Pricing Financial Derivatives on Weather Sensitive Assets

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Outline

- We shall use temperature as a generic example of an important weather variable and assume that the asset price is a quadratic function of temperature.
- In the continuous time model, we derive a PDE of a European call price on the weather sensitive asset under the quadratic relationship between the price of the asset and the temperature. We also derive a binomial tree approximation, a finite difference method and a Monte Carlo simulation to numerically solve the derived PDE.
- In the discrete time model, we derive the distribution of the underlying asset and show a connection between the distribution and the closed formula of the PDE.

Weather Sensitive Asset

- Weather derivatives — a growing body of academic research into pricing of such contracts — Problems:
    1. the underlying asset is not tradeable, new methodology is required;
    2. good models of weather risks are needed.
- Rather than studying derivative instruments with weather as the underlying asset we propose to study *pricing of derivatives where the underlying asset is sensitive to the weather.*
- Roll (1984) showed that the temperature is the most important factor influencing price fluctuations for **frozen orange juice futures**.

Weather Variable

- As suggested by Alaton et al (2001) and Cao and Wei (2000), we consider the simplest model with the mean-reverting property, namely an AR(1) first-order autoregressive model and its continuous time counterpart, the mean-reverting Ornstein-Uhlenbeck process.
- For the present, we choose to work with raw temperature rather than a temperature index, such as mean summer or winter temperature, in an attempt to capture as much statistical information about the behavior of temperature as possible.

Models of Temperature Variability

In discrete time we suppose that the temperature at time t, T_t, satisfies the following recursive equation:

$$T_t = a_0 + a_1 T_{t-1} + \xi_t, \tag{1}$$

where T_0 is known and $\{\xi_t\}$ is a sequence of i.i.d. random variables with $\xi_t \sim N(0, \sigma^2)$, $t = 1, 2, \ldots$.

The continuous time counterpart of the above is as follows:

$$dT_t = \rho(\mu - T_t)dt + \sigma dW_t, \tag{2}$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion and $\rho > 0$ and σ are positive constants.

Model of Asset Price $S_t = T_t^2$

- For most weather-sensitive assets, extreme temperatures, for instance too low (freezing) or too high (very hot), have a negative impact on production. According to the law of supply and demand, a significant decrease in supply due to extreme weather would lead to an increase in demand and hence an increase in price.

- A quadratic form will allow us to approximate arbitrary utility functions up to at least the second order.

European Call Option PDE

In the present setting, the price $f(t, S_t)$ of a European call option with strike price K and time to maturity τ on the asset satisfies the following Partial Differential Equation (PDE):

$$\begin{aligned} & f_t + rxf_x + 2\sigma^2 x f_{xx} - rf = 0, \\ & f(\tau, x) = (x - K)^+, \; x \in \mathbb{R}, \end{aligned} \tag{3}$$

where r is a risk-free interest rate, for simplicity assumed to be a constant.

Binomial Tree Approximation

The diffusion process of the price of the underlying asset is as follows:

$$\begin{aligned} dS_t = & \quad (-2\rho S_t + 2\rho\mu\sqrt{S_t} + 2\sigma^2)dt + \quad 2\sigma\sqrt{S_t}dW_t \\ := & \quad \mu(S_t)dt + \quad \sigma(S_t)dW_t. \end{aligned} \tag{4}$$

Since the volatility of S_t is not constant but a function of S_t, in order to construct a recombining binomial tree, a transformation on the underlying process is needed:

$$X(s) = \int_0^s \frac{dz}{\sigma(z)} = \int_0^s \frac{dz}{2\sigma \times \sqrt{z}} = \frac{\sqrt{2s}}{2\sigma},$$

The volatility of the diffusion process $X(S_t)$ becomes a constant, and a recombing (simple) binomial tree for X is obtained (Nelson and Ramaswamy (1990)).

Finite Difference Method

- Divide $[0, \tau]$ into N equally spaced intervals of length $\Delta t = \tau/N$. A total of $N+1$ times are considered: $0,\ \Delta t,\ 2\Delta t,\ \ldots,\ \tau$.
- Define $\Delta S = S_{max}/M$ and consider a total of $M+1$ equally spaced stock prices: $0,\ \Delta S,\ 2\Delta S,\ \ldots,\ S_{max}$. The level S_{max} is chosen so that, when it is reached, the call will be valued $S_{max} - K$ and the current stock price is one of $0,\ \Delta S,\ 2\Delta S,\ \ldots,\ S_{max}$.
- A grid consisting of a total of $(M+1)(N+1)$ points is defined. The (i, j) point on the grid corresponding to time $i\Delta t$ and stock price $j\Delta S$.
- Denote by $f_{i,j}$ the value of the option at the (i, j) point, namely, $f_{i,j} := f(i\Delta t, j\Delta S)$.

Implicit Finite Difference Method

- Finite difference equation:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{5}$$

where,

$$a_j = \frac{rj\Delta t}{2} - \frac{2\sigma^2 j\Delta t}{\Delta S}, b_j = 1 + r\Delta t + \frac{4\sigma^2 j\Delta t}{\Delta S}, c_j = -\frac{rj\Delta t}{2} - \frac{2\sigma^2 j\Delta t}{\Delta S}.$$

- The terminal condition:

$$f_{N,j} = \max(j\Delta S - K, 0),\ j = 0, 1, \ldots, M \tag{6}$$

- Two boundary conditions:

$$f_{i,0} = 0,\ i = 0, 1, \cdots, N \tag{7}$$

$$f_{i,M} = \max(M\Delta S - K, 0),\ i = 0, 1, \cdots, N \tag{8}$$

Monte Carlo Simulation

1. Divide interval $[0, \tau]$ into N equal spaces and set $h = \frac{\tau}{N}$. For $j = 0, \ldots, N-1$ we have:

$$T_{(j+1)h} = a_0 + a_1 T_{jh} + \xi_h, \tag{9}$$

where T_0 is known, $a_0 = \mu \cdot \rho$, $a_1 = 1 - \rho$ and $\xi_h \sim N(0, (\sigma\sqrt{h})^2)$.

2. Simulate the sample path according to equation (9) M times to obtain M terminal temperatures $T_\tau^i, i = 1, \ldots, M$.

3. Calculate the price of the underlying $S_\tau^i = (T_\tau^i)^2, i = 1, \ldots, M$.

4. The price of the European call with a strike K and time to maturity τ is:

$$C = e^{-r\tau} \frac{1}{M} \sum_{i=1}^{M} (S_\tau^i - K)^+.$$

Example

- Use Adelaide daily mean temperature record from Bureau of Meteorology from 1/07/1999 to 28/06/2002, which gives us $T_0 = 10.94C^\circ$.
- The price per share today of the underlying weather-sensitive asset is then $S_0 = \$119.63$.
- For the European call option considered, the time to expiration is one month ($\tau = 1/12$) and the strike price $K = S_0 = \$119.63$ so that the call is *at-the-money*.
- The risk-free annual interest rate is taken to be $r = 0.05$.

Volatility Estimation

- Fit a linear regression model $\hat{T}_t = \hat{a}_0 + \hat{a}_1 T_{t-1}$ based on the data one month earlier than when the contract starts, namely from 29/05/2002 to 28/06/2002.
- The estimates of the intercept and slope are $\hat{a}_0 = 6.29$ and $\hat{a}_1 = 0.45$ respectively.
- Calculate the sample standard deviation of the residuals $(T_t - \hat{T}_t)$, which gives a historical daily volatility estimate $\sigma = 1.44$.
- We use an an n-step binomial tree with $n = 3$, Finite Difference Method and Monte Carlo Simulation to calculate the call option price.

			135.897 (16.268)
		130.359 (1.896)	
	124.937 (6.880)		124.937 (5.307)
119.629 (4.179)		119.629 (2.705)	
	114.437 (1.377)		114.437 (0)
		109.359 (0)	
			104.397 (0)

Table 1: Results comparison

Method	Parameters	Call Price
BTA	$n = 400$	3.851
IFDM	$M = 200, N = 200, S_{max} = 146.79$	3.846
EFDM	$M = 200, N = 200, S_{max} = 146.79$	3.850
MC	$M = 10,000, N = 100$	3.818
Theoretical	Remark 1	3.874

Time Series Model of Asset Price

In discrete time, the temperature follows an AR(1) process

$$T_t = a_0 + a_1 T_{t-1} + \xi_t, \tag{10}$$

where T_0 is known and $\{\xi_t\}$ is a sequence of i.i.d. random variables with $\xi_t \sim N(0, \sigma^2)$, $t = 1, 2, \ldots$.

A quadratic form is assumed, namely

$$S_t = T_t^2, \; t = 0, 1, \ldots. \tag{11}$$

Distribution of the Weather-Sensitive Asset Price

Lemma 1 *For each $t \geq 1$, T_t follows a normal distribution with mean μ_t and standard deviation σ_t, where*

$$\mu_t = a_1^t T_0 + c(t), \quad \sigma_t^2 = \sigma^2 \sum_{i=0}^{t-1} a_1^{2i} = \sigma^2 \frac{1 - a_1^{2t}}{1 - a_1^2},$$

while S_t/σ_t^2 has noncentral χ^2 distribution with 1 degree of freedom and the non-centrality parameter

$$q_t = \frac{\mu_t^2}{\sigma_t^2}. \tag{12}$$

The mean and variance of S_t are as follows:

$$E(S_t) = \sigma_t^2(1+q_t) = \sigma_t^2+\mu_t^2, \quad Var(S_t) = 2(1+2q_t)\sigma_t^4 = 2\sigma_t^2(\sigma_t^2+2\mu_t^2).$$

Remark 1 *The non-central χ^2 distribution gives a closed form solution of the derived PDE:*

$$f(t,S) = Se^{-q(\tau-t)}[1-\chi^2(a;b+2,c)] - Ke^{-r(\tau-t)}\chi^2(c;b,a) \qquad (13)$$

where

$$a = \frac{4Ke^{-(r-q)(\tau-t)}}{v},\ b = 2,\ c = \frac{4S}{v},\ \textit{and}\ v = \frac{4\sigma^2}{r-q}[1-e^{-(r-q)(\tau-t)}].$$

where $\chi^2(z;k,v)$ is the cumulative probability that a variable with a noncentral χ^2 distribution with non-centrality parameter v and k degrees of freedom is less than z.