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Model Free Hedge Ratios and Scale Invariant Models

Carol Alexander and Leonardo Nogueira

ICMA Centre, University of Reading, UK

www.icmacentre.rdg.ac.uk

Outline

1. Classification of Scale Invariant Models
2. Hedging Options on Scale Invariant Processes
3. Minimum Variance Hedging
4. Empirical Results

1. Classification of Scale Invariant Models

- In this section we show that
    - o Most models for pricing options on tradable assets have the scale invariance property
    - o One does not even have to specify the price process, or the returns density, to decide whether an option pricing model has a scale invariant price process
- Thus many new models not previously recognised as scale invariant are proved to be scale invariant or otherwise

Merton (1973)

A price process is *scale invariant*
if the returns density has constant returns to scale

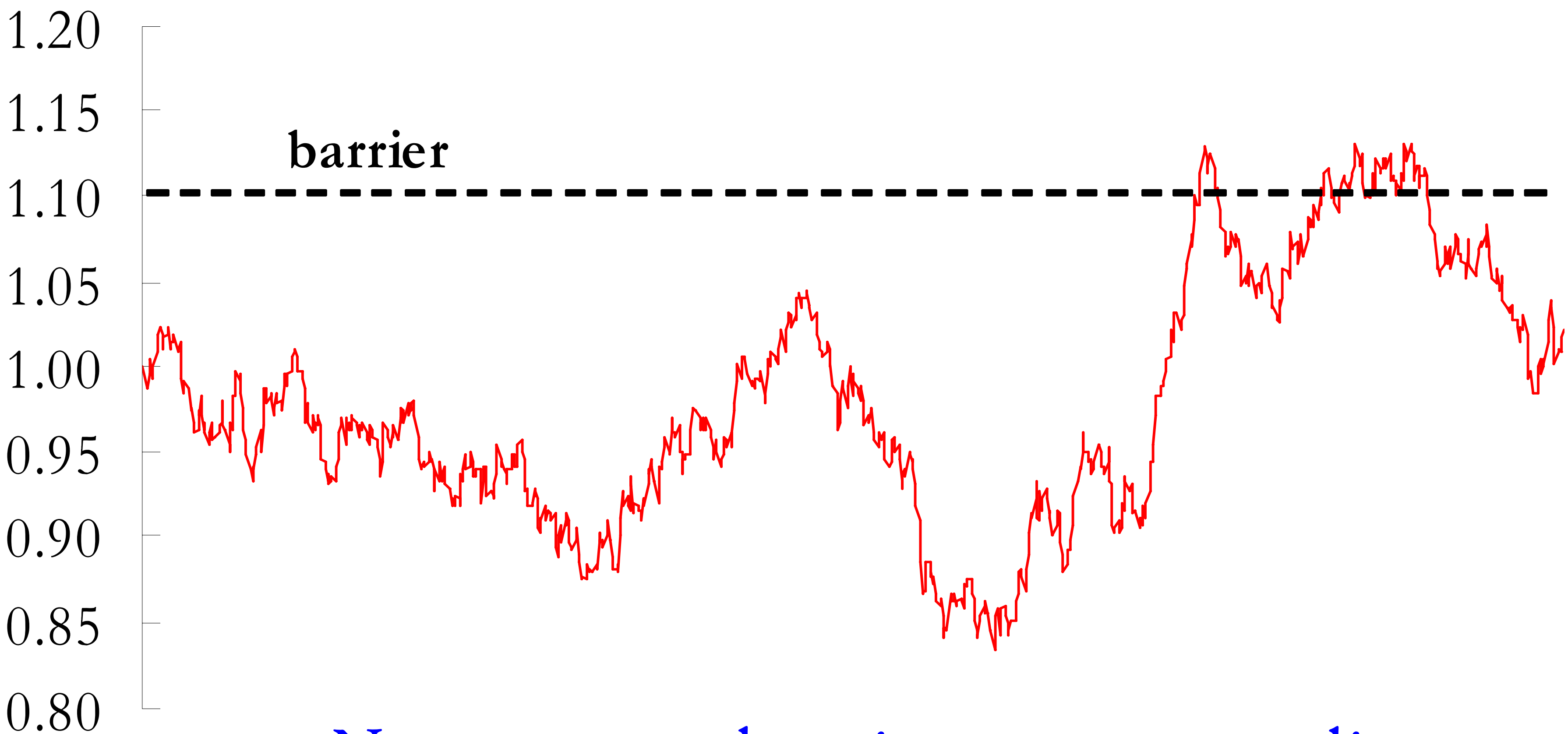
- So it does not matter whether price is measured in cents or dollars … the value of a standard option just scales with the *unit of measurement* of the price

- He concludes that:

  - If the price S is generated by a scale invariant process then the value f of a *standard call or put* on S with strike K is *homogeneous of degree one* in S and K

Extension to Other Claims

One simulated path for S

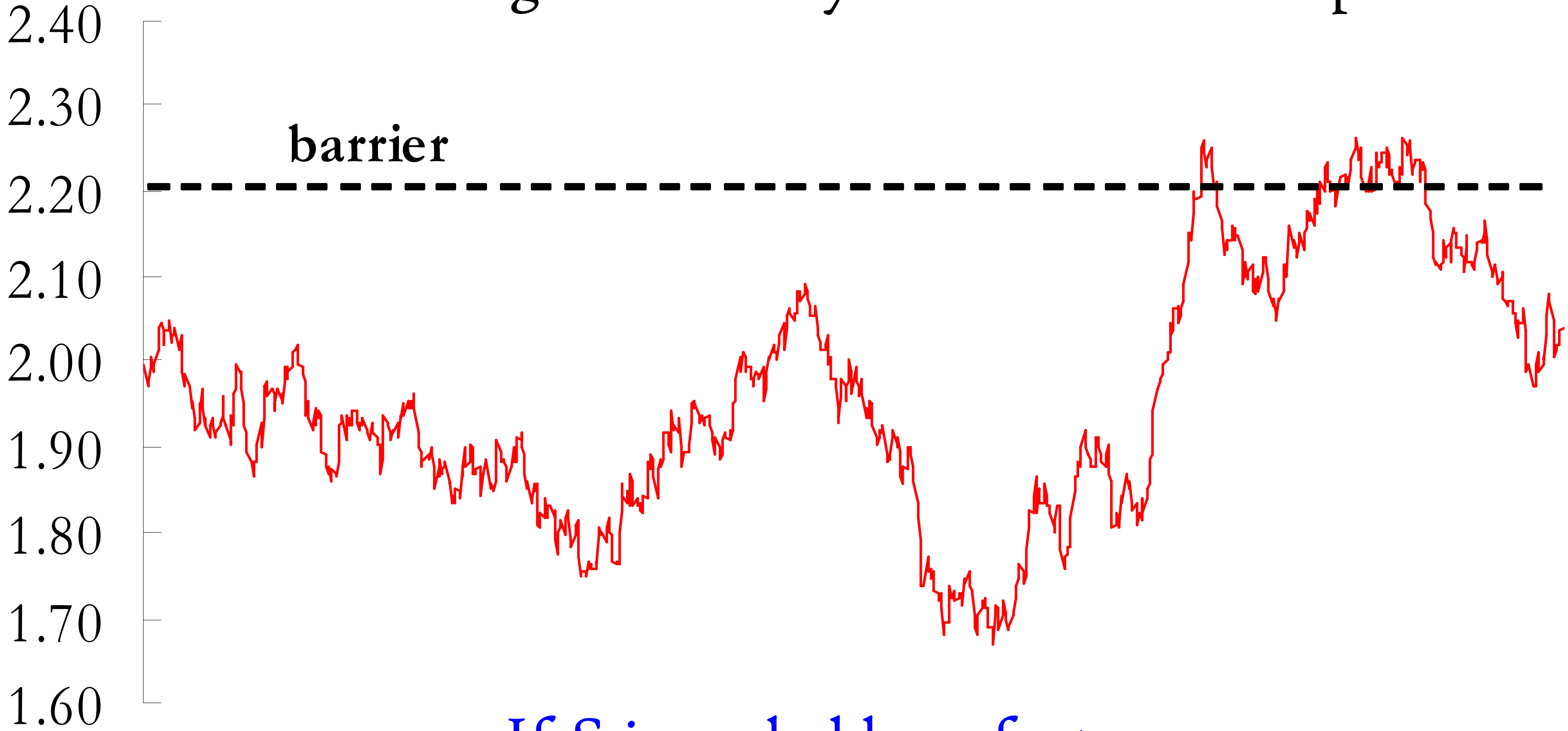


Now suppose there is a 2:1 reverse split

i.e. $S \rightarrow 2S$ and $B \rightarrow 2B$ and $K \rightarrow 2K$ ...

A Scale Invariant Process

Same simulation
when S is generated by a scale invariant process

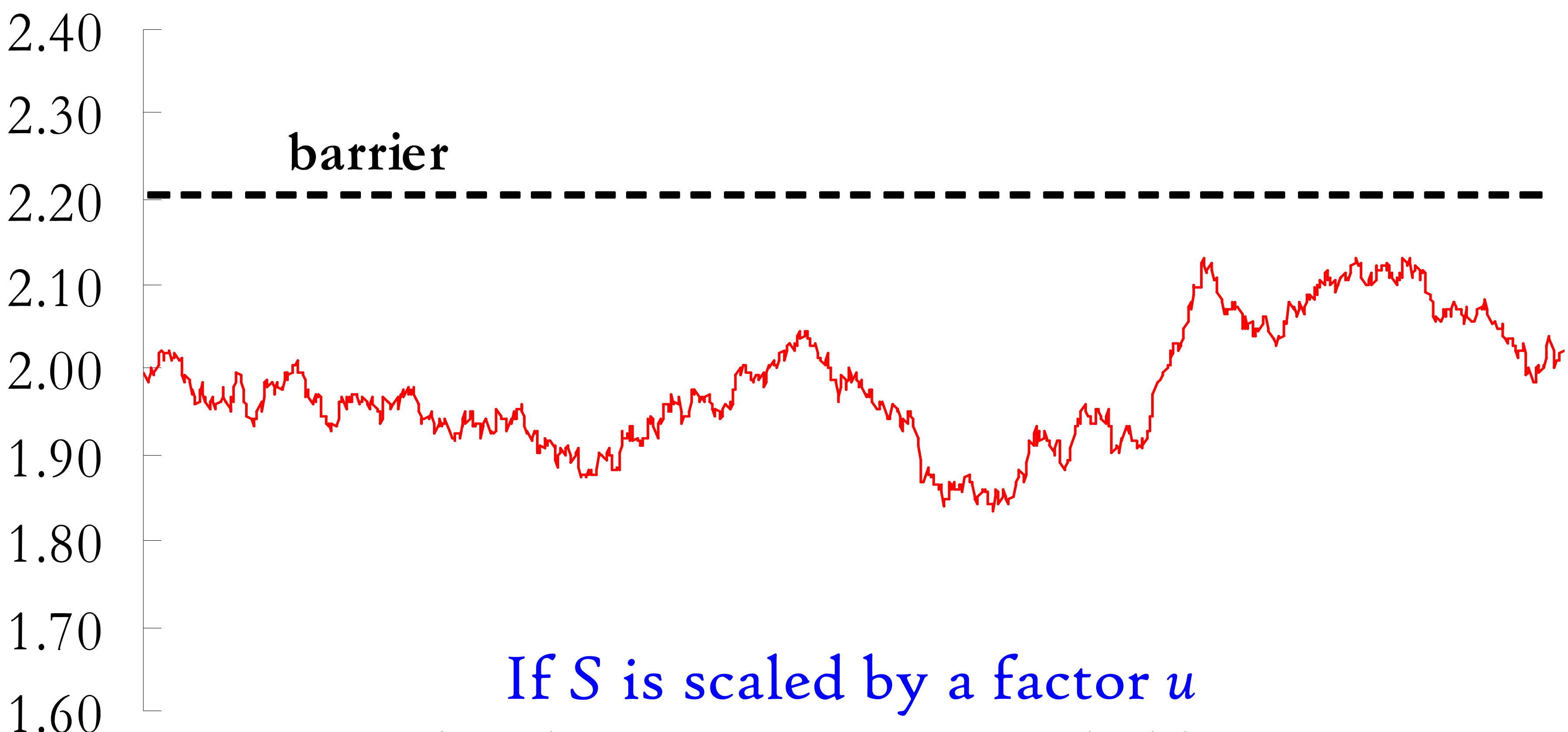


If S is scaled by a factor u
the claim price is also scaled by u

A Non–Scale Invariant Process

Same simulation

when S is generated by a *non* scale invariant process



If S is scaled by a factor u
the claim price is <u>not</u> scaled by u

Theorem 1

A price process $\{S_t\}$ is scale-invariant *iff* it is a semi-martingale and can be written in the form:

$$\boxed{\frac{dS}{S} = \mathbf{\Theta}' d\mathbf{\Lambda}}$$

Θ is a vector of random or deterministic *coefficients* that are independent of the unit of price measurement;

Λ is a vector of *factors* driving the asset price that contains the time t, Wiener processes and/or jump processes.

This theorem allows the classification of scale invariant processes *without knowing the returns density*

Corollary

Properties $(i) - (iii)$ are equivalent:

(i) S is generated by a scale invariant process

(ii) $\theta(uK, T; uS) = \theta(K, T; S) \quad u \in \mathbb{R}^+$

(iii) $\sigma(t, uS; uS_o) = \sigma(t, S; S_o) \quad u \in \mathbb{R}^+$

where θ and σ are the model implied volatility and local volatility

Hence the model's implied and local volatilities are homogeneous of degree *zero* in S and K

This result allows the classification of scale invariant processes without knowing either the price process or the returns density – *knowing only the local volatility*

Scale Invariant Models

- Geometric Brownian motion (Black-Scholes 1973)
- Jump diffusion (Merton 1976)
- Stochastic volatility (e.g. Hull & White 1987, Heston 1993)
- 'Sticky delta' local volatility (Derman, 1999)
- Mixture models (e.g. Brigo and Mercurio 2002)
- Lévy processes (Schoutens, 2003)
- Double jump models (Duffie et al., 2000; Eraker, 2004)
- Uncertain parameter models (e.g. Avellaneda *et al*. 1995)

So *most* models for traded assets are scale invariant

Non–Scale Invariant Models

- Arithmetic Brownian motion (Bachelier, 1900)
- 'Implied tree' local volatility (Dupire, 1994)
- Constant elasticity of variance (Cox, 1975)
- Short rate models (e.g. Vasicek, 1977; Cox–Ingersoll–Ross, 1985; and others)
- Stochastic Alpha–Beta–Rho (Hagan et.al, 2002)

So many models for *non-tradable* assets are *not* scale invariant

2. Hedging Options on Scale Invariant Processes

- Let g(*S*,K,…) be a claim with an expiry pay-off that is a homogeneous function of degree *k* in *S* and K
- We show that such a claim has *model free* price hedge ratios in the class of scale invariant models
- That is, it does not matter which scale invariant model we use to hedge the claim!
- Any two scale invariant models have theoretically identical price hedge ratios
- Any difference between estimates of delta, gamma, … is only due to different fits to market data

Bates (2005)

- Suppose
  - o S is generated by a scale invariant process
  - o At time t the model price of an option with strike K is homogeneous of degree one in S and K
- Then at time t the delta and gamma are *model free*

The following theorems extend this result ….

Theorem 2

- A price process is scale-invariant if and only if it *preserves the homogeneity* of a claim's pay-off at expiry throughout the life of the claim

- Note that most claims have homogeneous pay-offs:
  - Degree *zero* – digital, log contract
  - Degree *one* – almost all claims
  - Degree *two* or greater – power options

Theorem 3

- Suppose
  - o S is generated by a scale-invariant process
  - o A claim has pay-off at expiry T that is homogeneous of degree k in S and **K**
  - o **K** is a vector of *any* claim characteristics in the price dimension
- Then *all* partial derivatives of the claim price with respect to S at *any* time $t < T$ are model free

Model Free Hedge Ratios

$$\boxed{\begin{aligned} g_S &= S^{-1}\left(kg - \mathbf{K}' g_{\mathbf{K}}\right) \\ g_{SS} &= S^{-2}\left[\mathbf{K}' g_{\mathbf{KK}} \mathbf{K} + (k-1)\left(kg - 2\mathbf{K}' g_{\mathbf{K}}\right)\right] \end{aligned}}$$

where $g = g(T, \mathbf{K}; t, S)$ is the claim price

- Note that g_{K} and g_{KK} are observable if g is observable

Relationship with BS hedge ratios

- Let f^{model} be the model price of a European call or put, so that it can also be priced with a Black-Scholes (BS) type model

- Calibration to market prices $\Rightarrow$

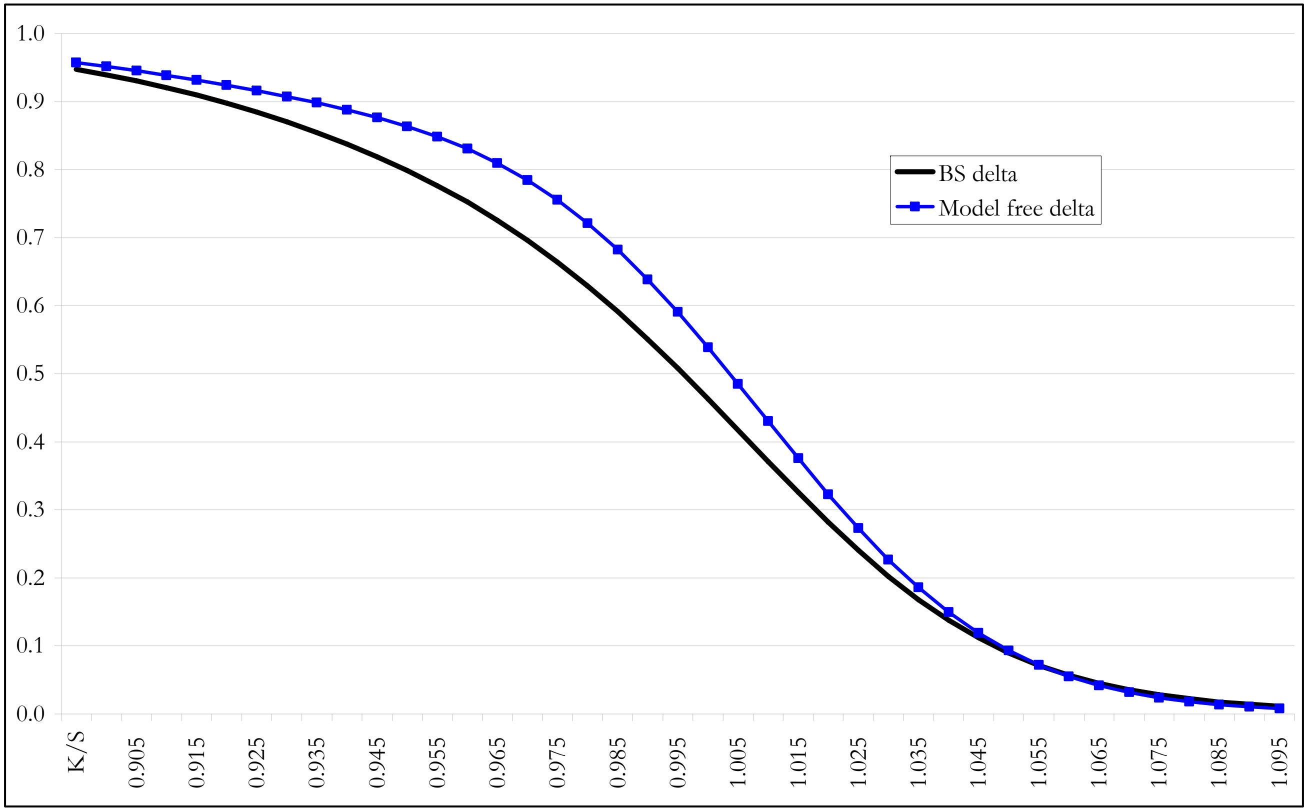
$$f^{market} \approx f^{model} \equiv f^{bs}(\theta)$$

- Using the chain rule and the fact that the model implied volatility is homogeneous of degree zero:

$$\boxed{\begin{aligned} f_S^{model} &= f_S^{bs} + f_\theta^{bs}\theta_S = f_S^{bs} - KS^{-1} f_\theta^{bs}\theta_K \\ f_{SS}^{model} &= f_{SS}^{bs} + K^2 S^{-2}\left(f_\theta^{bs}\theta_{KK} + 2 f_\theta^{bs}\theta_K + f_{\theta\theta}^{bs}\theta_K^2\right) \end{aligned}}$$

Scale Invariant *vs* BS Delta

- When the implied volatility skew is negative (positive) the model free delta will be greater than (less than) the BS delta



3. Minimum Variance Hedging

- The minimum variance (MV) delta is the amount of the underlying asset that minimizes the instantaneous variance of a delta-hedged portfolio
- So if δ_{mv} is the MV delta for a claim g on S then the portfolio $\Pi = g - \delta_{mv} S$ has minimum variance
- In many models (e.g. BS, Hull-White):

$$\delta_{mv} = g_S$$

- But if *any* parameter (e.g. volatility or interest rates) is *correlated* with the asset price

$$\delta_{mv} \neq g_S$$

MV Delta

- In a *stochastic volatility model without jumps* the MV hedge ratio is the ratio of the instantaneous covariance (between increments in the claim price and the underlying price) and the instantaneous variance (of the increments in the underlying price)

- Intuitively, this is like a *total* derivative

$$\boxed{\delta_{mv}^{sv} = \frac{dg}{dS} = g_S + g_\sigma \frac{d\sigma}{dS}}$$

- The second term is zero *iff* the two Brownians driving the price and the volatility are independent

MV Gamma

- Similarly

$$\boxed{\gamma_{mv}^{sv} = \frac{d^2 g}{dS^2} = g_{SS} + \left(2 g_{S\sigma} \frac{d\sigma}{dS} + g_{\sigma\sigma} \left(\frac{d\sigma}{dS} \right)^2 + g_{\sigma} \frac{d^2\sigma}{dS^2} \right)}$$

- So the MV gamma is equal to the standard gamma g_{SS} unless the two Brownians driving the price and the volatility are correlated

MV Hedge Ratios in Heston Model

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{V_t}\,dW_t$$

$$dV_t = a\left(m - V_t\right)dt + b\sqrt{V_t}\,dZ_t$$

$$\left\langle dW_t, dZ_t \right\rangle = \varrho\,dt$$

$$\boxed{\begin{aligned}\delta_{mv} &= g_S + g_V\left(\frac{\varrho b}{S}\right)\\ \gamma_{mv} &= g_{SS} + \frac{\varrho b}{S}\left(\left(\frac{\varrho b}{S}\right)g_{VV} + 2g_{SV} - \left(\frac{1}{S}\right)g_V\right)\end{aligned}}$$

MV Hedge Ratios in CEV Model

- SI CEV ≡ *Scale invariant* form of CEV

$$\frac{dS_t}{S_t} = r\,dt + \varphi\left(\frac{S_t}{S_0}\right)^{\beta} dW_t \qquad (\text{set } \alpha = \varphi S_0^{-\beta})$$

$$\boxed{\begin{aligned}\delta_{mv} &= g_S + g_\varphi\left(\frac{\varphi\beta}{S}\right)\\ \gamma_{mv} &= g_{SS} + \frac{\varphi\beta}{S}\left(\left(\frac{\varphi\beta}{S}\right)g_{\varphi\varphi} + 2g_{S\varphi} + \left(\frac{\beta-1}{S}\right)g_\varphi\right)\end{aligned}}$$

- NB: MV ratios of SI CEV ≡ standard ratios of CEV

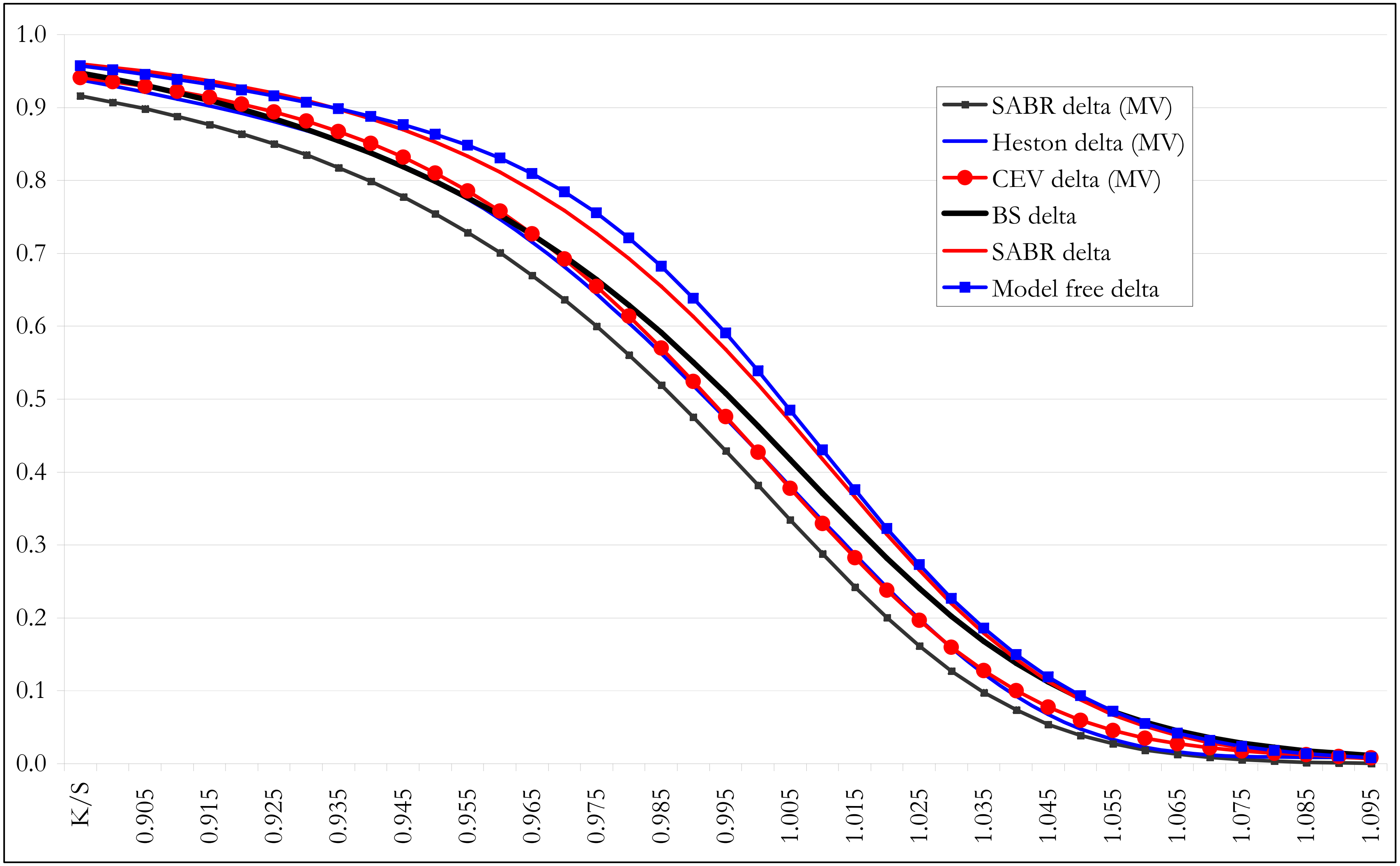
MV Hedge Ratios in SABR Model

If F_t is the forward price at time t the SABR model is:

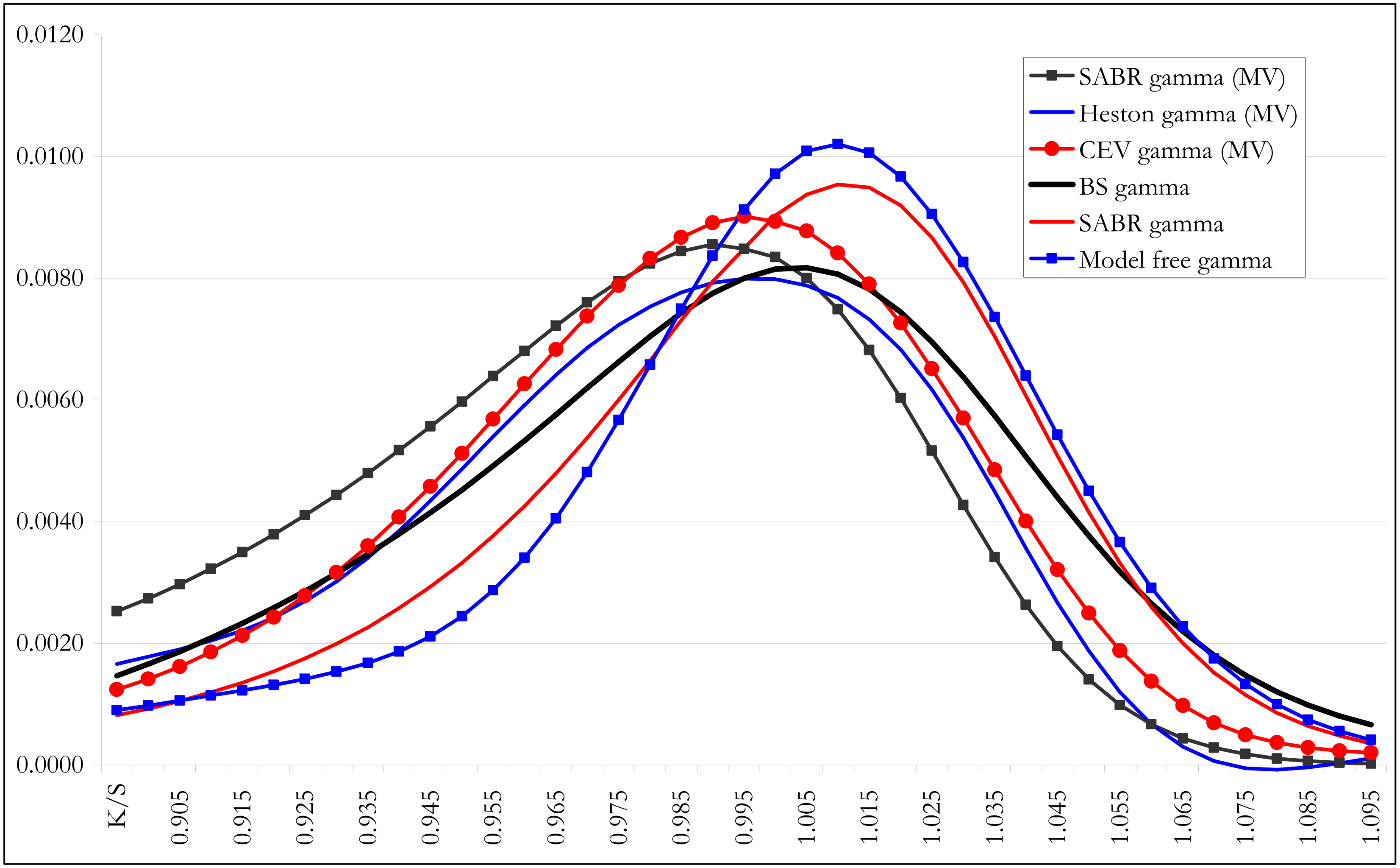
$$dF_t = \alpha_t F_t^{\beta} dW_t$$
$$d\alpha_t = \upsilon \alpha_t dZ_t$$
$$\langle dW_t, dZ_t \rangle = \varrho\, dt$$

$$\boxed{\begin{aligned} \delta_{mv} &= g_F + g_\alpha \frac{\varrho\upsilon}{F^\beta} \\ \gamma_{mv} &= g_{FF} + \frac{\varrho\upsilon}{F^\beta}\left(\left(\frac{\varrho\upsilon}{F^\beta}\right) g_{\alpha\alpha} + 2 g_{F\alpha} - \left(\frac{\beta}{F}\right) g_\alpha\right) \end{aligned}}$$

Standard *vs* MV deltas



Standard *vs* MV gammas



4. Empirical Results

Hedging Race

- *Model calibrations*

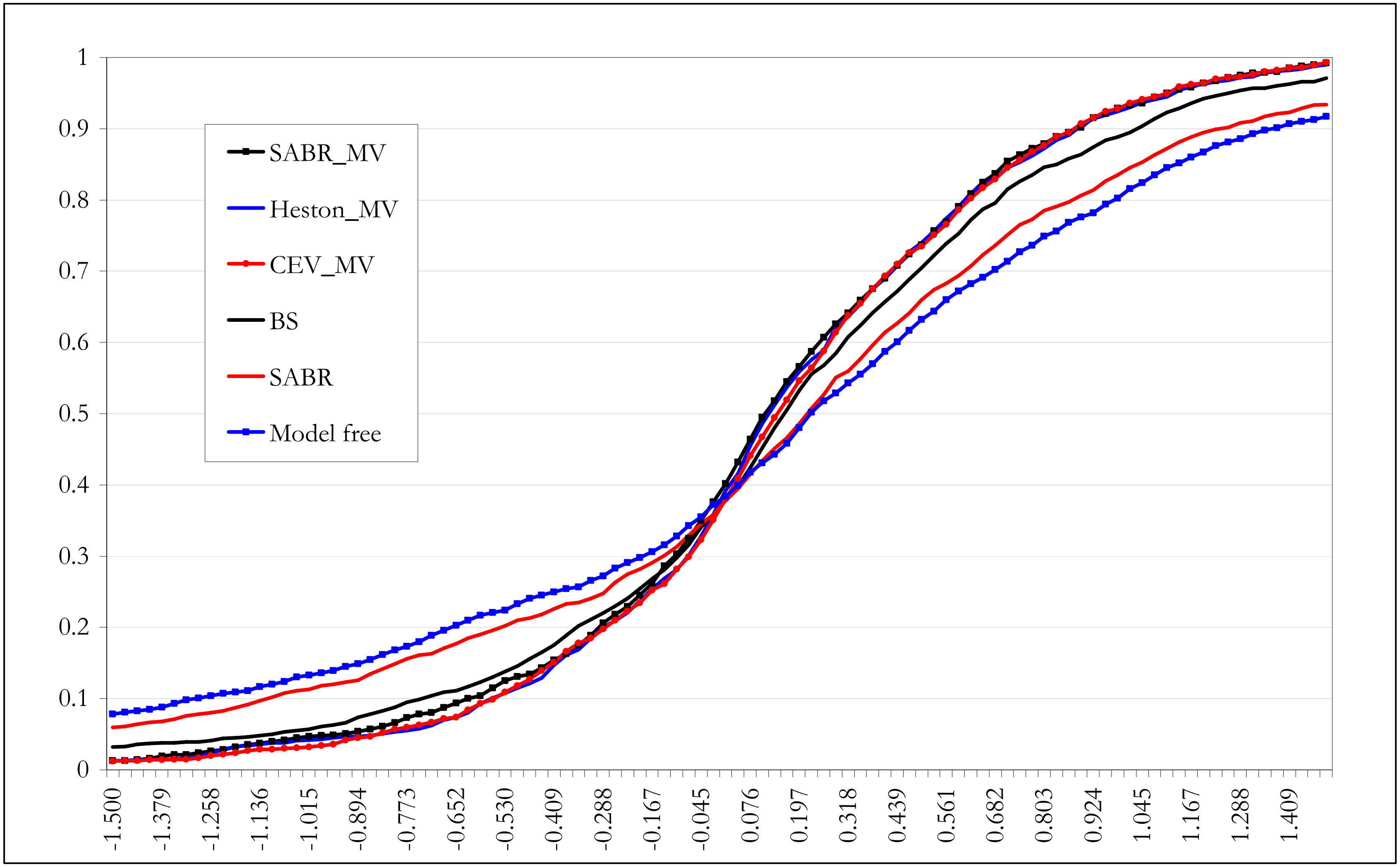
    - European options on SP500
    - Daily close prices from 02 Jan 2004 to 15 June 2004
    - Up to 34 strikes from 1005 to 1200
    - All strikes within ±10% of index level used for daily calibration

Delta-Gamma Hedging

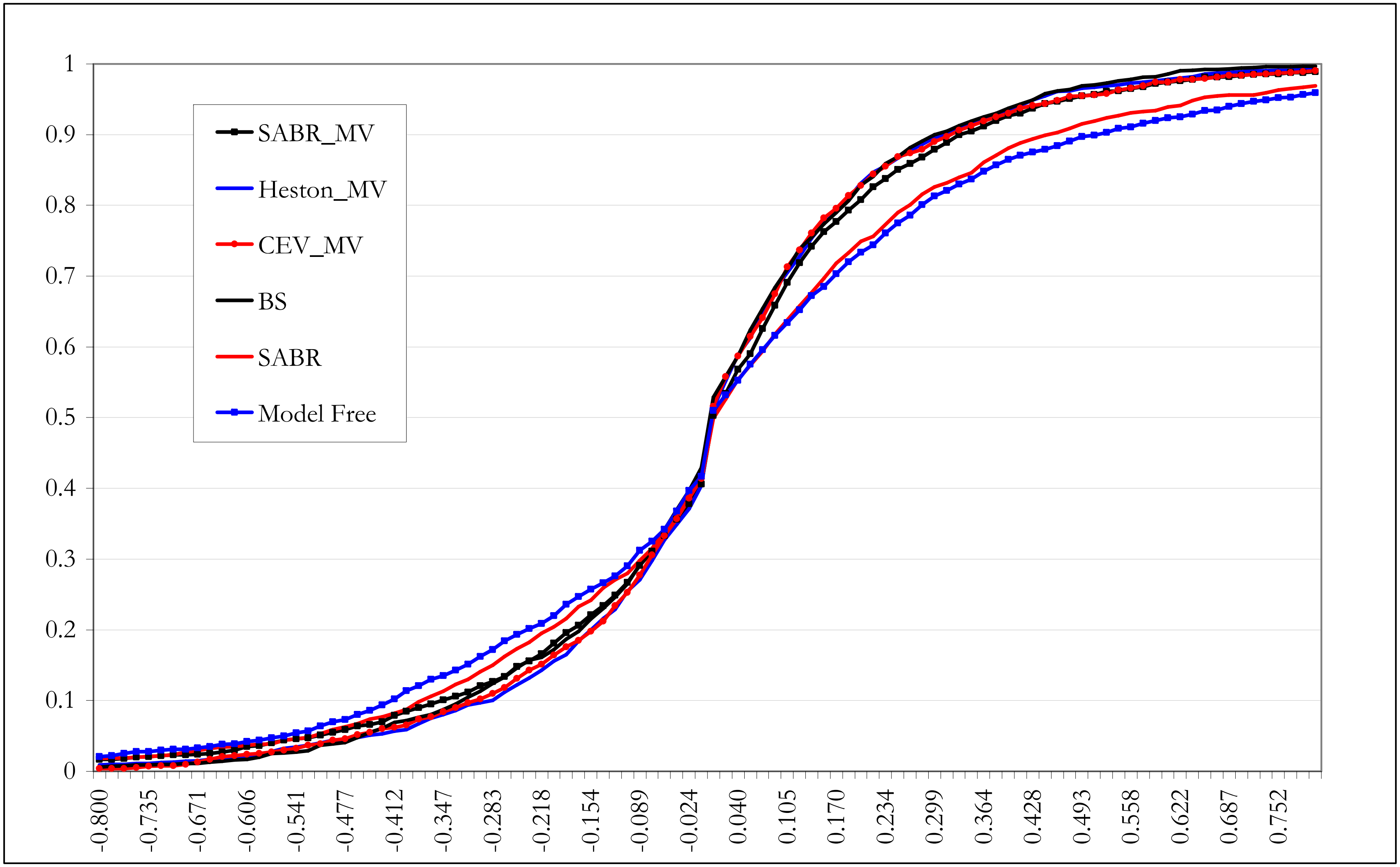
Two Hedging Strategies

- *Delta neutral*: Short one call on each strike and delta hedge with *S*
- *Delta-gamma neutral*: Buy 1125 strike to zero gamma then delta hedge
  - o Rebalance daily from 16 Jan 2004 onwards
  - o Transactions costs assumed zero

CDF of Delta Hedged P&L



CDF of Delta-Gamma Hedged P&L



Standard Deviations of Hedging P&L

Delta Hedging

K/S	0.90-0.95		0.95-1.00		1.00-1.05		1.05-1.10		1.10-1.15	
Best	SABR_MV	0.3657	CEV_MV	0.5740	CEV_MV	0.6372	CEV_MV	0.6051	Heston_MV	0.5507
	Heston_MV	0.3714	SABR_MV	0.5988	Heston_MV	0.6629	SABR_MV	0.6178	CEV_MV	0.5602
	CEV_MV	0.3854	Heston_MV	0.6161	SABR_MV	0.6729	Heston_MV	0.6202	SABR_MV	0.5673
	BS	**0.5652**	**BS**	**0.7876**	**BS**	**0.7844**	**BS**	**0.6921**	**BS**	**0.5917**
	SABR	0.6099	SABR	1.0106	SABR	1.1251	SABR	0.9301	SABR	0.7077
Worst	Model Free	0.7357	Model Free	1.2055	Model Free	1.2691	Model Free	1.0283	Model Free	0.7746
# options	141		476		435		217		55	

Delta-Gamma Hedging

K/S	0.90-0.95		0.95-1.00		1.00-1.05		1.05-1.10		1.10-1.15	
Best	Heston_MV	0.1801	CEV_MV	0.2358	**BS**	**0.2531**	Heston_MV	0.2907	Heston_MV	0.3134
	CEV_MV	0.1853	SABR_MV	0.2431	CEV_MV	0.3040	CEV_MV	0.2923	CEV_MV	0.3222
	SABR_MV	0.1984	**BS**	**0.2561**	Heston_MV	0.3132	**BS**	**0.2929**	**BS**	**0.3597**
	BS	**0.2012**	Heston_MV	0.2594	SABR_MV	0.3562	SABR_MV	0.3444	SABR_MV	0.3617
	SABR	0.2187	SABR	0.3202	SABR	0.4015	SABR	0.5022	SABR	0.4871
Worst	Model Free	0.3214	Model Free	0.3695	Model Free	0.4271	Model Free	0.5277	Model Free	0.5175
# options	141		476		435		217		55	

Summary of Results

- Delta Hedging
  - MV hedging is *best*
  - Model free is *worst*
- Delta-Gamma Hedging
  - BS is *best* for ATM options only
  - Otherwise MV hedging is *best*
  - Model free is *worst*

Conclusions

1. Most models for pricing options on tradable assets are based on scale invariant processes

2. All homogeneous claims on tradable assets have *model free* price hedge ratios!

3. In our results these hedge ratios are *bad* for hedging

4. However, if price and volatility are correlated then MV hedge ratios *differ* from standard hedge ratios

5. MV hedge ratios perform *better than BS* for delta hedging and for delta-gamma hedging OTM options

Main References & Acknowledgements

- Alexander, C. and L. Nogueira Model-Free Hedge Ratios and Scale-Invariant Models *Journal of Banking and Finance*, forthcoming

- Alexander, C. and L. Nogueira Model-Free Price Hedge Ratios for Homogeneous Claims on Tradable Assets *Quantitative Finance*, forthcoming

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