Credit contagion models with interacting intensities

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Default correlation – occurrence of multiple defaults in a portfolio of risky assets

Pricing of correlation-dependent products, like CDO tranches, requires modeling not only the credit process for each individual obligor but also the correlation of the credit processes.

Empirical facts

- Default of one firm may trigger the default of other related firms.
- Default times tend to concentrate in certain periods of time (clusters of default).

Sources of dependence between defaults

- common macro-economic factors and sectors
  - correlation of the individual credit quality process to the economic cycle and sector indices
- default contagion
  - (i) direct economic links between firms
  - (ii) information effect e.g. accounting scandal of WorldCom

Three approaches

- *Copula approach*

  The approach takes the marginal default probabilities of the different obligors as inputs and plugs them into a copula function. This provides the model with the default dependence structure and from which joint distribution of default probabilities can be derived.

- *Conditionally independent defaults*

  This approach introduces credit risk dependence through the dependence of the obligors' default intensities on a common set of state variables. The default rates are independent once we fix the realization of state variables.

- Contagion models

  (i) Infectious defaults (Davis and Lo, 2001)

  Once a firm defaults, it may bring down other firms with it.

  Z_i be the default indicator of Firm i

  Y_{ij} be an "infection" variable,which equals 1 when the default of Firm i triggers the default of Firm j.

$$Z_i = \underset{\substack{\uparrow \\ \text{direct default} \\ \text{of firm } i}}{X_i} + (1 - X_i)\left[1 - \prod_{j\neq i}\underset{\substack{\uparrow \\ \text{default of firm } j \\ \text{triggers default of firm } i}}{(1 - X_jY_{ji})}\right]$$

  (ii) Propensity model

  Upward jumps in the default intensity of non-defaulted firms at the default time of one of the default-correlated firms.

Default contagion with interacting intensities

Consider a portfolio of N firms, a random default time τ_i is associated with each firm. Define $H_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$. The default status of the assets in the portfolio is given by the default indicator process

$$\boldsymbol{H}_t = (H_t^1 \quad H_t^2 \cdots H_t^N) \in \{0, 1\}^N = S.$$

Jarrow and Yu (2001) characterize the default intensity of firm i by

$$\lambda_i(\boldsymbol{H}_t) = \lambda_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{N} \lambda_{ik} \mathbf{1}_{\{\tau_k \leq t\}},$$

where λ_{ij} are constants. The distributions of the default times are defined recursively through each other – looping default phenomenon.

The impact of defaults on the default intensities of surviving firms is exogenously specified, while the joint distribution of default times is then exogenously derived.

Various approaches of deriving joint probabilities of defaults

1. Total hazard construction (Yu, 2004)

   Define

$$\Lambda_i(t|m) = \int_{\widehat{\tau}_m}^{\widehat{\tau}_m + t} \lambda^i(u|m)\, du$$

   which is the total hazard accumulated by firm i between the m^{th} default and t, assuming no observed default between τ_m and t.

   The total hazards accumulated by the firms until they defaulted are iid unit exponential random variables. One can map a set of iid unit exponentials back to the original default stopping times

$$\Lambda_i^{-1}(x|n) = \inf\{t : \Lambda_i(t|n) \geq x\}, \quad x \geq 0.$$

   A simulation procedure can be constructed for defining a collection of default times based on unit exponentials.

2. Change of measure (Collins-Dufresne, Goldstein and Huggonier, 2002)

   Define a firm-specific probability measure P^i which puts zero probability on the paths where default occurs prior to maturity T.

   $$Z_T = \left.\frac{dP^i}{dP}\right|_{\mathcal{F}_T} = \mathbf{1}_{\{\tau^i > T\}} \exp\left(\int_0^T \lambda^i(s)\,ds\right).$$

   $\{\tau^i \leq T\}$ is the null set of the probability measure P^i. The intensity of the other Firm j is almost surely constant under this probability measure.

   This avoids the "looping default" phenomenon. Under the new measure, the impact of one party's default on the intensity of the other party can be "effectively" neglected.

Markovian framework (Frey and Backhaus, 2004)

The macroeconomic variables are described by a d-dimensional stochastic process

$$\mathbf{\Psi} = (\psi_t)_{t \in [0,T]}.$$

The overall state of the system is

$$\mathbf{\Gamma}_t = (\mathbf{\Psi}_t, \boldsymbol{H}_t).$$

The information available to the investor in the market at time t include the history of macroeconomic variables and default status of the portfolio up to time t.

$$\mathcal{F}_t = \mathcal{F}_t^{\mathbf{\Psi}} \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \cdots \vee \mathcal{F}_t^N$$

where the filtrations are generated collectively by the information contained in the state variables and the default processes

$$\begin{aligned} \mathcal{F}_t^{\mathbf{\Psi}} &= \sigma(\mathbf{\Psi}_s : 0 \leq s \leq t) \\ \mathcal{F}_t^i &= \sigma(H_s^i : 0 \leq s \leq t), \quad i = 1, 2, \cdots, N. \end{aligned}$$

Write $\lambda_i(\boldsymbol{\psi}_t, \boldsymbol{H}_t)$ as the martingale default intensity of Firm i, that is

$$H_t^i - \int_0^{t \wedge \tau_i} \lambda_i(\boldsymbol{\psi}_s, \boldsymbol{H}_s)\, ds$$

is a $\mathcal{F}_t$-martingale.

Let $y \in S$, where S is the state space of $\boldsymbol{H}$. Define the flipped state $\widetilde{y}^i \in S$ by

$$\widetilde{y}^i = 1 - y(i), \quad \widetilde{y}^i(j) = y(j), \quad j \neq i.$$

For example, take $y = (1 \quad 0 \quad 0) \in S = \{0, 1\}^3$, then

$$\widetilde{y}^2 = (1 \quad 1 \quad 0) \quad \text{and} \quad \widetilde{y}^3 = (1 \quad 0 \quad 1).$$

For $y_i, y_j \in S, y_j = \widetilde{y}_i^k$ for some k means y_j is obtained by flipping the k^{th} component in y_i.

Conditional on the given trajectory of $\boldsymbol{\Psi}$, the process $\boldsymbol{H}$ is a time-inhomogeneous Markov chain with initial value $y \in S$. For $y_i, y_j \in S$, the infinitesimal generator $\Lambda_{[\psi]}(t) = (\Lambda_{ij}(t|\psi))_{|S|\times|S|}$ for $\boldsymbol{H}$ at time t given $\boldsymbol{\Psi}_t = \psi$ is defined by

(a) for $i \neq j$

$$\Lambda_{ij}(t|\psi) = \begin{cases} [1 - y_i(k)]\lambda_k(\psi, y_j), & \text{if } y_j = \tilde{y}_i^k \text{ for some } k \\ 0 & \text{else} \end{cases} \quad ; \quad (1a)$$

(b) for $i = j$

$$\Lambda_{ii}(t|\psi) = -\sum_{j\neq i} \Lambda_{ij}(t|\psi) = -\sum_{k=1}^{N} [1 - y_i(k)]\lambda_k(\psi, y_i). \qquad (1b)$$

For $i \neq j$, the transition rate Λ_{ij} equals $\lambda_k(\psi, y_j)$ when y_j can be obtained from y_i by flipping its k^{th} element from 0 to 1.

Define the conditional transition density matrix on $\boldsymbol{\psi}_s = \psi'$ as

$$P(t, s|\psi') = (p_{ij}(t, s|\psi'))_{|S|\times|S|} = (p(t, s, y_i, y_j|\psi'))_{|S|\times|S|}.$$

Kolmogorov backward equation

$$\frac{dP(t, s|\psi')}{dt} = -\Lambda_{[\psi]}(t)P(t, s|\psi'), \quad P(s, s|\psi') = I.$$

The $(i, j)^{\text{th}}$ entry $p_{ij}(t, s|\psi')$ satisfies the following system of ODE:

$$\begin{cases} \dfrac{dp_{ij}(t, s, y_i, y_j|\psi')}{dt} = -\displaystyle\sum_{k=1}^{|S|} \Lambda_{ik}(t|\psi)p_{kj}(t, s, y_k, y_j|\psi') \\ p_{ij}(s, s, y_i, y_j|\psi') = \mathbf{1}_{\{y_j\}}(y_i) \end{cases} . \tag{2a}$$

Using the results in eqs (1a,b), eq. (2a) can be expressed as

$$\frac{dp_{ij}(t,s,y_i,y_j|\psi')}{dt}$$

$$+\sum_{k=1}^{N}[1-y_i(k)]\lambda_k(\psi,y_i)[p_{ij}(t,s,\tilde{y}_i^k,y_j|\psi')-p_{ij}(t,s,y_i,y_j|\psi')]=0$$

with auxiliary condition:

$$p_{ij}(s,s,y_i,y_j|\psi')=\mathbf{1}_{\{y_j\}}(y_i) \qquad (2b)$$

Kolmogorov forward equation

$$\frac{dP(t,s|\psi')}{ds} = P(t,s|\psi')\Lambda[\psi'](s), \quad P(t,t|\psi') = I. \tag{3a}$$

In a similar manner, we obtain

$$\begin{aligned}\frac{dp_{ij}(t,s,y_i,y_j|\psi')}{ds} &= \sum_{k=1}^{N} y(k)\lambda_k(\psi',y_j^k)p_{ij}(t,s,y_i,y_j^k|\psi') \\ &\quad - \sum_{k=1}^{N} [1-y(k)]\lambda_k(\psi',y_j)p_{ij}(t,s,y_i,y_j|\psi'),\end{aligned}$$

with auxiliary condition:

$$p_{ij}(t,t,y_i,y_j|\psi') = \mathbf{1}_{\{y_i\}}(y_j). \tag{3b}$$

Marginal distribution of the default time

The distribution function is defined as

$$F_i(t_i) = P_r(\tau_i \leq t_i) \quad \text{for } i = 1, 2, \cdots, N.$$

We have

$$F_i(t_i) = \int_D \sum_{y_j(i)=1} p_{1j}(0, t_i | \psi') \, dG_{\boldsymbol{\Psi}_s}(\psi')$$

where $G_{\boldsymbol{\Psi}_s}(\cdot)$ is the distribution of $\boldsymbol{\Psi}_s$.

Joint distribution of the default times

The joint distribution of the default times is defined as

$$F(t_1, t_2, \cdots, t_N) = P_r(\tau_1 \le t_1, \cdots, \tau_N \le t_N).$$

Consider the case $t_1 \le t_2 \le \cdots \le t_N$, we define

$$\mathcal{S}(n, m) = \left\{ y \in S : n \le \sum_{i=1}^{N} y(i) \le m \right\}, \quad n \le m.$$

Here, $\mathcal{S}(n, m)$ is the set of default states with total number of defaults lying between n and m, inclusively. Write $M = 2^N$, where M is the total number of possible states. We take $y_1(k) = 0$ and $y_M(k) = 1$ for all $k = 1, 2, \cdots, N$.

The distribution function can be expressed as

$$
\begin{aligned}
& F(t_1, t_2, \cdots, t_N) \\
= & \int_D \Bigg[p_{1M}(0, t_1|\psi') + \sum_{y_{j_1} \in S(1,N-1)} p_{1j_1}(0, t_1|\psi') p_{j_1M}(t_1, t_2|\psi) \\
& + \sum_{\substack{y_{j1} \in \mathcal{S}(1,N-1) \\ y_{j2} \in \mathcal{S}(2,N-1)}} p_{1j_1}(0, t_1|\psi') p_{j_1j_2}(t_1, t_2|\psi') p_{j_2M}(t_2, t_3|\psi') \\
& + \sum_{\substack{y_{j1} \in \mathcal{S}(1,N-1) \\ \vdots \\ y_{jN-1} \in \mathcal{S}(N-1,N-1)}} p_{1j1}(0, t_1|\psi') p_{j_1j_2}(t_1, t_2|\psi') \cdots p_{j_{N-1}M}(t_{N-1}, t_N|\psi') \Bigg] \\
& dG_{\mathbf{\Psi}_s}(\psi').
\end{aligned}
$$

Three-Firm Model

The inter-dependent default intensities of the 3 firms are defined as

$$\begin{aligned}
\lambda_t^A &= a_{10} + a_{12}\mathbf{1}_{\{\tau_B \le t\}} + a_{13}\mathbf{1}_{\{\tau_C \le t\}} + a_{14}\mathbf{1}_{\{\tau_B \le t, \tau_C \le t\}} \\
\lambda_t^B &= a_{20} + a_{21}\mathbf{1}_{\{\tau_A \le t\}} + a_{23}\mathbf{1}_{\{\tau_C \le t\}} + a_{24}\mathbf{1}_{\{\tau_A \le t, \tau_C \le t\}} \\
\lambda_t^C &= a_{30} + a_{31}\mathbf{1}_{\{\tau_A \le t\}} + a_{32}\mathbf{1}_{\{\tau_B \le t\}} + a_{34}\mathbf{1}_{\{\tau_A \le t, \tau_B \le t\}}.
\end{aligned}$$

We assume an extra jump in default intensity if the other two firms have defaulted, allowing the interaction between the default events on the intensity of surviving firms.

The state space S of $\boldsymbol{H} = (H_t^A, H_t^B, H_t^C)$ is given by

$S = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$

State 1	$(0,0,0)$	State 2	$(1,0,0)$	State 3	$(0,1,0)$	State 4	$(0,0,1)$
State 5	$(1,1,0)$	State 6	$(1,0,1)$	State 7	$(0,1,1)$	State 8	$(1,1,1)$

The infinitesimal generator Λ of the process $\boldsymbol{H}$ is given by

$$\Lambda = \begin{bmatrix} -(a_{10}+a_{20}+a_{30}) & a_{10} & a_{20} & a_{30} & 0 & 0 & 0 & 0 \\ 0 & \substack{-(a_{20}+a_{21} \\ +a_{30}+a_{31})} & 0 & 0 & a_{20}+a_{21} & a_{30}+a_{31} & 0 & 0 \\ 0 & 0 & \substack{-(a_{10}+a_{12} \\ +a_{30}+a_{32})} & 0 & a_{10}+a_{12} & 0 & a_{30}+a_{32} & 0 \\ 0 & 0 & 0 & \substack{-(a_{10}+a_{13} \\ +a_{20}+a_{21})} & a_{10}+a_{13} & a_{20}+a_{23} & 0 & \\ 0 & 0 & 0 & 0 & \substack{-(a_{30}+a_{31} \\ +a_{32}+a_{34})} & 0 & 0 & \substack{(a_{30}+a_{31} \\ +a_{32}+a_{33})} \\ 0 & 0 & 0 & 0 & 0 & \substack{-(a_{20}+a_{21} \\ +a_{23}+a_{24})} & 0 & \substack{-(a_{20}+a_{21} \\ +a_{23}+a_{24})} \\ 0 & 0 & 0 & 0 & 0 & 0 & \substack{-(a_{10}+a_{12} \\ +a_{13}+a_{14})} & \substack{-(a_{10}+a_{12} \\ +a_{13}+a_{14})} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For example, consider the transition rate from State 2 : $(1 \quad 0 \quad 0)$ to State 5 : $(1 \quad 1 \quad 0)$ and State 6 : $(1 \quad 0 \quad 1)$:

$$\begin{aligned} \Lambda_{25} &= a_{20} + a_{21} \\ \Lambda_{26} &= a_{30} + a_{31} \end{aligned}$$

$\Lambda_{21} = \Lambda_{23} = \Lambda_{24} = \Lambda_{27} = \Lambda_{28} = 0$ and $\Lambda_{22} = 1 - \Lambda_{25} - \Lambda_{26}$.

The transition density matrix $P(t)$ can be obtained from solving the Kolmogorov forward equation.

$$\frac{dP(t)}{dt} = P(t)\Lambda, P(0) = I.$$

The distribution functions of the default times are found to be

$$\begin{aligned} F_A(t) = P_r(\tau_A \leq t) &= P_{12}(t) + P_{15}(t) + P_{16}(t) + P_{18}(t) \\ F_B(t) = P_r(\tau_B \leq t) &= P_{13}(t) + P_{15}(t) + P_{17}(t) + P_{18}(t) \\ F_C(t) = P_r(\tau_C \leq t) &= P_{14}(t) + P_{16}(t) + P_{17}(t) + P_{18}(t) \end{aligned}$$

The joint distribution of default times τ_A, τ_B and τ_C is defined as

$$F(t_1, t_2, t_3) = P_r(\tau_A \leq t_1, \tau_B \leq t_2, \tau_C \leq t_3).$$

The distribution function takes different form under various scenarios of relative magnitudes of t_1, t_2 and t_3. For example, suppose $t_1 \leq t_2 \leq t_3$, then

$$\begin{aligned} F(t_1, t_2, t_3) \;=\; & P_{18}(t_1) + P_{12}(t_1)P_{28}(t_2 - t_1) + P_{15}(t_1)P_{58}(t_2 - t_1) \\ & + P_{16}(t_1)P_{68}(t_2 - t_1) + P_{12}(t_1)P_{25}(t_2 - t_1)P_{58}(t_3 - t_2) \\ & + P_{15}(t_1)P_{55}(t_2 - t_1)P_{58}(t_3 - t_2). \end{aligned}$$

$$\begin{tikzcd} \substack{\text{State } 1 \\ (0 \quad 0 \quad 0)} \arrow[r, bend left, "P_{18}(t_1)"] & \substack{\text{State } 8 \\ (1 \quad 1 \quad 1)} \end{tikzcd}$$

$$\begin{tikzcd} \substack{\text{State } 1 \\ (0 \quad 0 \quad 0)} \arrow[r, bend left, "P_{12}(t_1)"] & \substack{\text{State } 2 \\ (1 \quad 0 \quad 0)} \arrow[r, bend left, "P_{28}(t_2)"] & \substack{\text{State } 8 \\ (1 \quad 1 \quad 1)} \end{tikzcd}$$

$$\begin{tikzcd} \substack{\text{State } 1 \\ (0 \quad 0 \quad 0)} \arrow[r, bend left, "P_{12}(t_1)"] & \substack{\text{State } 2 \\ (1 \quad 0 \quad 0)} \arrow[r, bend left, "P_{25}(t_2-t_1)"] & \substack{\text{State } 5 \\ (1 \quad 1 \quad 0)} \arrow[r, bend left, "P_{58}(t_3-t_2)"] & \substack{\text{State } 8 \\ (1 \quad 1 \quad 1)} \end{tikzcd}$$

Joint density function of the default times

$$
f(t_1,t_2,t_3) =
\begin{cases}
\begin{array}{l} a_{10}(a_{20}+a_{21})(a_{30}+a_{31}+a_{32}+a_{34}) \\ e^{-(a_{10}-a_{21}-a_{31})t_1-(a_{20}+a_{21}-a_{32}-a_{34})t_2-(a_{30}+a_{31}+a_{32}+a_{34})t_3} \end{array} & t_1 \le t_2 \le t_3 \\
\begin{array}{l} a_{10}(a_{30}+a_{31})(a_{20}+a_{21}+a_{23}+a_{24}) \\ e^{-(a_{10}-a_{31}-a_{21})t_1-(a_{30}+a_{31}-a_{23}-a_{24})t_3-(a_{20}+a_{21}+a_{23}+a_{24})t_2} \end{array} & t_1 \le t_3 \le t_2 \\
\begin{array}{l} a_{20}(a_{10}+a_{12})(a_{30}+a_{31}+a_{32}+a_{34}) \\ e^{-(a_{20}-a_{12}-a_{32})t_2-(a_{10}+a_{12}-a_{31}-a_{34})t_1-(a_{30}+a_{31}+a_{32}+a_{34})t_3} \end{array} & t_2 \le t_1 \le t_3 \\
\begin{array}{l} a_{30}(a_{10}+a_{13})(a_{20}+a_{21}+a_{23}+a_{24}) \\ e^{-(a_{30}-a_{13}-a_{23})t_3-(a_{10}+a_{13}-a_{21}-a_{24})t_1-(a_{20}+a_{21}+a_{23}+a_{24})t_2} \end{array} & t_3 \le t_1 \le t_2 \\
\begin{array}{l} a_{20}(a_{30}+a_{32})(a_{10}+a_{12}+a_{13}+a_{14}) \\ e^{-(a_{20}-a_{12}-a_{32})t_2-(a_{30}+a_{32}-a_{13}-a_{14})t_3-(a_{10}+a_{12}+a_{13}+a_{14})t_1} \end{array} & t_2 \le t_3 \le t_1 \\
\begin{array}{l} a_{30}(a_{20}+a_{23})(a_{10}+a_{12}+a_{13}+a_{14}) \\ e^{-(a_{30}-a_{13}-a_{23})t_3-(a_{20}+a_{23}-a_{12}-a_{14})t_2-(a_{10}+a_{12}+a_{13}+a_{14})t_1} \end{array} & t_3 \le t_2 \le t_1
\end{cases} .
$$

Giesecke's exponential model for dependent defaults (2003)

- Default of an individual firm is triggered by the arrival of idiosyncratic shock and other economy wide shocks, and whose arrivals are modeled by independent Poisson processes.
- There are $m = \sum_{k=1}^{N} \binom{N}{k} = 2^N - 1$ shocks in a portfolio of N firms and each shock is governed by an independent Poisson process $N_j(t)$ with constant intensity $\widehat{\lambda}_j$ with $j \in \{1, 2, \cdots, m\}$. A given shock may affect only one firm, several firms or all firms.
- Let matrix $B = (b_{ij})_{N \times m}$ be defined by

  $$b_{ij} = \begin{cases} 1 & \text{if shock } j \in \{1, 2, \cdots, m\} \text{ leads to default of firm } i \\ 0 & \text{else} \end{cases} .$$

  We take the first N shocks to be firm specific, so $b_{ij} = \delta_{ij}$ for $j = 1, 2, \cdots, N$.

Take $n = 3$

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The first 3 shocks (corresponding to the first 3 columns) are firm specific. The 4th shock causes the default of Firms 1 and 2, and the 7th shock causes the default of all firms.

The default time τ_i of Firm i is given by

$$\tau = \inf \left\{ t \geq 0 : \sum_{k=1}^{m} b_{ik} N_k(t) > 0 \right\}.$$

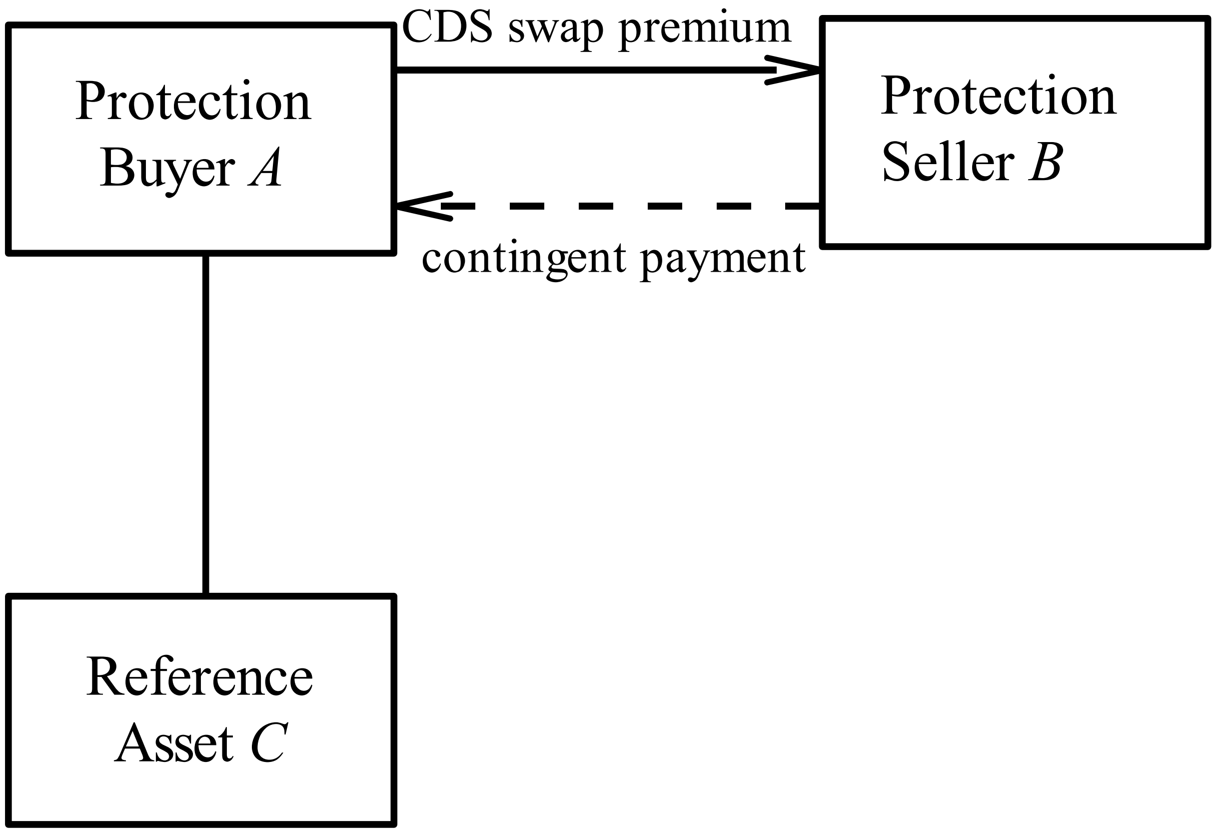
Any shock k with $b_{jk} = 1$ will lead to the default of Firm j.

Giesecke's model can be formulated as the N-firm model under Jarrow and Yu's framework by taking the default intensity of firm j as

$$\lambda_j(\boldsymbol{H}_t) = \widehat{\lambda}_j + \left(\sum_{k=N+1}^{m} b_{jk} a_{jk} \mathbf{1}_{\{N_k(t)>0\}} \right), \quad j = 1, 2, \cdots, N,$$

The indicator variable b_{jk} indicates whether the k^{th} shock, $N + 1 \leq k \leq m$, has impact on Firm j. If so ($b_{jk} = 1$), then the jump in intensity a_{jk} is infinite since the arrival of shock k [as revealed by $N_k(t) > 0$] is considered fatal for Firm j.

Counterparty risk of credit default swap



3 parties: Protection Seller, Protection Buyer, Reference Obligor

- In order that funding cost arbitrage works, the Protection Buyer should have higher credit rating than the Protection Seller. It is advantageous for the Protection Buyer to hold the risky asset to take advantage of the lower funding cost.

- Before the 1997 crisis in Korea, Korean financial institutions are willing to offer protection on Korean bonds. The financial melt down caused failure of compensation payment on defaulting Korean bonds by the Korean Protection Sellers.

How does the inter-dependent default risk structure between the Protection Seller and the Reference Obligor affect the swap rate?

1. *Replacement cost* (Seller defaults earlier)
    - If the Protection Seller defaults prior to the Reference Entity, then the Protection Buyer renews the CDS with a new counterparty.
    - Supposing that the default risks of the Protection Seller and Reference Entity are positively correlated, then there will be an increase in the swap rate of the new CDS.
2. *Settlement risk* (Reference Entity defaults earlier)
    - The Protection Seller defaults during the settlement period after the default of Reference Entity.

The inter-dependent default risk structure between Protection Seller B and Reference Entity C is characterized by the correlated default intensities:

$$\begin{aligned}\lambda_t^B &= b_0 + b_2 \mathbf{1}_{\{\tau^C \le t\}} \\ \lambda_t^C &= c_0 + c_2 \mathbf{1}_{\{\tau^B \le t\}}.\end{aligned}$$

The joint density of default time (τ^B, τ^C) is

$$f(t_1, t_2) = \begin{cases} c_0(b_0 + b_2)e^{-(b_0+b_2)t_1} - (c_0 - b_2)t_2, & t_2 \le t_1, \\ b_0(c_0 + c_2)e^{-(c_0+c_2)t_2} - (b_0 - c_2)t_1, & t_2 > t_1. \end{cases}$$

We compute the swap rate $S(T)$ by setting

$$\begin{aligned} & \text{expected present value of Protection Buyer payment} \\ = \; & \text{expected present value of compensation payment} \\ & \text{at } \tau^{C+\delta}, \text{with } \tau^C < T \text{ and } \tau^B > \tau^C + \delta, \end{aligned}$$

where δ is the settlement period.

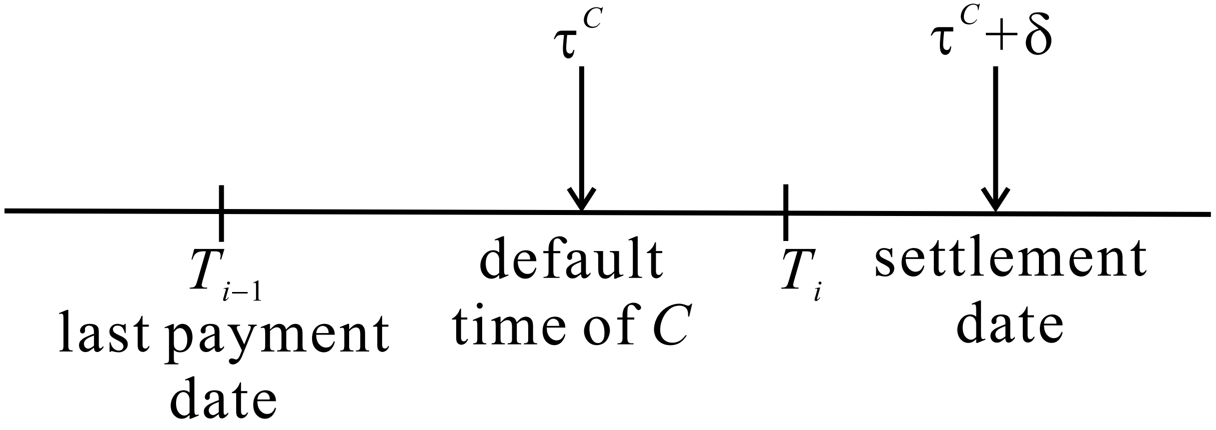
Buyer pays $S(T)$ at T_i, provided $\tau^B \wedge \tau^C > T_i, i = 1, 2, \cdots, n$.

The swap rate $S(T)$ under this two-firm model is determined by

$$\begin{aligned}\sum_{i=1}^{n} E[e^{-rT_i} S(T)\mathbf{1}_{\{\tau^B \wedge \tau^C > T_i\}}] + S(T)A(T) \\ = \; E\left[e^{-r(\tau^C+\delta)}\mathbf{1}_{\{\tau^C \leq T\}}\mathbf{1}_{\{\tau^B > r^C+\delta\}}\right],\end{aligned}$$

where $S(T)A(T)$ is the present value of the accured swap premium for the fraction of period between τ^C and the last payment date.

$$A(T) = \sum_{i=1}^{n} E\left[e^{-r\tau^C}\left(\frac{\tau^C - T_{i-1}}{\Delta T}\right)\mathbf{1}_{\{T_{i-1}<\tau^C<T_i\}}\mathbf{1}_{\{\tau^B>\tau^C\}}\right].$$



Settlement risk

Suppose the Protection Seller is default-free, the swap premium is then given by

$$\sum_{i=1}^{n} E[e^{-rT_i}\overline{S}(T)\mathbf{1}_{\{\tau^C > T_i\}}] + \overline{S}(T)\overline{A}(T) = E\left[e^{-r(\tau^C+\delta)}\mathbf{1}_{\{\tau^C \leq T\}}\right],$$

where

$$\overline{A}(T) = \sum_{i=1}^{n} E\left[e^{-r\tau^C}\left(\frac{\tau^C - T_{i-1}}{\Delta T}\right)\mathbf{1}_{\{T_{i-1}<\tau^C<T_i\}}\right].$$

To examine the effect of settlement risk on the swap premium, we define the swap premium spread as the difference of the swap premium with and without settlement risk.

Without default risk of the Protection Seller B, Protection Buyer faces

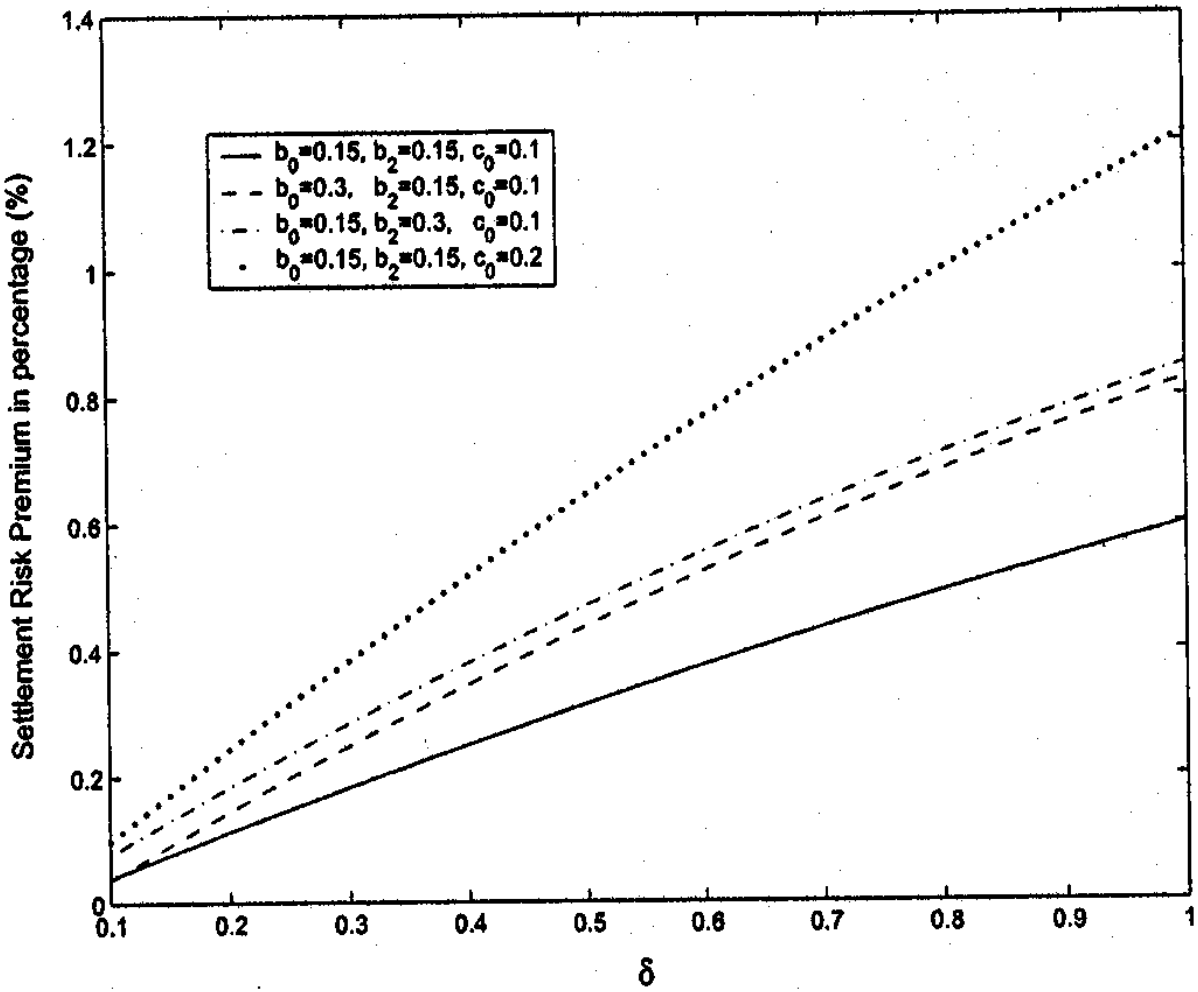
(i) paying higher expected present value of total swap payments to the Seller since

$$E\left[e^{-rT_i}\mathbf{1}_{\{\tau^C>T_i\}}\right] > E\left[e^{-rT_i}\mathbf{1}_{\{\tau^B\wedge\tau^C>T_i\}}\right];$$

(ii) receiving higher expected present value of contingent payment from the Seller

$$E\left[e^{-r(\tau^C+\delta)}\mathbf{1}_{\{\tau^C\leq T\}}\right] \geq E\left[e^{-r(\tau^C+\delta)}\mathbf{1}_{\{\tau^C\leq T\}}\mathbf{1}_{\{\tau^B>\tau^C+\delta\}}\right],$$

since there is no settlement risk.



Dependence of settlement risk premium on δ. The base parameter values are: $r = 0.05, \Delta T = 0.25, b_0 = 0.15, b_2 = 0.15, c_0 = 0.1, c_2 = 0.1$.

Replacement cost

With positive correlation between the Seller and Reference Entity, the Buyer has to pay a higher swap rate to a new Protection Seller – replacement cost.

This occurs when $\tau_B < \min(\tau_C, T)$. Suppose $S(T)$ represents the fair swap rate charged by the seller party B, the replacement cost would not be included in the calculation of the swap premium.

Dependence of settlement risk premium on δ

Seller: $\lambda_t^B = b_0 + b_2 \mathbf{1}_{\{\tau^C < t\}}$

Reference Entity: $\lambda_t^C = c_0 + c_2 \mathbf{1}_{\{\tau^B < t\}}$

- The underlying intensity value of λ_t^C, c_0, has the strongest influence on the settlement risk premium.

- The intensity values of λ_t^B have less influence.

- The default correlation between B and C, proxied by b_2, is slightly more important than b_0.

Summary

- Markov chain formulation of credit contagion with interacting intensities is discussed.
- Derivations of the joint distribution of the default times of obligors with interacting intensities using the Markov chain techniques are presented.
- Application to the analysis of counterparty risk in credit default swaps.