# Analysis of Monte Carlo Methods for Option Pricing under Stochastic Volatility Models

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# Multifactor Stochastic Volatility Model

Under a risk-neutral prob. meas. $\mathbb{P}^{\star(\Lambda)}$, a multifactor SV model is of the form:

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t, Z_t) S_t dW_t^{(0)\star}, \\ dY_t &= c_1(Y_t, Z_t) dt + g_1(Y_t, Z_t) dW_t^{(1)\star}, \\ dZ_t &= c_2(Y_t, Z_t) dt + g_2(Y_t, Z_t) dW_t^{(2)\star} \end{aligned}$$

$$\begin{aligned} d\langle W^{(0)}, W^{(1)} \rangle_t &= \rho_1 dt \\ d\langle W^{(0)}, W^{(2)} \rangle_t &= \rho_2 dt \\ d\langle W^{(1)}, W^{(2)} \rangle_t &= \rho_{12} dt. \end{aligned}$$

# Monte Carlo Pricing with Control Variate

$$P(0, S_0, \sigma_0) \approx \frac{1}{N} \sum_{i=1}^{N} \left[ e^{-r(\tau \wedge T)} H(S^{(i)}_{\tau \wedge T}) - \mathcal{M}^{(i)}(\tilde{P}) \right],$$

where $\tau$ is a stopping time and

$$\mathcal{M}(\tilde{P}) = \int_0^{\tau \wedge T} e^{-rs} \frac{\partial \tilde{P}}{\partial x}(s, S_s, \sigma_s) \sigma_s S_s dW^\star_s$$

is a *martingale* with $\tilde{P}$ being an approximation of $P$.

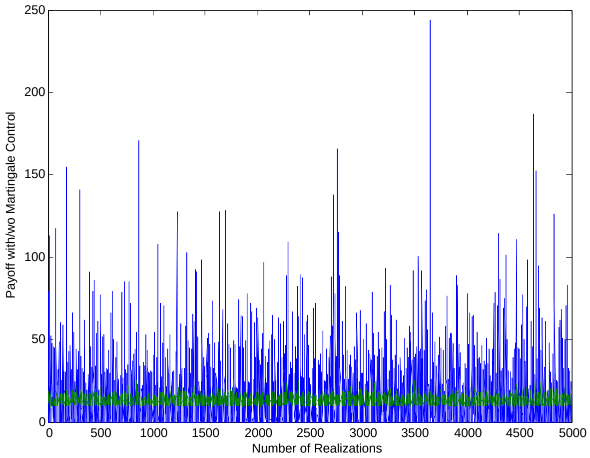
# History of Martingale Control Variates

**M1**: Clewlow and Carverhill(1994): $\tilde{P} = P_{BS}(t, S_t; \bar{\sigma})$, $\bar{\sigma}$ is a **constant** (the long-run mean).

**M2**: Heath and Platen(2002): $\tilde{P} = P_{BS}(t, S_t; \widehat{\sigma}(t))$, $\widehat{\sigma}_t$ is a **deterministic** volatility.

**M3**: Fouque and Han(2004): $\tilde{P} = P_{BS}(t, S_t; \bar{\sigma}_t)$, $\bar{\sigma}_t$ is an averaged **stochastic** volatility process.

# Sampled Payoffs of European Call Option



blue: Basic MC green: MC+CV

Q: $Var(e^{-rT}H(S_T) - \mathcal{M}(\tilde{P}))$?

# Asymptotic Formulation
# Rescale Stochastic Volatility model

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_{0t}^\star \\
\sigma_t &= f(Y_t, Z_t) \\
dY_t &= \left[\frac{1}{\varepsilon} c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t)\right] dt \\
&+ \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \left(\rho_1 dW_{0t}^\star + \sqrt{1-\rho_1^2}\, dW_{1t}^\star\right) \\
dZ_t &= \left[\delta c_2(Z_t) + \sqrt{\delta} g_2(Z_t) \Lambda_2(Y_t, Z_t)\right] dt + \sqrt{\delta} g_2(Z_t) \\
& \cdot \left(\rho_2 dW_{0t}^\star + \rho_{12} dW_{1t}^\star + \sqrt{1-\rho_2^2-\rho_{12}^2}\, dW_{2t}^\star\right)
\end{aligned}
$$

# Variance Decomosition
## Case: No Correlation between BMs

$$
\begin{aligned}
&Var\left(e^{-rT}H(S_T) - \mathcal{M}(P_{BS})\right) = I\!\!E^{\star}{}_{0,x,y,z}\Big\{ \\
&\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s, Z_s) f^2(Y_s, Z_s) S_s^2 \\
&+ \frac{1}{\varepsilon}\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial y}\right)^2 (s, S_s, Y_s, Z_s) g_1^2(Y_s) ds \\
&+ \delta \int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)^2 (s, S_s, Y_s, Z_s) g_2^2(Z_s) ds \Bigg\}.
\end{aligned}
$$

## Variance Analysis

Under some smooth and bound conditions,

1.$\mathbb{E}^\star \left\{ \int_0^T e^{-2rs} \left( \frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x} \right)^2 f^2(Y_s, Z_s) S_s^2 ds \right\} < C \max\{\varepsilon, \delta\}$

2.$\mathbb{E}^\star \left\{ \int_0^T e^{-2rs} \left( \frac{\partial P^{\varepsilon,\delta}}{\partial y} \right)^2 g_1^2(Y_s) ds \right\} \leq C \, \varepsilon^2$

3.$\mathbb{E}^\star \left\{ \int_0^T e^{-2rs} \left( \frac{\partial P^{\varepsilon,\delta}}{\partial z} \right)^2 g_2^2(Z_s) ds \right\} \leq C$

such that

$$Var \ \ \left( e^{-rT} H(S_T) - \mathcal{M}(P_{BS}) \right) \leq C \max\{\varepsilon, \delta\}.$$

# First Inequality vs Delta Approximation

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_t}(t, S_t, Y_t, Z_t) = \mathbb{E}^{\star}{}_t \left\{ e^{-r(T-t)} \mathbf{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_t} \right\}$$

$$= \tilde{E}_t \left\{ \mathbf{I}_{\{S_T > K\}} \right\}$$

where the Radon-Nikodym derivative is

$$\frac{d\tilde{P}}{d\mathbb{P}^{\star}} = e^{-\int_0^T \frac{\sigma_t^2}{2} dt + \int_0^T \sigma_t dW_t^{(0)\star}}.$$

Digital option approximation yields

$$\left| \left( \frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x} \right) (t, S_t, Y_t, Z_t) \right| \leq C \max\{\sqrt{\varepsilon}, \sqrt{\delta}\}.$$

# Second Inequality vs Stability of Perturbed Dynamical System

Rescaling $\tilde{Y}_s^\varepsilon = Y_{s\varepsilon}$ and $\tilde{Z}_s^\varepsilon = Z_{s\varepsilon}$, we deduce

$$\frac{d}{ds}\begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix}$$

$$+ \quad \sqrt{\varepsilon} \cdot \begin{pmatrix} \nu_1\sqrt{2}\frac{\partial \tilde{\Lambda}_1}{\partial y}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) & \frac{\partial \tilde{\Lambda}_1}{\partial z}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \\ \sqrt{\delta}\nu_2\frac{\partial \tilde{\Lambda}_2}{\partial y}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) & -\delta + \sqrt{\delta}\nu_2\sqrt{2}\frac{\partial \tilde{\Lambda}_2}{\partial z}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix} \text{ with } \left(\frac{\partial \tilde{Y}_0^\varepsilon}{\partial y}, \frac{\partial \tilde{Z}_0^\varepsilon}{\partial y}\right)^T = (1,0)^T.$$

# Replication Error and Variance Reduction

## Numerical Results I: European Options

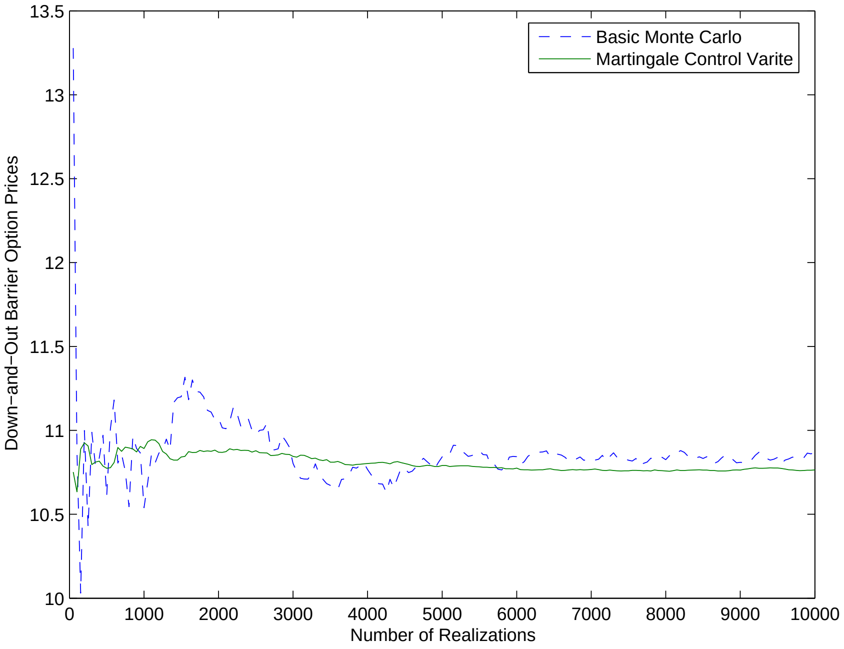
## Double Heston Model

| $1/\varepsilon$ | $\delta$ | $Std^{BMC}$ | $Std^{MCV}$ | Ratio |
|---|---|---|---|---|
| 75 | 0.01 | 0.1103 (7.03) | 0.0068 (7.09) | **265** |
| 50 | 0.1 | 0.1102 (6.97) | 0.0073 (7.08) | **230** |
| 10 | 0.5 | 0.1085 (6.94) | 0.0103 (7.03) | **111** |
| 5 | 1 | 0.1063 (6.91) | 0.0113 (6.99) | **89** |

# Numerical Results II: Barrier Options

| $1/\varepsilon$ | $\delta$ | $Std^{BMC}$ | $Std^{MCV}$ | Ratio |
|---|---|---|---|---|
| 100 | 0.01 | 0.2822 (10.82) | 0.0304 (10.85) | **86** |
| 75 | 0.1 | 0.2047 (10.77) | 0.0306 (10.76) | **45** |
| 50 | 1 | 0.2455 (11.21) | 0.0474 (11.10) | **27** |
| 25 | 10 | 0.2604 (12.62) | 0.0417 (12.44) | **39** |

# Variance Reduction: a down and out Barrier Option



# American Option Pricing Problem

Given the risk-neutral prob. space $\left(\Omega, \mathcal{F}, I\!\!P^\star, \mathcal{F}_{[0:T]}\right)$, an Americcan option pricing problem is formulated as an optimal stopping problem:

$$P_{am}(0, S_0) = \sup_{0 \le \tau \le T} I\!\!E^\star \left\{ e^{-r\tau} H(S_\tau) | \mathcal{F}_0 \right\},$$

where $\tau$ is any $\mathcal{F}_t$-adapted stopping time and $S$ is the underlying asset price (diffusion) process.

# Recent Development on Efficient Monte Carlo Pricing Algorithms

(1) Primal Approach: approximate optimal stopping rule
Tsitsiklis and Van Roy (2001), Longstaff and Schwartz (2001)

(2) Dual Approach: approximate (super-)martingales
Rogers (2002) used martingale approximation.
Haugh and Kogan (2004) used super-martingale approximation.

# Primal Approach (I)

## A Low-Biased Solution

For any stopping rule $\tilde{\tau}$, a lower solution is deduced

$$\begin{aligned} P_{am}(0, S_0) &= \sup_{0 \le \tau \le T} \mathbb{E}^\star \left\{ e^{-r\tau} H(S_\tau) | \mathcal{F}_0 \right\}, \\ &\ge \mathbb{E}^\star \left\{ e^{-r\tilde{\tau}} H(S_{\tilde{\tau}}) | \mathcal{F}_0 \right\} \equiv P_{am}^{low}(0, S_0) \end{aligned}$$

$\tilde{\tau}$ can be estimated from least squares methods from dynamic programming.

# Primal Approach (II)

## Variance Reduction

By martingale control variate methods*,

$$\mathbb{E}^{\star}\left\{e^{-r\tilde{\tau}}H(S_{\tilde{\tau}})\right\} = \mathbb{E}^{\star}\left\{e^{-r\tilde{\tau}}H(S_{\tilde{\tau}}) - \mathcal{M}_{\tilde{\tau}}\right\},$$

where $\tilde{P} \approx P_{am}$ and

$$\mathcal{M}_{\tilde{\tau}} = \int_0^{\tilde{\tau}} e^{-rs}\frac{\partial \tilde{P}}{\partial x}(s, S_s)\sigma_s S_s dW_s.$$

Given $\tilde{\tau}$ an American option problem becomes a **Barrier option problem.**

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*Fouque and H. (2006)

# Dual Approach (I)
# A Direct Monte Carlo Simiulation

Rogers (2002) derived the duality of the American option problem by

$$P_{am}(0, S_0) = \inf_{\mathcal{M}\in H_0^1} \mathbb{E}^\star \left\{ \sup_{0\le t\le T} \left( e^{-rt} H(S_t) - \mathcal{M}_t \right) \right\},$$

$H_0^1$=\{ all integrable martingales but vanish at time 0\}

**proof:**

$\geq$: trivial

$\leq$: use Doob-Meyer decomposition of a supermartingale. (infemum can be attained)

# Dual Approach (II)

## A High-Biased Solution

Given any martingale $\tilde{M} \in H_0^1$, one deduces an upper solution

$$\begin{aligned} P_{am}^{high}(0, S_0) &\equiv \mathbb{E}^\star \left\{ \sup_{0 \le t \le T} \left( e^{-rt} H(S_t) - \tilde{M}_t \right) \right\} \\ &\ge P_{am}(0, S_0) \end{aligned}$$

Given $\tilde{M}$ an American option problem becomes a **Lookback option problem.**

# Dual Approach (III)

## Error Bound Estimate

**Lemma 1:** For any given martingale $\tilde{M} \in H_0^1$, $P_{am}^{high}(0, S_0) \leq P_{am}(0, S_0) + 2\sqrt{\mathbb{E}^\star\left\{\left(M_T^\star - \tilde{M}_T\right)^2\right\}}$.

Proof:

$$
\begin{aligned}
& P_{am}^{high}(0, S_0) \\
& = \mathbb{E}^\star\left\{\sup_{0 \leq t \leq T}\left(e^{-rt}H(S_t) - M_t^\star + M_t^\star - \tilde{M}_t\right)\right\} \\
& \leq P_{am}(0, S_0) + \mathbb{E}^\star\left\{\sup_{0 \leq t \leq T}\left(M_t^\star - \tilde{M}_t\right)\right\} \\
& \leq P_{am}(0, S_0) + 2\sqrt{\mathbb{E}^\star\left\{\left(M_T^\star - \tilde{M}_T\right)^2\right\}}
\end{aligned}
$$

# Variance Bound

**Lemma 2:** For low-biased solution, the variance of its MCV estimator is

$$
\begin{aligned}
& Var\left(e^{-r\underline{\tau}}H(S_{\underline{\tau}}) - \mathcal{M}(\tilde{P};\underline{\tau})\right) = Var\left(\mathcal{M}\left(P_{am}^{low} - \tilde{P};\underline{\tau}\right)\right. \\
& = E\left(\int_0^{\underline{\tau}} e^{-2rs}\left(\frac{\partial P_{am}^{low}}{\partial x} - \frac{\partial \tilde{P}}{\partial x}\right)^2 \sigma_s^2 S_s^2 ds\right) \\
& \leq E\left(\int_0^{T} e^{-2rs}\left(\frac{\partial P_{am}^{low}}{\partial x} - \frac{\partial \tilde{P}}{\partial x}\right)^2 \sigma_s^2 S_s^2 ds\right) \\
& = Var\left(\mathcal{M}\left(P_{am}^{low} - \tilde{P}\right)\right)
\end{aligned}
$$

# Numerical Results III: Low-Biased Solution

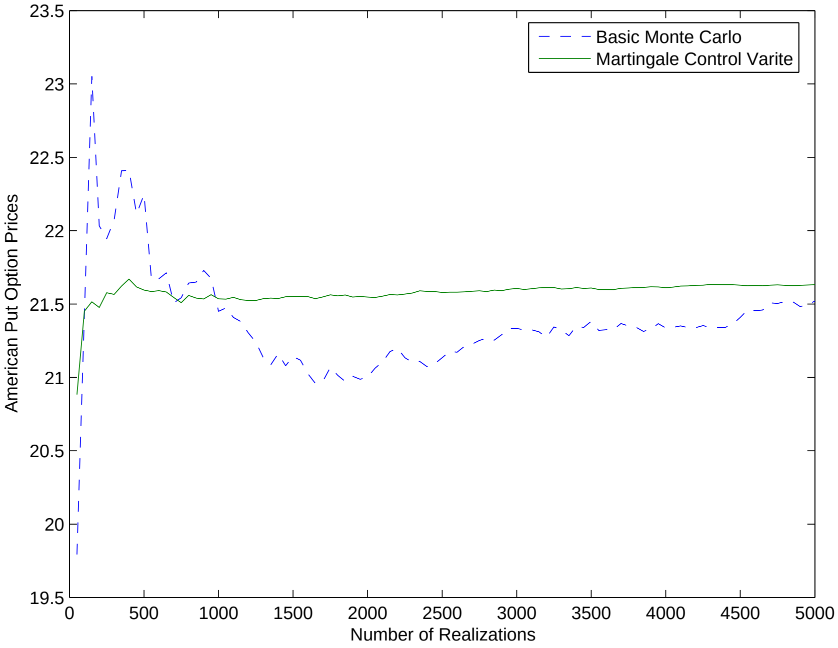
## Primal Approach

Use the Least-squares method, which provides a biased **lower** bound solution.

| $1/\varepsilon$ | $\delta$ | $Std_{low}^{Pri_{MC}}$ | $Std_{low}^{Pri_{CV}}(P_{BAW})$ | Ratio |
|---|---|---|---|---|
| 100 | 0.01 | 0.235 (21.43) | 0.024 (21.59) | **96** |
| 75 | 0.1 | 0.256 (21.48) | 0.028 (21.80) | **81** |
| 50 | 1 | 0.257 (21.52) | 0.035 (21.63) | **54** |
| 25 | 10 | 0.260 (21.96) | 0.045 (21.32) | **32** |

Ref: G. Barone-Adesi and R. E. Whaley, “Efficient Analytic Approximation of American Option Values,” The Journal of Finance, Vol. XLII, No. 2, June 1987.

# Variance Reduction: American Options



# Numerical Results IV:
# Low-Biased Solution VS High-Biased Solution

Use our control martingale. (Usual duality approach is not easy to generalized to SV models.)

| $1/\varepsilon$ | $\delta$ | $Std_{low}^{Pri_{CV}}$ | $Std_{high}^{Dul}$ |
|---|---|---|---|
| 100 | 0.01 | 0.0240 (21.59) | 0.0239 (22.29) |
| 75 | 0.1 | 0.0286 (21.80) | 0.0271 (22.33) |
| 50 | 1 | 0.0350 (21.63) | 0.0334 (22.37) |
| 25 | 10 | 0.0453 (21.32) | 0.0433 (22.29) |