Remarks on the Dynamic Capital Asset Pricing Model in discrete time

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Introduction

The Capital Asset Pricing Model is a major theory in modern Finance. The intertemporal capital asset pricing model has been introduced by R. MERTON [3]. The concept of price of state of nature is a concept of stochastic economics introduced by K. ARROW and G. DEBREU [1]. A related concept in Finance is that of risk-neutral probability. When there are only a finite number of states of nature and when the assumption of complete markets is valid, these prices play a significant role in obtaining optimal consumption and portfolio investment policies in a general set up. This fact is well known for a single period problem.

In a dynamic set up, the theory has been developed in continuous time, using properties of martingale generated by Wiener processes, see I. KARATZAS and S.E. SHREVE [2]. In discrete time, although natural, the corresponding treatment is not widely available in the literature, see however S.E. SHREVE [4] for a treatment in the binomial model. We present here a detailed theory of what can be done in discrete time.

Static case

We consider an economy with $n-1$ assets and cash. There is a single period. We denote by Y^0 the $n-1$ dimensional vector of prices of assets at the beginning of the period. We consider a probabilistic framework Ω, P with only a finite number of events $\omega_1, \cdots, \omega_n$ and corresponding probabilities $p_1, \cdots, p_n$. We denote by $Y(\omega)$ the prices of the assets at the end of the period. The price of cash evolves deterministically, from 1 at the beginning of the period to R at the end, whatever be the outcome of randomness.

Prices of nature

Each of the events ω_j has a price $\psi(\omega_j)$. We can use these prices at the beginning of the period to buy all possible outcomes for the assets and the cash at the end of the period. Economic equilibrium implies that we can recover in this way the prices at the beginning of the period. So we must have the equilibrium equations

$$\sum_{j=1}^{n} \psi(\omega_j) Y(\omega_j) = Y^0$$

$$R \sum_{j=1}^{n} \psi(\omega_j) = 1$$

Risk Neutral probability

Considering the preceding system as a set of equations determining the prices of nature, the assumption of complete markets tells that there exists a unique positive solution.Defining simply

$$\hat{p}_j = R\psi(\omega_j)$$

we obtain a probability with the property

$$\hat{E}Y = RY^0$$

and the value at 0 is simply the discounted expected value of the random value at the end of the period, with this probability called the
risk neutral probability.

Radon Nikodym

We define the Radon Nikodym derivative $Z(\omega)$ as

$$Z(\omega_j) = \frac{\hat{p}_j}{p_j}$$

Binomial model

For $n = 2$ there is only one asset. it is convenient to write

$$Y(\omega_1) = uY^0, \;\; Y(\omega_2) = dY^0, \; u \geq d$$

The assumption of complete markets is equivalent to

$$u > R > d$$

and we have

$$\hat{p}_1 = \frac{R - d}{u - d}, \;\; \hat{p}_2 = \frac{u - R}{u - d}$$

Replication portfolio

Any scalar random variable $V(\omega)$ such that $\hat{E}V = 0$ can be represented as

$$V(\omega) = \pi.(Y(\omega) - RY^0)$$

where π is a $n-1$ dimensional deterministic vector, called the replicating portfolio. In the binomial model, π is scalar which can be expressed as

$$\pi = \frac{V(\omega_1) - V(\omega_2)}{Y(\omega_1) - Y(\omega_2)}.$$

Optimal Consumption and Portfolio Problem

An individual having a wealth X^0 at the beginning of the period will decide a consumption and investment policy. The latter is composed of a portfolio made of a vector π of risky assets and π_f of cash.So we have the budget equation

$$X^0 = C + \pi . Y^0 + \pi_f$$

It becomes at the end of the period

$$X(\omega) = \pi.Y(\omega) + \pi_f R$$

He must decide according to the objective

$$\max_{\{C,\pi\}}[U_1(C) + \frac{EU_2(X)}{R}]$$

where U_1, U_2 are concave increasing utility functions.

This problem can be solved very simply using the Radon-Nikodym derivative $Z(\omega)$. The budget equation can be written as

$$X^0 = E(X\frac{Z}{R}) + C$$

The idea is to find an optimal $\hat{C}, \hat{X}(\omega)$ instead of an optimal pair $\hat{C}, \hat{\pi}$.

From Lagrange multiplier theory, there is a number λ such that

$$U_1'(\hat{C}) = \lambda, \;\; U_2'(\hat{X}) = \lambda Z$$

inverting these relations we get

$$\hat{C} = I_1(\lambda), \;\; \hat{X} = I_2(\lambda Z)$$

and λ is defined by the budget equation

$$X^0 = E[I_2(\lambda Z)\frac{Z}{R}] + I_1(\lambda)$$

Having defined $\hat{\lambda}$, $\hat{C}$, $\hat{X}$, we define

$$\hat{V} = \hat{X} - R(X^0 - \hat{C})$$

for which there exists a replicating portfolio

$$\hat{V} = \hat{\pi}.(Y - RY^0)$$

which is the optimal investment portfolio.

This framework extends to dynamic problems.

A Probability set-up

The time is described by $0, 1 \cdots, T$. We consider a probability space Ω made of a finite number of single events

$$\omega = (\omega^1_{j_1}, \cdots, \omega^T_{j_T})$$

where

$$j_1, \cdots, j_T \in \{1, \cdots, n\}.$$

We interpret $\omega^t_{j_t}, j_t \in \{1, \cdots, n\}$ as the states of nature which can occur at time t. There are n new possibilities at each time. Globally there are n^T single events ω. To each single event is attached a probability

$$p_{j_1, \cdots, j_T} = \text{Prob}\{(\omega^1_{j_1}, \cdots, \omega^T_{j_T})\}.$$

We assume

$$p_{j_1,\cdots,j_T} > 0, \forall j_1,\cdots,j_T \in \{1,\cdots,n\} \qquad (1)$$

We naturally have

$$\sum_{j_1,\cdots,j_T=1}^{n} p_{j_1,\cdots,j_T} = 1.$$

This probability distribution on Ω is denoted by $P = P^T$. We introduce next

$$\tilde{\omega}^t = (\omega^1_{j_1},\cdots,\omega^t_{j_t}) = \bigcup_{j_{t+1},\cdots,j_T=1}^{n} (\omega^1_{j_1},\cdots,\omega^T_{j_T})$$

Define next

$\mathcal{F}^t = \sigma -$ algebra on Ωgenerated by unions of sets $\tilde{\omega}^t$.

Events belonging to $\mathcal{F}^t$ are the only ones which can be observed at time t. Clearly $\mathcal{F}^t$ is generated by the coordinates

$$\omega \longrightarrow \omega^s_{j_s}, s = 1 \cdots t, j_s = 1, \cdots n.$$

We note that $\mathcal{F}^T$ is simply the σ-algebra made of all the subsets of Ω and $\mathcal{F}^0 = \{\Omega, \emptyset\}$.

Define the numbers

$$p_{j_1,\cdots,j_t} = \sum_{j_{t+1},\cdots,j_T=1}^{n} p_{j_1,\cdots,j_t,j_{t+1},\cdots,j_T}$$

which form a probability distribution on $(\Omega, \mathcal{F}^t)$. We denote this probability by P^t. It coincides with P on the σ-algebra $\mathcal{F}^T$. These concepts reduce for $t = 0$ to a single event

$$\tilde{\omega}^0 = \Omega; \quad p_0 = 1.$$

We can compute the conditional probability

$$\mathrm{Prob}((\omega^1_{j_1},\cdots,\omega^{t+1}_{j_{t+1}})|(\omega^1_{j_1},\cdots,\omega^t_{j_t}))=$$
$$=\theta(\omega^{t+1}_{j_{t+1}}|\omega^1_{j_1},\cdots,\omega^t_{j_t})=\frac{p_{j_1,\cdots,j_{t+1}}}{p_{j_1,\cdots,j_t}}$$

Obviously

$$\sum_{j_{t+1}=1}^{n}\theta(\omega^{t+1}_{j_{t+1}}|\omega^1_{j_1},\cdots,\omega^t_{j_t})=1,\forall\omega^1_{j_1},\cdots,\omega^t_{j_t}.$$

These conditional probabilities generate the full probabilistic set-up by induction

Binomial Model

As a particular case, we consider the binomial model developed in S. SHREVE [4]. We have $n = 2$ and we assume

$$\theta(\omega_1^{t+1}|\omega_{j_1}^1, \cdots, \omega_{j_t}^t) = p$$

$$\theta(\omega_2^{t+1}|\omega_{j_1}^1, \cdots, \omega_{j_t}^t) = q = 1 - p$$

for all values $j_1, \cdots, j_t \in \{1, 2\}$. Therefore

$$p_{j_1, \cdots, j_t} = p^{\#1(j_1, \cdots, j_t)} q^{\#2(j_1, \cdots, j_t)}$$

where

$$j_1, \cdots, j_t \in \{1, 2\}$$

$$\begin{aligned} \#1(j_1, \cdots, j_t) &= \text{ number of 1 in } (j_1, \cdots, j_t) \\ \#2(j_1, \cdots, j_t) &= \text{ number of 2 in } (j_1, \cdots, j_t). \end{aligned}$$

Prices of States of nature

The market is made of $n-1$ assets, whose prices at time t are represented by

$$Y(t;\omega) = Y(t;\omega^1_{j_1},\cdots,\omega^t_{j_t}), \mathcal{F}^t \text{ measurable }.$$

The price of cash at time t is simply R^t. The price of cash at time 0 is 1. At time 0 the prices of assets $Y(0)$ are deterministic. We make an assumption of complete markets.

For each t, there exists a unique positive random variable

$$\psi(t;\omega) = \psi(t;\omega^1_{j_1},\cdots,\omega^t_{j_t})$$

such that

$$\sum_{j_{t+1}=1}^{n} \psi(t+1;\omega^1_{j_1},\cdots,\omega^t_{j_t},\omega^{t+1}_{j_{t+1}})Y(t+1;\omega^1_{j_1},\cdots,\omega^{t+1}_{j_{t+1}}) =$$
$$= Y(t;\omega^1_{j_1},\cdots,\omega^t_{j_t}), \forall \omega^1_{j_1},\cdots,\omega^t_{j_t}$$

$$\sum_{j_{t+1}=1}^{n} \psi(t+1;\omega^1_{j_1},\cdots,\omega^{t+1}_{j_{t+1}}) = \frac{1}{R} \tag{2}$$

We call $\psi(t;\omega)$ the *prices of the states of nature* at time t. They are $\mathcal{F}^t$ measurable real random variables. These prices correspond to the states of nature

$$\omega^t_{j_t}, \quad j_t \in \{1, \cdots, n\}$$

Consider the binomial model. There is a single asset for which we assume the following evolution of prices

$$Y(t+1;\omega_{j_1}^1,\cdots,\omega_{j_t}^t,\omega_1^{t+1}) = uY(t;\omega_{j_1}^1,\cdots,\omega_{j_t}^t)$$

$$Y(t+1;\omega_{j_1}^1,\cdots,\omega_{j_t}^t,\omega_2^{t+1}) = dY(t;\omega_{j_1}^1,\cdots,\omega_{j_t}^t)$$

with

$$d < R < u.$$

The prices of the states of nature are given by

$$
\begin{aligned}
\psi(t+1;\omega_{j_1}^1,\cdots,\omega_{j_t}^t,\omega_1^{t+1}) &= \frac{R-d}{R(u-d)} = \frac{\tilde{p}}{R} \\
\psi(t+1;\omega_{j_1}^1,\cdots,\omega_{j_t}^t,\omega_2^{t+1}) &= \frac{u-R}{R(u-d)} = \frac{\tilde{q}}{R}
\end{aligned}
\qquad (3)
$$

Risk-Neutral Probability

We define next a new probability on $\Omega, \mathcal{F}^T$ by

$$\hat{p}_{j_1,\cdots,j_T} = R^T \prod_{t=1}^{T} \psi(t; \omega_{j_1}^1, \cdots, \omega_{j_t}^t) \tag{4}$$

We call this new probability $\hat{P}$.

We associate probabilities on $\Omega, \mathcal{F}^t$, called $\hat{P}^t$ defined by

$$\hat{p}_{j_1,\cdots,j_t} = R^t \prod_{s=1}^{t} \psi(s; \omega^1_{j_1}, \cdots, \omega^s_{j_s}) \tag{5}$$

Clearly

$$\sum_{j_{t+1}=1}^{n} \hat{p}_{j_1,\cdots,j_{t+1}} = \hat{p}_{j_1,\cdots,j_t}$$

As P^t for P, $\hat{P}^t$ and $\hat{P}$ coincide on the σ-algebra $\mathcal{F}^t$.

Finally we define the process

$$Z(t) = Z(t;\omega) = Z(t;\omega^1_{j_1},\cdots,\omega^t_{j_t}) = \frac{\hat{p}_{j_1,\cdots,j_t}}{p_{j_1,\cdots,j_t}} \qquad (6)$$

We set $Z(0) = 1$. The process $Z(t)$ can be viewed as the Radon Nikodym derivative

$$\frac{d\hat{P}}{dP}|_{\mathcal{F}^t} = Z(t).$$

By construction $Z(t)$ is a $P, \mathcal{F}^t$ martingale.

Martingale property

We can assert

PROPOSITION 1. *The following relation holds*

$$\widehat{E}[Y(t+1)|\mathcal{F}^t] = RY(t)$$

which can be expressed as: The process

$$\frac{Y(t)}{R^t} \text{ is a } \hat{P}, \mathcal{F}^t \text{ martingale}$$

The proof is an easy consequence of the definitions of $\hat{P}$ and the prices of states of nature $\psi(t;\omega)$. The probability $\hat{P}$ is called the risk-neutral probability. So, under the risk-neutral probability the discounted asset prices process is a martingale.

Replication Portfolio

Suppose we have a process scalar $V(t)$ which adapted to the filtration $\mathcal{F}^t$. So

$$V(t) = V(t;\omega) = V(t;\omega^1_{j_1},\cdots,\omega^t_{j_t}).$$

In addition we assume

$$\widehat{E}[V(t+1)|\mathcal{F}^t] = 0 \tag{7}$$

A **replication portfolio** is a $n-1$ dimensional process $\pi(t)$ which is adapted to the filtration $\mathcal{F}^t$ and satisfies

$$V(t+1) = \pi(t).(Y(t+1) - RY(t)),\ \forall t = 0,\cdots,T-1 \tag{8}$$

We have the
PROPOSITION 2. *We assume market completeness.For any process* $V(t)$ *such that 7 is satisfied, there exists a unique replication portfolio.*

In the case of the binomial model we have the formula

$$\pi(t) = \frac{V(t+1;2) - V(t+1;1)}{Y(t+1;2) - Y(t+1;1)}.$$

Optimal Portfolio and Consumption Problem

We define two Utility functions $U_1(x), U_2(x)$ satisfying

$$\begin{array}{l} U_1, U_2 \text{ nondecreasing and concave.} \\ U_i'(0) = \infty; \quad U_i'(\infty) = 0 \end{array} \tag{9}$$

These functions will be differentiable in $(0, \infty)$.

A consumption process $C(t)$ is a positive $\mathcal{F}^t$ adapted stochastic process. Similarly a portfolio $\pi(t)$ is a $\mathcal{F}^t$ adapted stochastic process with values in R^{n-1}. There is also a $\mathcal{F}^t$ adapted real stochastic process representing the amount of cash at time t. We call it $\pi_f(t)$. This process will in fact cancel out from budget considerations. There is no outside income flux in this model.

Wealth Process

The wealth process $X(t)$ satisfies the relations

$$X(t+1) = \pi(t).Y(t+1) + \pi_f(t)R^{t+1}$$

$$X(t) = C(t) + \pi(t).Y(t) + \pi_f(t)R^{t}$$

with a given initial wealth $X(0)$ which is a given deterministic number. The first one tells that the wealth at time $t+1$ corresponds to the portfolio decided at time t and the amount of cash available at t, with changes of values coming from the market. The second relation is the consequence of the decisions taken at time t. One allocates the wealth which is available between consumption expenses and investment decisions.

We can next eliminate $\pi_f(t)$ and we obtain the evolution relation

$$X(t+1) = R(X(t)-C(t))+\pi(t).(Y(t+1)-RY(t)). \tag{10}$$

The problem is stated as follows: Maximize

$$J_{X(0)}(C(.),\pi(.)) = E[\sum_{t=0}^{T-1} \frac{U_1(C(t))}{R^t} + \frac{U_2(X(T))}{R^T}] \tag{11}$$

Martingale Considerations

Recall the definition of $Z(t)$ by equation (6). We introduce the process

$$M(t) = \frac{X(t)Z(t)}{R^t} + \sum_{s=0}^{t-1} C(s)\frac{Z(s)}{R^s} \qquad (12)$$

We have

$$M(0) = X(0).$$

We have the Proposition

PROPOSITION 3. *The process* $M(t)$ *is a* $P, \mathcal{F}^t$ *martingale.*

From the martingale property we derive

$$E[\frac{X(T)Z(T)}{R^T} + \sum_{t=0}^{T-1} C(t)\frac{Z(t)}{R^t}] = X(0) \qquad (13)$$

Optimality Conditions

From equation (13) we can consider the optimization problem with constraints

Maximize

$$E[\sum_{t=0}^{T-1} \frac{U_1(C(t))}{R^t} + \frac{U_2(X(T))}{R^T}] \tag{14}$$

with the constraint

$$E[\frac{X(T)Z(T)}{R^T} + \sum_{t=0}^{T-1} C(t)\frac{Z(t)}{R^t}] = X(0)$$

If we consider in the problem (14) $X(T)$ and the consumptions $C(t)$ as the variables to be optimized and as unrelated quantities, we get a larger maximum, since we do not pay attention to the relation describing the wealth evolution, see (10). This problem is a standard maximization problem with concave payoff and linear constraint. It is easily solved by introducing a Lagrange multiplier.

We deduce easily

$$\begin{aligned} U_2'(\hat{X}(T)) &= \lambda Z(T) \\ U_1'(\hat{C}(t)) &= \lambda Z(t) \end{aligned} \tag{15}$$

Define I_1, I_2 to be the inverses of U_1', U_2' respectively. They are decreasing functions, since U_1, U_2 are concave. The relations (15) imply

$$\begin{aligned} \hat{X}(T) &= I_2(\lambda Z(T)) \\ \hat{C}(t) &= I_1(\lambda Z(t)) \end{aligned} \tag{16}$$

To get the value of λ we shall use the constraint.

$$E[I_2(\lambda Z(T))\frac{Z(T)}{R^T} + \sum_{t=0}^{T-1} I_1(\lambda Z(t))\frac{Z(t)}{R^t}] = X(0) \tag{17}$$

From the assumptions on $U_i'(0), U_i'(\infty)$, we can assert that the left-hand side of equation (17) is a monotone decreasing function of λ , is equal to ∞ when $\lambda = 0$ and to 0 when $\lambda = \infty$. Therefore the equation has a unique solution $\hat{\lambda}$.

Solution

The consumption $\hat{C}(t)$ computed by the second equation (16), with $\lambda = \hat{\lambda}$ will be optimal if one can find a portfolio $\hat{\pi}(t)$ which achieves the wealth $\hat{X}(T)$ at time t. Knowing $\hat{M}(T)$ we compute $\hat{M}(t)$ by the martingale property

$$\hat{M}(t) = E[\hat{M}(T)|\mathcal{F}^t]$$

This relation implies

$$\hat{M}(t;\omega_{j_1}^1,\cdots,\omega_{j_t}^t) = \sum_{j_{t+1}=1}^{n} \hat{M}(t+1;\omega_{j_1}^1,\cdots,\omega_{j_t}^t,\omega_{j_{t+1}}^{t+1})\theta(\omega_{j_{t+1}}^{t+1}|\omega_{j_1}^1,\cdots,\omega_{j_t}^t) \tag{18}$$

Once we know $\hat{M}(t)$ and the optimal consumptions $\hat{C}(t)$ we can compute the expression of $\hat{X}(t)$, the optimal wealth process. We then define

$$\hat{V}(t+1) = \hat{X}(t+1) - R(\hat{X}(t) - \hat{C}(t)).$$

To find the optimal portfolio we then solve the replication equations (8)

$$\hat{V}(t+1) = \hat{\pi}(t).(Y(t+1) - RY(t))$$

which has a unique solution from proposition 2.

Dynamic Programming

We introduce the family of problems

$$J_{x,t}(C(.),\pi(.)) = E[\sum_{s=t}^{T-1} \frac{U_1(C(s))}{R^{s-t}} + \\ + \frac{U_2(X(T))}{R^{T-t}} | X(t) = x, \mathcal{F}^t] \tag{19}$$

This payoff depends only on $C(t), \pi(t); \cdots ; C(T-1), \pi(T-1)$ and of the events $\omega_{j_1}^1, \cdots, \omega_{j_t}^t$.

Our original problem corresponds to the case $X(0) = x$, and $Y(0)$ is a deterministic quantity. We are interested in the random function

$$W(x,t) = \max J_{x,t}(C(.), \pi(.)) \tag{20}$$

Note that $W(x,t)$ is $\mathcal{F}^t$ measurable.

Bellman Equation

It is convenient to denote by $X_{x\,t}(s), s \geq t$ the evolution of the wealth process, for a given initial wealth at time t equal to x. An optimal policy of consumption and investment depends also on x, t and is denoted by

$$\hat{C}_{x\,t}(s); \quad \hat{\pi}_{x\,t}(s)$$

and the corresponding wealth process is denoted by $\hat{X}_{x\,t}(s)$. Bellman equation writes

$$\begin{aligned} W(x,t) &= \max_{C(t),\pi(t)} \{U_1(C(t)) + \\ &\frac{1}{R} E[W(X_{x\,t}(t+1), t+1) | \mathcal{F}^t]\} \qquad (21) \\ W(x,T) &= U_2(x) \end{aligned}$$

Derivative of the Value function

The derivative $W_x(x,t)$ with respect to x will have an interesting interpretation, reminiscent of the Lagrange Multiplier .We first differentiate equation (21) to obtain

$$W_x(x,t) = E[\hat{W}_x(X_{x\,t}(t+1), t+1)|\mathcal{F}^t] \tag{22}$$

Next we recall that the wealth process satisfies a martingale property as follows

$$E[X_{x\,t}(t+1)\frac{Z(t+1)}{R}|\mathcal{F}^t] + Z(t)C(t) = xZ(t) \tag{23}$$

This constraint holds for any pair $C(t), \pi(t)$.

We look for optimal $\hat{C}_{x\,t}(t), \hat{X}_{x\,t}(t+1)$ in the right hand side of Bellman equation by minimizing

$$U_1(C(t)) + \frac{1}{R} E[W(X_{x\,t}(t+1), t+1)|\mathcal{F}^t]$$

under the constraint (23). Therefore we introduce a Lagrange multiplier $\lambda_{x\,t}$ which is $\mathcal{F}^t$ measurable and such that the optimum is obtained by minimizing

$$\frac{1}{R} E[W(X_{x\,t}(t+1), t+1) - \lambda_{x\,t} X_{x\,t}(t+1) Z(t+1)|\mathcal{F}^t] \\ + U_1(C(t)) - \lambda_{x\,t} Z(t) C(t)$$

in $C(t)$, $X_{x\,t}(t+1)$.

The optimal $\pi(t)$ is obtained afterwards using the replication equations (8). We thus write the conditions

$$\begin{aligned} U_1'(\hat{C}_{x\,t}(t)) &= \lambda_{x\,t} Z(t) \\ W_x(\hat{X}_{x\,t}(t+1), t+1) &= \lambda_{x\,t} Z(t+1) \end{aligned} \tag{24}$$

Taking into account (22) we get also

$$W_x(x,t) = Z(t)\lambda_{x\,t} \tag{25}$$

which connects the Lagrange multiplier to the gradient in x of the value function.

Obtaining the Derivative of the Value function

We can get an explicit formula for $W_x(x,t)$. We first prove the following result

PROPOSITION 4. *The following relations hold*

$$U_1'(\hat{C}_{x\,t}(s)) = \frac{Z(s)W_x(x,t)}{Z(t)}, \forall s \geq t$$

$$U_2'(\hat{X}_{x\,t}(T)) = \frac{Z(T)W_x(x,t)}{Z(t)} \tag{26}$$

From relations (26) we deduce

$$\begin{aligned} \hat{C}_{x\,t}(s) &= I_1\left(\frac{Z(s)W_x(x,t)}{Z(t)}\right), \forall s \geq t \\ \hat{X}_{x\,t}(T) &= I_2\left(\frac{Z(T)W_x(x,t)}{Z(t)}\right) \end{aligned} \tag{27}$$

We finally write the martingale property

$$E[I_2\left(\frac{Z(T)W_x(x,t)}{Z(t)}\right)\frac{Z(T)}{Z(t)R^{T-t}}+ \\ \sum_{s=t}^{T-1} I_1\left(\frac{Z(s)W_x(x,t)}{Z(t)}\right)\frac{Z(s)}{Z(t)R^{s-t}}|\mathcal{F}^t]=x. \tag{28}$$

This equation defines uniquely the random variable $\mathcal{F}^t$ measurable $W_x(x,t)$. We can complete the definition of the optimal feedback

$$\hat{C}_{x\,t}(t)=I_1(W_x(x,t))$$

and $\hat{\pi}_{x\,t}(t)$ by using the replication formulas.

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