Valuing American-Style Lookback Options of Floating Strike Type

A Laplace Transform Approach

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Lookback Options

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- Lookback options can be broadly classified into two types.
  - fixed strike
  - floating strike
    - standard $\subset$ fractional

Lookback Options (cont.)

Let $(S_t)_{t\geq 0}$ be the price process of the underlying asset, and let

$$m_t = \min_{0\leq u\leq t} S_u \quad \text{and} \quad M_t = \max_{0\leq u\leq t} S_u$$

Assume that the price process is monitored continuously.

Standard Floating Strike

A floating-strike lookback call (put) depends on the processes $(S_t)_{t\geq 0}$ and $(m_t)_{t\geq 0}$ $((M_t)_{t\geq 0})$, and it always gives the option holder the right to buy (sell) at the lowest (highest) realized price.

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For European floating-strike lookback call and put with maturity date T, their standard terminal payoffs are given by

$$(S_T - m_T)^+ = S_T - m_T \quad \text{for call}$$

$$(M_T - S_T)^+ = M_T - S_T \quad \text{for put}$$

Fractional Floating Strike

Clearly, standard floating-strike lookback options are not genuine option contracts since they are always exercised until the maturity, finishing in-the-money. This means that high premiums are charged for the standard floating-strike lookback options, being less attractive to investors.

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Conze and Viswanathan (1991) introduced a more general and less expensive variant called a fractional lookback option, where the strike is fixed at some fraction over (for a call) or below (for a put) the extreme value.

Fractional Floating Strike (cont.)

Specifically, the payoffs for European lookback with fractional floating strikes and maturity date T are given by

$$\begin{aligned} &(S_T - \alpha m_T)^+ \quad \text{for call} \\ &(\beta M_T - S_T)^+ \quad \text{for put} \end{aligned}$$

where α and β are positive constants, allowing flexible adjustment of option premiums.

Fractional Floating Strike (cont.)

Specifically, the payoffs for European lookback with fractional floating strikes and maturity date T are given by

$$\begin{aligned} &(S_T - \alpha m_T)^+ \quad \text{for call} \\ &(\beta M_T - S_T)^+ \quad \text{for put} \end{aligned}$$

where α and β are positive constants, allowing flexible adjustment of option premiums.

To reduce option premiums, we assume that $\alpha \geq 1$ and $0 < \beta \leq 1$.

Aims of This Talk

Our target is an American-style fractional floating-strike lookback option, for which we

- obtain valuation formulas in the domain of Laplace transforms;
- prove asymptotic behaviors of early exercise boundaries; and
- develop a fast and accurate algorithm for computing early exercise boundaries as well as option values.

For brevity we deal with the call case.

Previous Research

- Dai (2000)

  who derives a closed-form solution for a perpetual American standard floating-strike lookback option via a partial differential equation (PDE) approach.

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- Yu, Kwok and Wu (2001)

  who characterize an early exercise boundary of the standard American floating-strike lookback option, by combining the PDE approach and a finite difference method.

Previous Research (cont.)

- Dai and Kwok (2005)

  who analyzed asymptotic behaviors of the early execise boundary at time close to expiration and infinite time to expiration.

Previous Research (cont.)

- Dai and Kwok (2005)

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- Lattice methods

  Hull and White (1993), Kat (1995), Barraquand and Pudet (1996), Cheuck and Vorst (1997), Babbs (2000), Dai (2000), Lai and Lim (2004)

Basic Framework

Assume that $(S_t)_{t\geq 0}$ is a risk-neutralized diffusion process described by the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0$$

where

r : risk-free rate of interest

δ : continuous dividend rate

σ : volatility coefficient

W : standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

Basic Framework (cont.)

Let $C \equiv C(t, S_t, m_t)$ denote the value of the American floating-strike lookback call option at time $t \in [0, T]$.

Basic Framework (cont.)

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In the absence of arbitrage opportunities, the value $C(t, S_t, m_t)$ is a solution of an optimal stopping problem

$$C(t, S_t, m_t) = \operatorname*{ess\,sup}_{\mathcal{T}_t \in [t,T]} \mathbb{E}\left[e^{-r(\mathcal{T}_t - t)}(S_{\mathcal{T}_t} - \alpha m_{\mathcal{T}_t})^+ \,|\, \mathcal{F}_t\right]$$

for $t \in [0, T]$, where $\mathcal{T}_t$ is a stopping time of the filtration $(\mathcal{F}_t)_{t \geq 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$.

Basic Framework (cont.)

Solving the optimal stopping problem is equivalent to finding the points (t, S_t) for which early exercise is optimal.

Let $\mathcal{S}$ and $\mathcal{C}$ denote the stopping region and continuation region, respectively. The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is referred as the early exercise boundary, which is defined by

$$\bar{S}_t = \sup\left\{S \in [m, +\infty) \,|\, C(t, S, m) > (S - \alpha m)^+\right\}$$

for $t \in [0, T]$, at which the American lookback call option would be optimally exercised.

Free Boundary Problem

It has been known that $C(t, S, m)$ is obtained by solving a *free boundary problem*;
see, *e.g.*, Kwok (1998) and Wilmott *et al.* (1993).

Define the differential operator $\mathcal{A}$ by

$$\mathcal{A} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta) S \frac{\partial}{\partial S} - r$$

Free Boundary Problem (cont.)

Then, the free boundary problem can be written in linear complimentary form as

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathcal{A}C \leq 0, \qquad C - (S - \alpha m)^+ \geq 0 \\ \left(\dfrac{\partial C}{\partial t} + \mathcal{A}C\right) \cdot \left(C - (S - \alpha m)^+\right) = 0, \quad (t, S) \in \mathcal{C} \end{cases}$$

together with auxiliary conditions

$$\left| \begin{array}{l} C(T, S, m) - (S - \alpha m)^+ = 0 \\ \lim\limits_{S \downarrow m} \dfrac{\partial C}{\partial m} = 0 \end{array} \right.$$

PDE Formulation

For the free boundary $(\bar{S}_t)_{t\in[0,T]}$, this problem is equivalent to solving the Black-Scholes-Merton PDE

$$\frac{\partial C}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r-\delta)S\frac{\partial C}{\partial S}-rC=0,\quad m<S<\bar{S}_t$$

with the terminal condition $C(T,S,m)=(S-\alpha m)^+$ and the boundary conditions

$$\lim_{S\uparrow\bar{S}_t} C(t,S,m)=\bar{S}_t-\alpha m$$
$$\lim_{S\uparrow\bar{S}_t}\frac{\partial C}{\partial S}=1,\quad \lim_{S\downarrow m}\frac{\partial C}{\partial m}=0$$

LCTs of Option Values

For the remaining time to maturity $\tau = T - t$, define the time-reversed value

$$\widetilde{C}(\tau, S, m) = C(T - \tau, S, m), \quad \tau \geq 0$$

and its Laplace-Carlson transform (LCT) as

$$\begin{aligned} C^*(\lambda, S, m) &= \mathcal{LC}[\widetilde{C}(\tau, S, m)] \\ &= \int_0^\infty \lambda e^{-\lambda\tau} \widetilde{C}(\tau, S, m) \mathrm{d}\tau \end{aligned}$$

for $\mathrm{Re}(\lambda) > 0$.

LCTs of Option Values (cont.)

No doubt, there is no essential difference between the LCT and the Laplace transform (LT)

$$\widehat{C}(\lambda, S, m) = \mathcal{L}[\widetilde{C}(\tau, S, m)] = \int_0^\infty e^{-\lambda\tau}\widetilde{C}(\tau, S, m)\mathrm{d}\tau$$

LCTs of Option Values (cont.)

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The principal reason why we prefer LCT to LT is that LCT generates relatively simpler formulas than LT for option pricing problems because constant values are invariant after taking transformation.

See Carr (1998) for the use of LCTs in the randomization of an American vanilla option.

Theorem 1

$$C^*(\lambda, S, m) = \begin{cases} S - \alpha m, & S \geq \bar{S}^* \\ S \displaystyle\sum_{i=1}^{2} A_i \left(\frac{\alpha m}{S}\right)^{\theta_i} + \frac{\lambda}{\lambda + \delta} S - \frac{\lambda}{\lambda + r} \alpha m, & \\ & \alpha m < S < \bar{S}^* \\ S \displaystyle\sum_{i=1}^{2} A_{i+2} S \left(\frac{\alpha m}{S}\right)^{\theta_i}, & m < S \leq \alpha m \end{cases}$$

Thorem 1 (cont.)

$$A_1 = \frac{\theta_2}{\theta_2 - \theta_1}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\frac{\alpha m}{\bar{S}^*}\right)\left(\frac{\bar{S}^*}{\alpha m}\right)^{\theta_1}$$

$$A_2 = \frac{\theta_1}{\theta_1 - \theta_2}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_1}{\theta_1}\frac{r}{\lambda+r}\frac{\alpha m}{\bar{S}^*}\right)\left(\frac{\bar{S}^*}{\alpha m}\right)^{\theta_2}$$

$$A_3 = A_1 + \frac{\theta_2}{\theta_2 - \theta_1}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\right)$$

$$A_4 = A_2 + \frac{\theta_1}{\theta_1 - \theta_2}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_1}{\theta_1}\frac{r}{\lambda+r}\right)$$

Thorem 1 (cont.)

The parameters $\theta_1 \equiv \theta_1(\lambda) > 0$ and $\theta_2 \equiv \theta_2(\lambda) < 0$ are two real roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (\delta - r - \frac{1}{2}\sigma^2)\theta - (\lambda + \delta) = 0$$

Thorem 1 (cont.)

In addition, $\bar{S}^* \equiv \bar{S}^*(\lambda)$ is defined by $\bar{S}^* = m/\xi^*$, of which $\xi^* \equiv \xi^*(\lambda)$ satisfies the functional equation

$$\begin{aligned}
&\lambda\left\{\frac{\alpha^{\theta_2}}{\theta_1} - \frac{\alpha^{\theta_1}}{\theta_2} + (\alpha^{\theta_2} - \alpha^{\theta_1})\frac{r-\delta}{\lambda+\delta}\right\}(\xi^*)^{\theta_1+\theta_2} \\
= \;& \delta\frac{\lambda+r}{\lambda+\delta}\left\{(\xi^*)^{\theta_2} - (\xi^*)^{\theta_1}\right\} \\
&+ \alpha r\left\{\frac{1-\theta_2}{\theta_2}(\xi^*)^{\theta_2+1} - \frac{1-\theta_1}{\theta_1}(\xi^*)^{\theta_1+1}\right\}
\end{aligned}$$

for $0 < \xi^* < \alpha^{-1} \leq 1$.

Functional Equation

The equation for $\xi^*(\lambda)$ can be rewritten as

$$\xi^* = f_\lambda(\xi^*)$$

where

$$f_\lambda(\xi^*) \equiv \frac{\lambda g(\lambda)(\xi^*)^{\theta_1+\theta_2} - \delta\dfrac{\lambda + r}{\lambda + \delta}\left\{(\xi^*)^{\theta_2} - (\xi^*)^{\theta_1}\right\}}{\alpha r\left\{\dfrac{1-\theta_2}{\theta_2}(\xi^*)^{\theta_2} - \dfrac{1-\theta_1}{\theta_1}(\xi^*)^{\theta_1}\right\}}$$

$$g(\lambda) \equiv \frac{\alpha^{\theta_2}}{\theta_1} - \frac{\alpha^{\theta_1}}{\theta_2} + (\alpha^{\theta_2} - \alpha^{\theta_1})\frac{r-\delta}{\lambda+\delta}$$

Theorem 2

For the early exercise boundary $(\bar{S}_t)_{t\in[0,T]}$ of the perpetual fractional lookback call option with $T=\infty$, we have

$$\bar{S}_t = \bar{S}_\infty \equiv \frac{m}{\xi_\infty}$$

for all $t \geq 0$. If $\delta > 0$, the constant $\xi_\infty \in (0, \alpha^{-1})$ is given by the unique root of the equation

$$x = f_0(x)$$

If $\delta = 0$, then

$$\bar{S}_\infty = \infty$$

Remark 1

The functional equation agrees with Equation (13) in Proposition 3(a) of Lai and Lim (2004), which is given by

$$\alpha\theta_2^\circ(1-\theta_1^\circ)x^{\theta_1^\circ-\theta_2^\circ+1}+\theta_1^\circ\theta_2^\circ x^{\theta_1^\circ-\theta_2^\circ}-\alpha\theta_1^\circ(1-\theta_2^\circ)x-\theta_1^\circ\theta_2^\circ=0$$

where $\theta_i^\circ = \lim_{\lambda\to 0}\theta_i(\lambda)$ $(i=1,2)$.

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where $\theta_i^\circ = \lim_{\lambda\to 0}\theta_i(\lambda)$ $(i=1,2)$.

Also, it is consisitent with Proposition 2.2 of Dai and Kwok (2005), where the constant ξ_∞ is represented as the unique root of the equation

$$x^{\theta_1^\circ-\theta_2^\circ} = \frac{\theta_1^\circ}{\theta_2^\circ}\,\frac{\alpha(1-\theta_2^\circ)x+\theta_2^\circ}{\alpha(1-\theta_1^\circ)x+\theta_1^\circ}$$

Remark 2

The uniquness of the root suggests us a recursive root-finding scheme such as

$$x_n = f_0(x_{n-1}), \quad n \geq 1$$

for an appropriately selected initial value x_0. Actually, we see from some numerical experiments that this recursion with $x_0 = 0$ generates a quick convergent sequence for the fractional lookback call.

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For put, however, we need a careful treatment.

Table 1

Convergence of the sequence $\{x_n\}$ ($r = 0.05, \delta = 0.03, \sigma = 0.2$)

n	$\beta = 0.75$	$\beta = 1.0$	$\alpha = 1.0$	$\alpha = 1.5$
1	3.20874	2.87791	0.36754	0.24503
2	2.58182	2.08828	0.35541	0.24280
3	2.61655	2.27077	0.35664	0.24286
4	2.62352	2.26600	0.35652	0.24286
5	2.62467	2.26679	0.35653	—
6	2.62485	2.26666	—	—
7	2.62488	2.26668	—	—
8	2.62489	—	—	—

Theorem 3

For the early exercise boundary $(\bar{S}_t)_{t\in[0,T]}$ of the fractional lookback call option with $T < \infty$, we have

$$\lim_{t\to T} \bar{S}_t = \max\left(\frac{r}{\delta}, 1\right)\alpha m$$

Early Exercise Premium

For American vanilla options, it has been well known that the value of an American option can be represented as the sum of the value of the corresponding European option and the so-called early exercise premium.

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Here, we will derive closed-form LCTs of early exercise premiums for the American fractional lookback call and put options.

Proposition 1

For the time-reversed value $\widetilde{c}(\tau, S, m) = c(T - \tau, S, m)$ ($\tau \geq 0$), define its LCT as $c^*(\lambda, S, m) = \mathcal{LC}[\widetilde{c}(\tau, S, m)]$. Then, we have

$$c^*(\lambda, S, m) = \begin{cases} a_1 S \left(\dfrac{\alpha m}{S} \right)^{\theta_1} + \dfrac{\lambda}{\lambda + \delta} S - \dfrac{\lambda}{\lambda + r} \alpha m, \\ \hfill S > \alpha m \\ a_3 S \left(\dfrac{\alpha m}{S} \right)^{\theta_1} + a_4 S \left(\dfrac{\alpha m}{S} \right)^{\theta_2}, \\ \hfill m < S \leq \alpha m \end{cases}$$

Proposition 1 (cont.)

$$
\begin{aligned}
a_1 &= \frac{\theta_2}{\theta_1 - \theta_2}\left\{(\alpha^{\theta_1} - \alpha^{\theta_2})\frac{\lambda}{\lambda+\delta}\right. \\
&\quad \left. + \left(\frac{1-\theta_2}{\theta_2}\alpha^{\theta_1} - \frac{1-\theta_1}{\theta_1}\alpha^{\theta_2}\right)\frac{\lambda}{\lambda+r}\right\}\alpha^{-\theta_1} \\
a_2 &= 0 \\
a_3 &= \frac{\theta_2}{\theta_2 - \theta_1}\left(\frac{\lambda}{\lambda+\delta} + \frac{1-\theta_1}{\theta_1}\frac{\lambda}{\lambda+r}\right)\alpha^{\theta_2-\theta_1} \\
a_4 &= \frac{\theta_1}{\theta_1 - \theta_2}\left(\frac{\lambda}{\lambda+\delta} + \frac{1-\theta_1}{\theta_1}\frac{\lambda}{\lambda+r}\right)
\end{aligned}
$$

Theorem 4

$$C^*(\lambda, S, m) = \begin{cases} c^*(\lambda, S, m) + e_c^*(\lambda, S, m), & m < S < \bar{S}^* \\ S - \alpha m, & S \geq \bar{S}^* \end{cases}$$

where $e_c^*(\lambda, S, m)$ is the early exercise premium

$$e_c^*(\lambda, S, m) = \begin{cases} S \sum_{i=1}^{2} (A_i - a_i) \left(\frac{\alpha m}{S}\right)^{\theta_i}, & \alpha m < S < \bar{S}^* \\ S \sum_{i=1}^{2} (A_{i+2} - a_{i+2}) \left(\frac{\alpha m}{S}\right)^{\theta_i}, & \\ & m < S \leq \alpha m \end{cases}$$

Numerical Implementation

Inversion algorithm consists of

1. finding the root $\xi^*(\lambda)$ for a given complex number λ
    - Newton method
    - recursion methods for solving $\xi^* = f_\lambda(\xi_*)$
    - initial estimate $\Rightarrow$ approximations for $\xi^*(\lambda)$

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1. finding the root $\xi^*(\lambda)$ for a given complex number λ
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2. inverting Laplace transforms
   - Gaver-Stefest method
   - Euler method

Computational Results

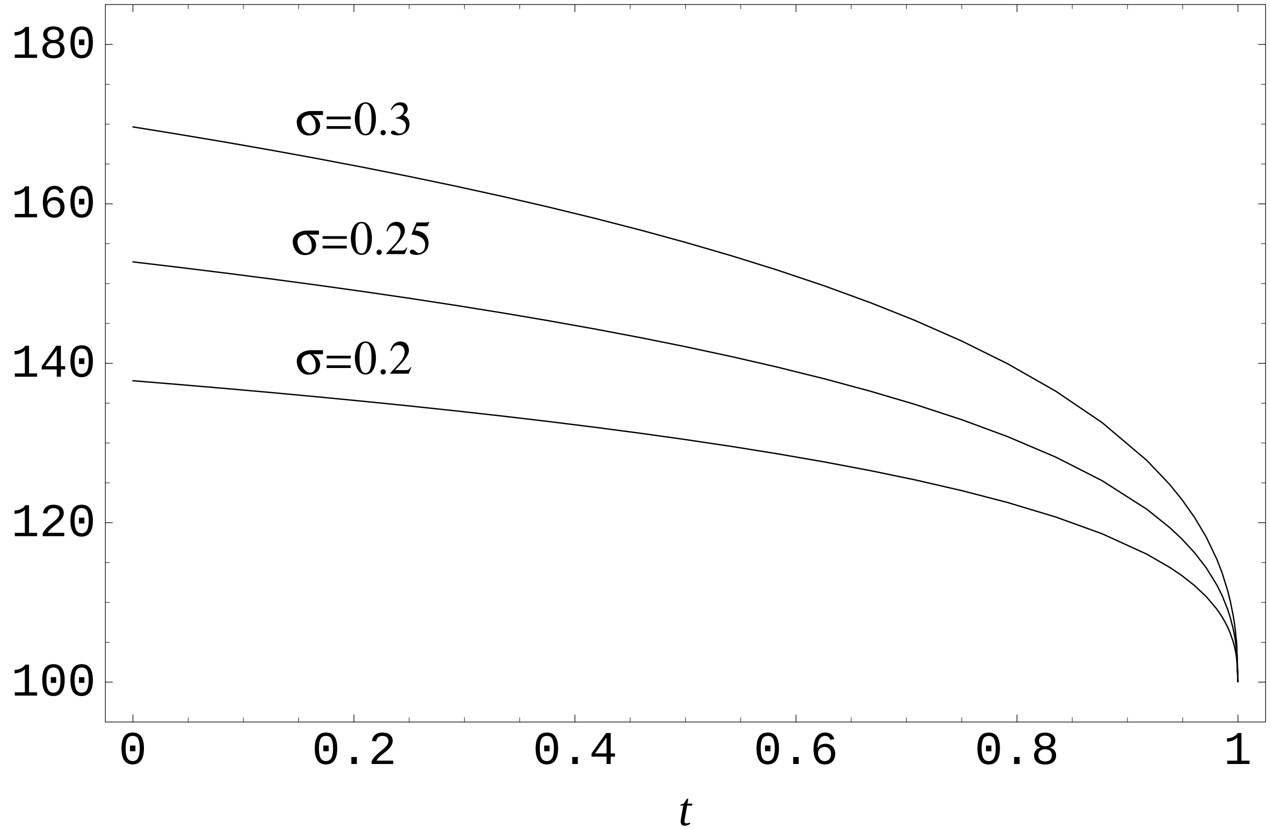

Figure 1(a): Early exercise boundaries $\bar{S}_t$ ($t \in [0, 1]$) for call

$$(T = 1, m = 100, \alpha = 1, r = 0.03, \delta = 0.07)$$

Computational Results (cont.)

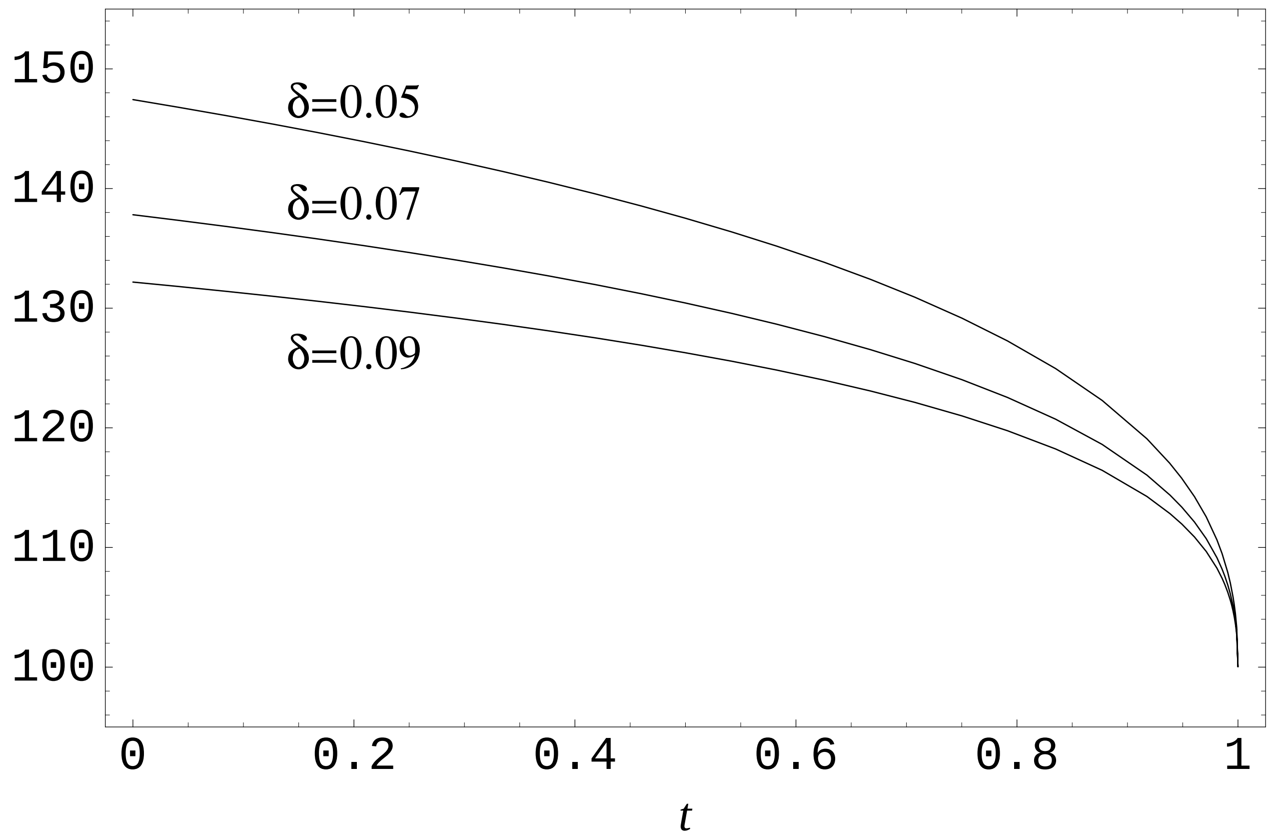

Figure 1(b): Early exercise boundaries $\bar{S}_t$ ($t \in [0, 1]$) for call

$(\boldsymbol{T = 1, m = 100, \alpha = 1, r = 0.03, \sigma = 0.2})$

Computational Results (cont.)

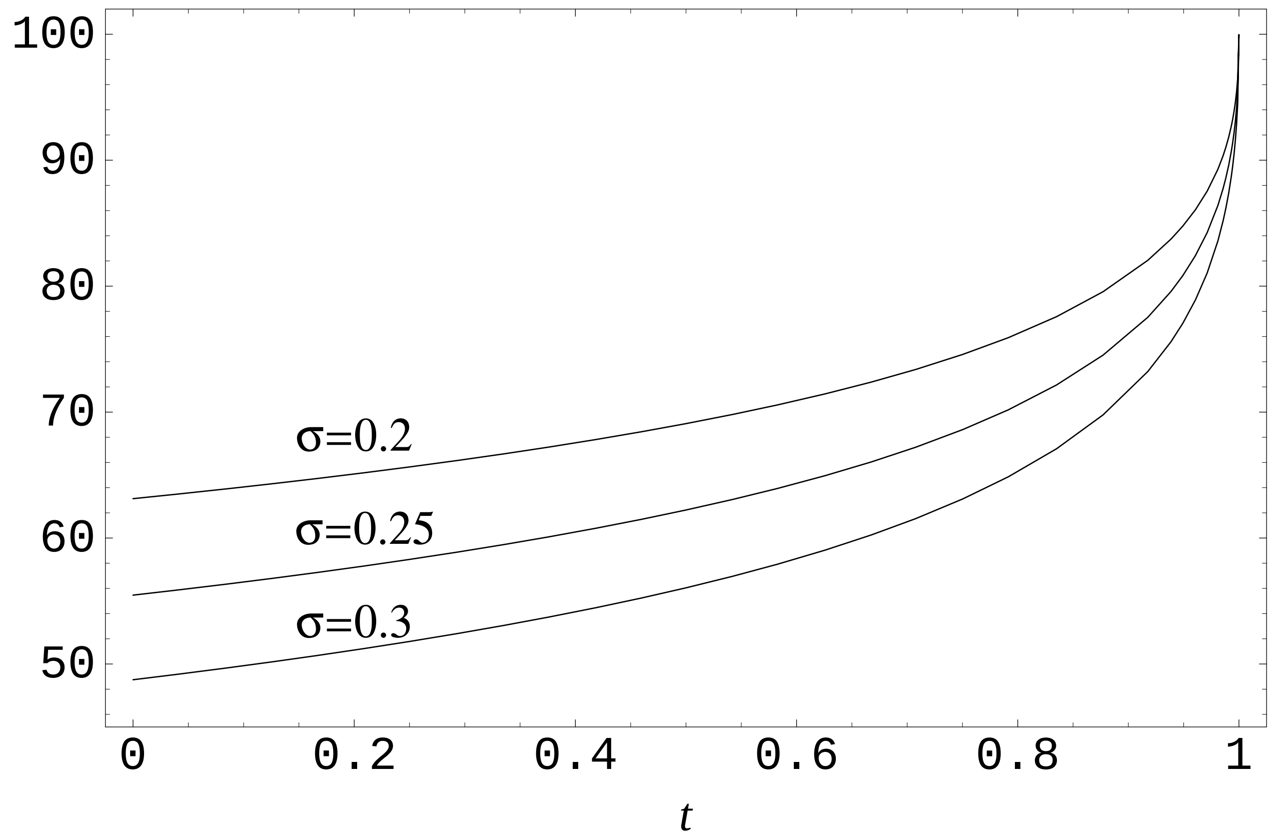

Figure 2(a): Early exercise boundaries $\underline{S}_t$ ($t \in [0, 1]$) for put ($T = 1, M = 100, \beta = 1, r = 0.03, \delta = 0.02$)

Computational Results (cont.)

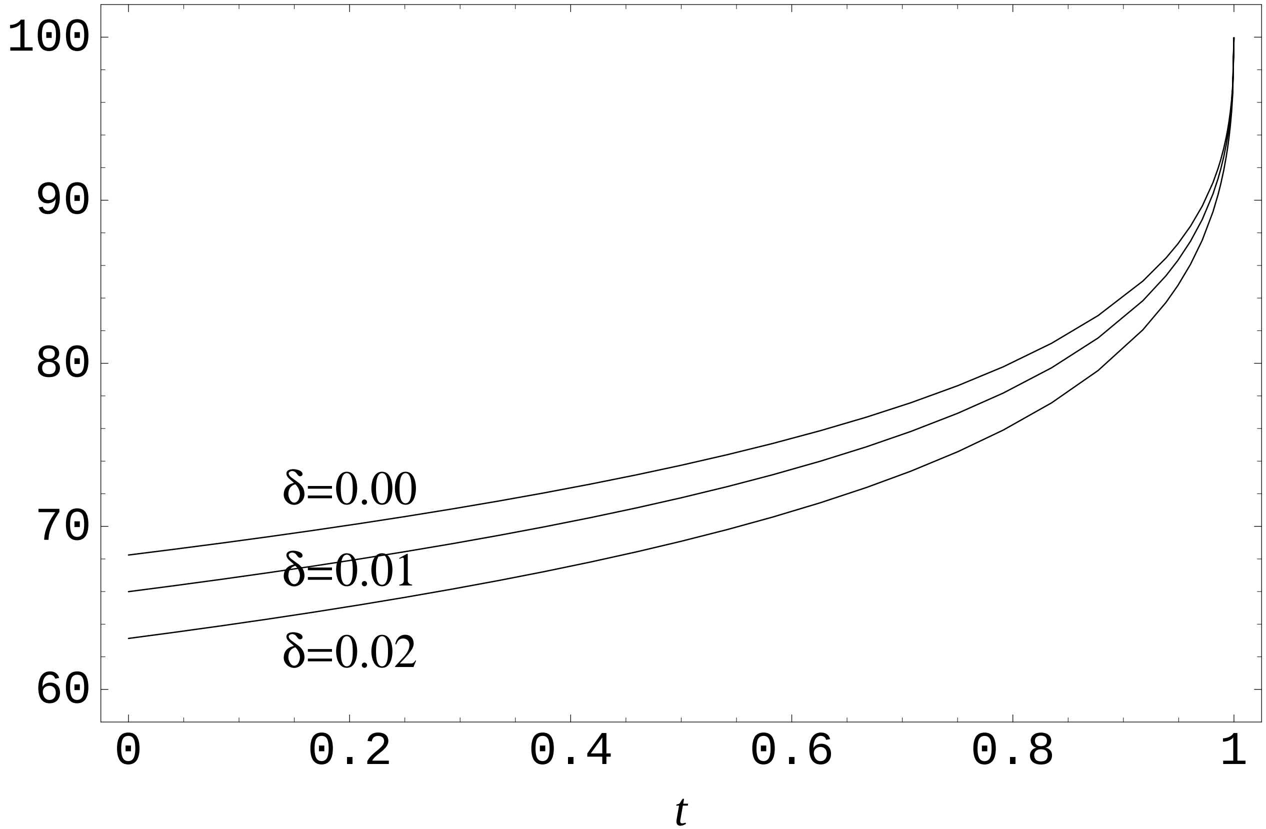

Figure 2(b): Early exercise boundaries $\underline{S}_t$ $(t \in [0, 1])$ for put
$(T = 1, M = 100, \beta = 1, r = 0.03, \sigma = 0.2)$

Concluding Remarks

- The Gaver-Stefest inversion method combined with the Newton method works well for almost all cases if we use an appropriate
  - form of the equation to be solved
  - initial estimate of $\xi^*(\lambda)$

  according to the values of r and δ.

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  - form of the equation to be solved
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  according to the values of r and δ.

- We will apply our method to
  - American exotic options such as a barrier option;
  - a case that the underlying stock price follows a jump diffusion process.