Valuation of Convertible Bonds in an Abstract Set-Up

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Based on

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Related works

J. Kallsen and C. Kühn: Convertible Bonds: Financial Derivatives of Game Type. In *Exotic Option Pricing and Advanced Lévy Models*. Edited by Kyprianou, A., Schoutens, W., and Wilmott, P., Wiley, 2005.

Outline

1. Generic Convertible Bond
2. Decomposition of the Cash Flows
3. Dynkin Games and Price Decomposition

1 Generic Convertible Bond

1.1 Covenants of a Convertible Bond

Convertible Bonds (CBs) can be seen as defaultable bonds supplemented with options to exchange this defaultable bond for a given number κ of stock shares. Denoting by S_t the value per share of the firm's equity at date t, typical clauses include the promised payment of a nominal $\bar{N}$ at the maturity T of the CB, and of coupons c_is at coupon dates T_is, plus optional clauses as follows:

- A holder put or conversion clause, that allows the holder to redeem the CB for an amount of cash (*put payement*) $\bar{P}$ or to convert the CB into κ stock shares κS_t, at the holder's convenience at any time t where the CB is still alive (namely, has not been called, put or converted, and has not defaulted either). Since the bond holder is entitled to receive a relevant accrued interest payment A_t, the effective put payement received by the holder is $P_t = (\bar{P} \vee \kappa S_t) + A_t$. The bondholder may still convert at time T, so we define the effective nominal $N_T = (\bar{N} \vee \kappa S_T) + A_T$.

Covenants of a Convertible Bond (continued)

- An issuer call, that allows the issuer to force the holder to redeem the CB for an amount of cash (*call payement*) $\bar{C}$ or to convert the CB into κ stock shares κS_t, at the issuer's convenience at any time t where the CB is still alive, provided that so-called call protection conditions have been lifted. The effective call payement received by the holder is $C_t = (\bar{C} \vee \kappa S_t) + A_t$

Convertible Bonds are often advertized as products with upside potential and limited downside risk.

After years of steady growth, the market of convertible bonds has suffered an unprecedented drawback in April–May 2005.

1.2 Main Notation & Standing assumptions

$(\Omega, \mathbb{G}, \mathbb{P})$ – Objective filtered probability space on which stochastic processes are defined,

β_t – Discount factor process on $\mathbb{R}_+$, a $\mathbb{G}$-adapted, positive and non-increasing process,

t_0, t – the inception date of a CB, the current time ($0 \leq t_0 \leq t$),

S_t – Market price process of the underlying equity, assumed to be a $\mathbb{G}$-semimartingale with RCLL trajectories. The underlying market may be incomplete, but it is arbitrage-free.

$\mathcal{M}$ – set of the $\mathbb{P}$-equivalent martingale measures on the underlying market

τ_d – the default time of the bond issuer, an $\mathbb{R}_+$-valued $\mathbb{G}$-stopping time,

R_t, X_t – Recovery rate and Default claim processes of the CB, assumed to be bounded and $\mathbb{G}$-predictable,

η – Fractional loss on the underlying equity upon default ($0 \leq \eta \leq 1$).

Main Notation & Standing assumptions (continued)

$\bar{P}$, $\bar{C}$ – put and call prices, such that $\bar{P} \leq \bar{N} \leq \bar{C}$,

τ_p – time of put or conversion decision by the bond holder,

$\bar{\tau}$ – the lifting time of a call protection,

τ_c – time of call decision by the bond issuer,

$\tau = \tau_p \wedge \tau_c$ – the minimum of τ_c and τ_p,

$\mathcal{G}^t$ – the set of $[t, \infty]$-valued $\mathbb{G}$-stopping times, where $t_0 \leq t$,

$\mathcal{G}_T^t$ – the set of $[t, T]$-valued $\mathbb{G}$-stopping times,

$\bar{\mathcal{G}}^t$ – $\{\tau \in \mathcal{G}^t\,;\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$ where $\bar{\tau}$ belongs to $\mathcal{G}_T^{t_0}$,

$\bar{\mathcal{G}}_T^t$ – $\{\tau \in \mathcal{G}_T^t\,;\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$.

1.3 Call notice period

Typically there is a fixed call notice period $\delta \geq 0$ (such as one month) such that if the issuer calls the bond at time τ_c, then the bond holder has to convert the bond into κ shares of stock or redeem the bond at price $\bar{C}$, at a time u at his/her convenience in $[\tau_c, \tau_c^\delta]$, where $t^\delta \equiv (t + \delta) \wedge T$.

Definition. Let π_t denote the market value of the convertible bond upon call at time t.

Assumption A. π is an RCLL process such that

$$C_t \leq \pi_t, \quad \forall\, 0 \leq t < \tau_d. \tag{1}$$

This assumption is always satisfied when there is no notice period (i.e. when $\delta = 0$) since in this case we have that $\pi_t = C_t$. Also, if (1) is not satisfied then an obvious arbitrage opportunity arises for the holder; it suffices to redeem the CB instantaneously upon any call at time t such that (1) fails to hold. Note that in the context of the Standard Market Model for CBs, the above Assumption can be formally established, rather than postulated.

2 Decomposition of the Cash Flows

2.1 Discounted cash flows

- Let us first specify the cash flows from the perspective of the bondholder.
- We are given a CB, including a default time τ_d modeled as a $\mathbb{G}$-stopping time, a lifting time for call protection $\bar{\tau} \in \mathcal{G}_T^{t_0}$. Let also be given $t \in [t_0, T]$, and stopping times $(\tau_p, \tau_c) \in \mathcal{G}^t \times \bar{\mathcal{G}}^t$.
- Recall that we denote $\tau_c \wedge \tau_p$ by τ.

We temporarily define the cumulative discounted future cash flows of the CB at time t as follows:

$$\begin{aligned}\beta_t \varphi_t^{temp}(\tau_d, \tau_p, \tau_c) \quad = \quad & \sum_{t < T_i \leq \tau \wedge T,\, T_i < \tau_d} \beta_{T_i}\, c_i + \mathbf{1}_{t < \tau_d \leq \tau \wedge T}\, \beta_{\tau_d} (R_{\tau_d} X_{\tau_d} \vee \kappa S_{\tau_d}) \\ & + \mathbf{1}_{\tau_d > \tau \wedge T}\, \beta_\tau \Big(\mathbf{1}_{\tau = \tau_p < T} P_{\tau_p} + \mathbf{1}_{\tau = \tau_c < \tau_p, \tau_c < T} \pi_{\tau_c} + \mathbf{1}_{\tau \geq T} N_T \Big).\end{aligned}$$

Note that if $\bar{C} = \infty$ or $\bar{\tau} = T$, then we effectively deal with a non-callable CB (PCB).

Discounted cash flows (continued)

Observe that $\varphi_t^{temp}(x, y, z) \equiv \varphi_t^{temp}(x, y \wedge T, z \wedge T)$ for any $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+ \cup \{\infty\}$, and $y, z \in [t, \infty]$. Moreover, if, additionally, $z \wedge x \geq \bar{t} \wedge x$ for some $\bar{t} \leq T$, then $(z \wedge T) \wedge x \geq \bar{t} \wedge x$.

So all CB covenants can be properly taken into account using stopping times $(\tau_p, \tau_c) \in [t, T]$, rather than $(\tau_p, \tau_c) \in [t, T] \cup \infty$, provided that we agree on the payoff N_T if $\tau_p = \tau_c = T < \tau_d$. Finally, we shall use the following definition of the cash flows, for any given $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$.

Definition. The cumulative discounted cash flows $\varphi \equiv \varphi_t(\tau_d, \tau_p, \tau_c)$ associated with the CB over the time horizon $[t, T]$ is given by

$$\begin{aligned} \beta_t \varphi_t(\tau_d, \tau_p, \tau_c) \quad = \quad & \sum_{t < T_i \leq \tau,\, T_i < \tau_d} \beta_{T_i} c_i + \mathbf{1}_{t < \tau_d \leq \tau} \, \beta_{\tau_d} (R_{\tau_d} X_{\tau_d} \vee \kappa S_{\tau_d}) \\ & + \mathbf{1}_{\tau_d > \tau} \, \beta_\tau \Big(\mathbf{1}_{\tau = \tau_p < T} P_{\tau_p} + \mathbf{1}_{\tau = \tau_c < \tau_p} \pi_{\tau_c} + \mathbf{1}_{\tau = T} N_T \Big). \end{aligned}$$

2.2 Ex Dividend, Cum Dividend and Hybrid Cash Flows

Traditionally in finance, ex-dividend cash flows are considered, such as

$$\beta_t \varphi_t^{ex}(\tau_d, \tau_p, \tau_c) \equiv \sum_{t < T_i \leq \tau,\, T_i < \tau_d} \beta_{T_i}\, c_i + \mathbf{1}_{t < \tau_d \leq \tau}\, \beta_{\tau_d} (R_{\tau_d} X_{\tau_d} \vee \kappa S_{\tau_d})$$

$$+ \mathbf{1}_{t < \tau < \tau_d}\, \beta_\tau \Big(\mathbf{1}_{\tau = \tau_p < T} P_{\tau_p} + \mathbf{1}_{\tau = \tau_c < \tau_p} \pi_{\tau_c} + \mathbf{1}_{\tau = T} N_T \Big) \,.$$

However, in the context of Dynkin game optimization problems, ex-dividend cash flows are inappropriate. So one can check that $\varphi_t^{ex}(x, y, t)$ is equal to 0, for any $t \in [t_0, T]$, $x \in \mathbb{R}_+ \cup \{\infty\}$ and $y \in [t, \infty]$. Therefore ex-dividend cash flows would imply that a possibility for the issuer is to call the CB instantaneously (or at $t = \bar{\tau}$, if there are call protections) at no cost, which of course is not the case.

2.3 Embedded defaultable bond representation

Discounted cash flows can be decomposed as follows

$$\beta_t \varphi_t(\tau_d, \tau_p, \tau_c) = \beta_t \phi_t(\tau_d) + \beta_t \psi_t^{temp}(\tau_d, \tau_p, \tau_c)$$

where ϕ defines the embedded defaultable bond

$$\beta_t \phi_t(\tau_d) = \sum_{t<T_i \le T,\, T_i<\tau_d} \beta_{T_i} c_i + \mathbf{1}_{t<\tau_d\le T}\, \beta_{\tau_d} R_{\tau_d} X_{\tau_d} + \mathbf{1}_{\tau_d>T}\, \beta_T \bar{N}$$

and ψ^{temp} satisfies

$$\begin{aligned}\beta_t \psi_t^{temp}(\tau_d, \tau_p, \tau_c) = {} & \mathbf{1}_{t<\tau_d\le\tau}\, \beta_{\tau_d} \Big(\kappa S_{\tau_d} - R_{\tau_d} X_{\tau_d}\Big)^+ + \\ & \mathbf{1}_{\tau_d>\tau}\, \beta_\tau \Big(\mathbf{1}_{\tau=\tau_p<T} \left(P_{\tau_p} - \phi_{\tau_p}(\tau_d)\right) + \mathbf{1}_{\tau=\tau_c<\tau_p} \left(\pi_{\tau_c} - \phi_{\tau_c}(\tau_d)\right)\Big) \\ & \mathbf{1}_{\tau_d>\tau}\, \beta_\tau \mathbf{1}_{\tau=T} \left(\kappa S_T - \bar{N}\right)^+ .\end{aligned}$$

Embedded defaultable bond representation (continued)

In the case of the cum-dividend convention, the cash flows decomposition would become:

$$\beta_t \varphi_t^{cum}(\tau_d, \tau_p, \tau_c) = \beta_t \phi_t^{cum}(\tau_d) + \beta_t \psi_t^{cum}(\tau_d, \tau_p, \tau_c) \tag{2}$$

where

$$\beta_t \phi_t^{cum}(\tau_d) = \sum_{t \le T_i \le T, T_i < \tau_d} \beta_{T_i} c_i + \mathbf{1}_{t \le \tau_d \le T}\, \beta_{\tau_d} R_{\tau_d} X_{\tau_d} + \mathbf{1}_{\tau_d > T}\, \beta_T \left(\bar{N} + A_T \right)$$

and

$$\beta_t \psi_t^{cum}(\tau_d, \tau_p, \tau_c) = \mathbf{1}_{t \le \tau_d \le \tau}\, \beta_{\tau_d} \Big(\kappa S_{\tau_d} - R_{\tau_d} X_{\tau_d} \Big)^+ + \mathbf{1}_{\tau_d > \tau}\, \beta_\tau \times$$
$$\left(\mathbf{1}_{\tau = \tau_p < T} \left(P_{\tau_p} - \phi_{\tau_p}^{ex}(\tau_d) \right) + \mathbf{1}_{\tau = \tau_c < \tau_p} \left(\pi_{\tau_c} - \phi_{\tau_c}^{ex}(\tau_d) \right) + \mathbf{1}_{\tau = T} \left(\kappa S_T - \bar{N})^+ \right) \right) .$$

This is not a consistent decomposition, as far as the choice of a convention regarding “ex” or “cum” is concerned.

3 Dynkin Games and Price Decomposition

3.1 Pricing theorem for the CB

Theorem A. $\Pi = (\Pi_t)_{t_0 \le t \le T}$ is an arbitrage price process for the CB iff there exists $\mathbb{Q} \in \mathcal{M}$ for which Π is a $\mathbb{G}$-semimartingale and Π is the value process of the related family of Dynkin games:

$$\begin{aligned} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \Big(\varphi_t(\tau_d, \tau_p, \tau_c) \,\Big|\, \mathcal{G}_t \Big) &= \Pi_t \\ = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}} \Big(\varphi_t(\tau_d, \tau_p, \tau_c) \,\Big|\, \mathcal{G}_t \Big), &\quad t_0 \le t \le T. \end{aligned}$$

Proof. Kallsen & Kühn (2004) □

This theorem applies, in particular, to:

- a non-callable CB (PCB), in which case the saddle point value collapses to a $\mathbb{Q}$-Snell envelope with respect to stopping times $\tau_p \in \mathcal{G}_T^t$,
- a defaultable bond, in which case the saddle point value reduces to a $\mathbb{Q}$-expectation.

3.2 Application to the Embedded PCBs

Following Kwok and Lau (2004), we shall consider from now on that when a CB is called, it is actually replaced by a non callable convertible bond (PCB) with the same characteristics as the CB, except that (recall that $t^\delta = (t+\delta) \wedge T$):

- by definition, the PCB has no call clause,
- the maturity of the PCB is equal to τ_c^δ and its nominal is equal to C,
- the schedule of the coupons of the PCB is the trace on $(\tau_c, \tau_c^\delta]$ of the schedule of the coupons of the CB,
- the effective put price of the PCB is equal to the effective call price C_u of the original CB, at any time $u \in [\tau_c, \tau_c^\delta)$.

The market price processes of these PCBs will be denoted by $(\Pi_u^t)_{u \in [t, t^\delta]}$, $0 \le t \le T$. In particular, $\pi_t \equiv \Pi_t^t$, $0 \le t \le T$.

Since PCBs are special cases of CBs, all the results obtained for CBs apply in particular to the embedded PCBs. Moreover, as PCBs have no call clause, any assumptions regarding π in these results are trivially satisfied.

Theorem A'. Consider a CB satisfying our general assumptions, except maybe for $C_t \leq \pi_t$. We assume that the embedded PCB price process Π^t is arbitrage-free, $t_0 \leq t \leq T$. Then $C_t \leq \pi_t$. Moreover, there exists $\mathbb{Q}^t \in \mathcal{M}$ for which Π^t is a $\mathbb{G}$-semimartingale, and Π^t is the Snell envelope process of the family of related American claim problems, that is,

$$\Pi^t_u = \operatorname{esssup}_{\tau_p \in \mathcal{G}^u_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\widetilde{\varphi}_u(\tau_d, \tau_p) \,\Big|\, \mathcal{G}_u\Big), \quad t \leq u \leq t^\delta,$$

where

$$\begin{aligned} \beta_t \widetilde{\varphi}_t(\tau_d, \tau_p) \quad = \quad & \sum_{t < T_i \leq \tau_p,\, T_i < \tau_d} \beta_{T_i} c_i + \mathbf{1}_{t < \tau_d \leq \tau_p} \, \beta_{\tau_d} (R_{\tau_d} X_{\tau_d} \vee \kappa S_{\tau_d}) \\ & + \mathbf{1}_{\tau_d > \tau_p} \, \beta_{\tau_p} C_{\tau_p} . \end{aligned}$$

Note that it is not clear whether $\mathbb{Q}^t$ has to be equal to $\mathbb{Q}$, nor whether it has to be independent of t.

3.3 Game option representation

For any $\mathbb{Q} \in \mathcal{M}$, $t \in [t_0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, we define $\Phi_t = \mathbb{E}_{\mathbb{Q}}\big(\phi_t(\tau_d) \,\big|\, \mathcal{G}_t\big)$, and

$$\begin{aligned}\beta_t \psi_t(\tau_d, \tau_p, \tau_c) \quad = \quad & \mathbf{1}_{t<\tau_d \leq \tau}\, \beta_{\tau_d} \Big(\kappa S_{\tau_d} - R_{\tau_d} X_{\tau_d}\Big)^+ + \\ & + \mathbf{1}_{\tau_d > \tau}\, \beta_\tau \Big(\mathbf{1}_{\tau=\tau_p<T}\, \big(P_{\tau_p} - \Phi_{\tau_p}\big) + \mathbf{1}_{\tau=\tau_c<\tau_p}\, \big(\pi_{\tau_c} - \Phi_{\tau_c}\big)\Big) \\ & + \mathbf{1}_{\tau_d > \tau}\, \beta_\tau \mathbf{1}_{\tau=T}\, \big(\kappa S_T - \bar{N}\big)^+ .\end{aligned}$$

Recall that $\phi_t(\tau_d)$ represents embedded defaultable bond. The discounted ex-dividend cash flows $\beta_t \psi_t(\tau_d, \tau_p, \tau_c)$ represents the embedded $\mathbb{Q}$-game option to exchange this defaultable bond. Note that the cash flows $\psi_t(\tau_d, \tau_p, \tau_c)$ depends on $\mathbb{Q}$, via Φ.

3.4 Pricing theorem for the game option

Theorem A". Let a martingale measure $\mathbb{Q} \in \mathcal{M}$ be given. Then the process $\Psi = (\Psi_t)_{0 \leq t \leq T}$ is an arbitrage-free price process for the embedded $\mathbb{Q}$-game option if and only there exists $\widetilde{\mathbb{Q}} \in \mathcal{M}$ for which Ψ is a $\mathbb{G}$-semimartingale, and Ψ is the value process of the family of Dynkin games with cost criteria $\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\psi_t(\tau_d, \tau_p, \tau_c) \,\Big|\, \mathcal{G}_t\Big)$ on $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, $t_0 \leq t \leq T$.

Remark. The pricing measure $\widetilde{\mathbb{Q}}$ does not have to be the same as the pricing measure $\mathbb{Q}$ that appears in the definition of the embedded $\mathbb{Q}$-game option.

3.5 Price relationships

Theorem B. Given $\mathbb{Q} \in \mathcal{M}$, then Φ is an arbitrage-free price process for the embedded defaultable bond, associated with the pricing measure $\mathbb{Q}$. Moreover,

(i) If Π is an arbitrage-free price process for a CB, associated with the pricing measure $\mathbb{Q}$, then $\Psi \equiv \Pi - \Phi$ is an arbitrage-free price process for the embedded $\mathbb{Q}$-game option, associated with the pricing measure $\widetilde{\mathbb{Q}} = \mathbb{Q}$.

(ii) If Ψ is an arbitrage-free price process for the embedded $\mathbb{Q}$-game option, associated with the pricing measure $\widetilde{\mathbb{Q}} = \mathbb{Q}$ then $\Pi = \Phi + \Psi$ is an arbitrage-free price process for the CB, associated with the pricing measure $\mathbb{Q}$.

Corollary. Under the assumptions of Theorem B.(i) or (ii), we have:

(i) $\Pi \geq \Psi$;

(ii) In the special case of a zero-coupon CB we have $\Pi \geq \Phi$.

3.6 Market data for CBs and related options

CB	Stock Price	Nominal	CB Price	Credit Spread	CB IV
A	8.39	16.18	17.42	135 bp	30.2%
B	19.54	24.40	103.45	15 bp	24.0%
C	44.95	52.50	219.45	15 bp	22.8%

Table 1: *CB data on names of the CAC40 on May 10, 2005*

CB	CB Expected Life	Closest Option Strike/Expiry	Option IV
A Jan-11	Oct-10	13.0 Dec-09	30.7%
B Jan-07	Dec-06	24.0 Dec-06	22.3%
C Jan-07	Dec-06	52.5 Dec-06	20.9%

Table 2: *Closest listed options details*

3.7 CB spread and implied volatility

Assume that we are given the market price process of a CB, which happens to be the arbitrage price associated with the pricing measure $\mathbb{Q}$ (cf. Theorem A).

Then Theorem B can be used to give a definite meaning to the terms: "CB spread" and "CB implied volatility".

- One can define the CB spread as the credit spread of the embedded defaultable bond, and the CB implied volatility as the Black-Scholes implied volatility of the embedded $\mathbb{Q}$-game exchange option. However, it is not guaranteed that it is possible to map every possible arbitrage-free price process for the game exchange option, to a well-defined and unique Black-Scholes implied volatility process.

- Note that the embedded $\mathbb{Q}$-game exchange option can be thought of as an equity option, but with floating strike, equal at time t to the value Φ_t of the embedded defaultable bond at date t.

Conclusions & Open Problems

Conclusions

We derived some results regarding defaultable convertible bonds, in particular:
• the decomposition of the payoff process of a CB in terms of embedded defaultable bond and a game option,
• the decomposition of an arbitrage price of a CB (under $\mathbb{Q}$, say) in terms of an arbitrage price of the embedded defaultable bond under $\mathbb{Q}$ and an arbitrage price of the embedded $\mathbb{Q}$-game option under $\widetilde{\mathbb{Q}}$,
• the interpretation of market data, the CB spread and CB implied volatility, in terms of well-established pricing theory,

Open Problems

Relationships between various martingale measures, $\mathbb{Q}$, $\widetilde{\mathbb{Q}}$ and $\mathbb{Q}^t$, should be further investigated.