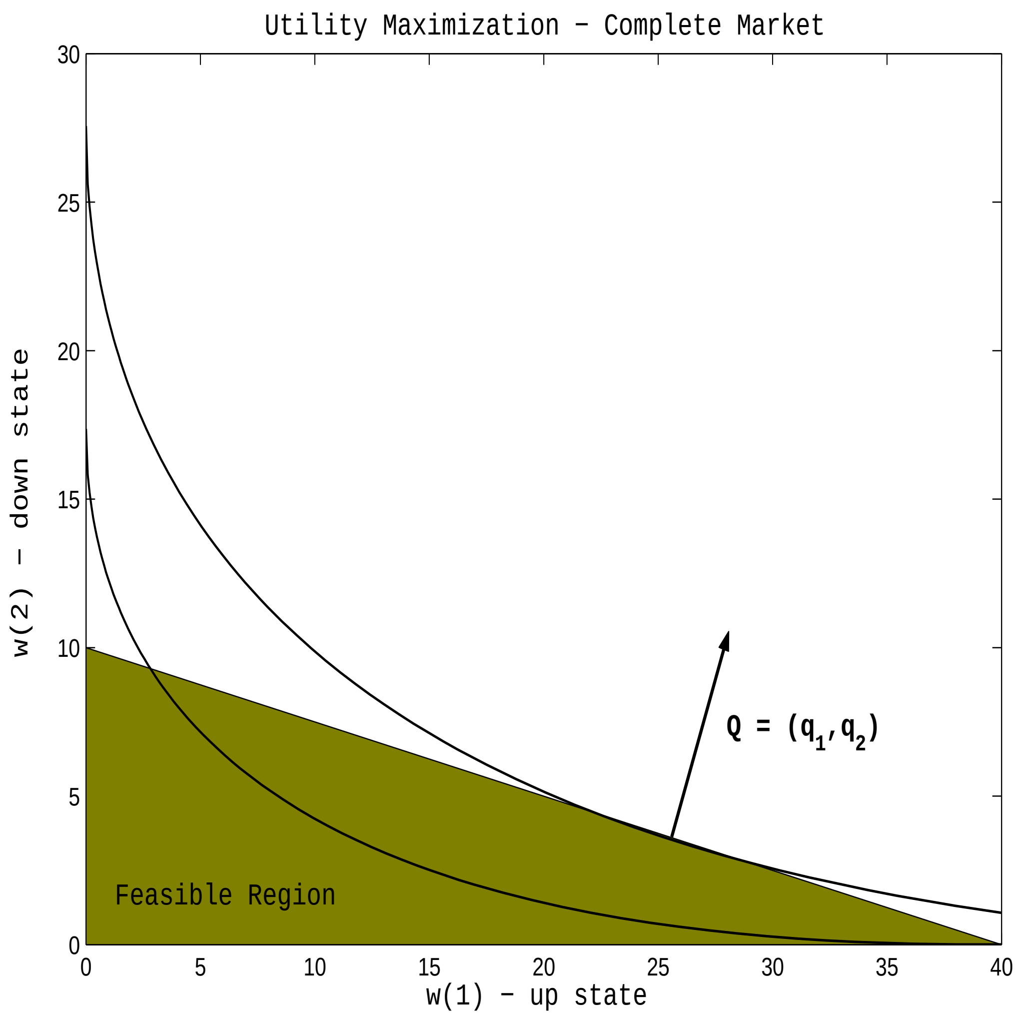
$$\max_{\theta} \mathbb{E}^{\mathbb{P}} U\big(x + (\theta \cdot S)_T\big)$$

- Instead of thinking about strategies θ, we can think about the outcomes $W \triangleq x + (\theta \cdot S)_T$
  - Representation theorems
- From Lagrange theory we get the first order condition

$$U'\Big(\hat{W}(\omega)\Big) = y\hat{Y}(\omega)$$

  - $\hat{Y}$ is a state-price density $\left(\approx \frac{d\mathbb{Q}}{d\mathbb{P}},\ \text{Arrow-Debrau}\right)$
  - y is the budget multiplier determined by x

Solution - Graphically



Example - Drift

- Zero interest rate ($S_t^0 \equiv 1$), and let ($\sigma \equiv 1$)

$$dS_t = S_t(\lambda_t dt + dB_t)$$

- The **unique** state-price density reads $\mathcal{E}\left(-\lambda \cdot B\right)$
- Recall the optimizer $\widehat{W}$ is given by

$$\widehat{W} = (U')^{-1}\Big(yY\Big) = (U')^{-1}\Big(y\mathcal{E}\left(-\lambda \cdot B\right)_T\Big)$$

  - Direct λ-dependence via the state-price density Y
  - Indirect λ-dependence via the budget constraint:

$$x = \mathbb{E}\left[Y(U')^{-1}\left(yY\right)\right]$$

Example - Drift (Cont)

- Let us look at the power case, $p < 1$

$$U(x) \triangleq \frac{x^p}{p}, \quad V(x) \triangleq \frac{x^{-q}}{q}$$

where V is U's conjugate, $q \triangleq \frac{p}{1-p}$.

- In this case the Lagrange multiplier y satisfies

$$x = y^{\frac{1}{p-1}} \mathbb{E}\left[Y^{\frac{p}{p-1}}\right] = q y^{\frac{1}{p-1}} \mathbb{E}\left[V\left(Y\right)\right].$$

- The state-price density Y can depend continuously on λ **without** the Lagrange multiplier y doing so

Main Theorem - Drift

I: S^n is modeled by a **continuous** martingale M:

$$dS_t^n = \lambda_t^n d\langle M\rangle_t + dM_t$$

II: $\lambda^n \to \lambda^0$ in the sense that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (\lambda_u^n - \lambda_u^0)^2 d\langle M\rangle_u\right] \to 0$$

III: The following family is uniform integrable

$$\left\{V^+\left(\mathcal{E}\left(-\lambda^n \cdot M\right)_T\right)\right\}_{n=1}^{\infty}$$

THM: Assume I,II and III. The optimal wealths converges in **probability**, $\widehat{W}(\lambda^n) \to \widehat{W}(\lambda^0)$, and the value functions converge $\mathbb{E}[U(\widehat{W}(\lambda^n))] \to \mathbb{E}[U(\widehat{W}(\lambda^0))]$.

Example - Preferences

- Zero interest rate ($S_t^0 \equiv 1$), and let ($\sigma \equiv 1$)

$$dS_t = S_t(\lambda_t dt + dB_t)$$

- Reasoning as before, we need to ensure continuity of the Lagrange multiplier y:

$$\mathbb{E}\left[Y((U^n)')^{-1}(yY)\right] \to \mathbb{E}\left[Y((U^0)')^{-1}(yY)\right]$$

- We can have $U^n \to U^0$ pointwise **without** the corresponding Lagrange multipliers y^n converging to y^0.

Main Theorem - Preferences

I: S is modeled by a **continuous** martingale M:

$$dS_t = \lambda_t d\langle M \rangle_t + dM_t$$

II: $U^n \to U^0$ pointwise and satisfy for all $x > 0$ (see JN04)

$$U^n(x) \leq \overline{U}(x)$$

III: The conjugate of $\overline{U}$, denoted by $\overline{V}$, satisfies

$$\mathbb{E}\overline{V}^{+}\left(\mathcal{E}\left(-\lambda \cdot M\right)_T\right) < \infty$$

THM: Assume I,II and III. The optimal wealths converges in **probability**, $\widehat{W}(U^n) \to \widehat{W}(U^0)$, and the value functions converge $\mathbb{E}[U^n(\widehat{W}(U^n))] \to \mathbb{E}[U^0(\widehat{W}(U^0))]$.