Quantitative Methods in Finance 2005

Sydney, 14-17 December 2005

Calibration of a nonlinear feedback option pricing model

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Outline

- The Black-Scholes model relies on the hypothesis that the market is *complete*, *frictionless* and *perfectly liquid*.
- We provide a microfoundation of the asset price dynamics where the price is defined *in equilibrium* with some traders employing a trading strategy to hedge a contingent claim.
- $\Rightarrow$ FULLY NONLINEAR PDE

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- analysis of the impact of hedging on market volatility;
- illiquid markets analysis;
- use of portfolio insurance strategies.

References:

- *Microeconomic equilibrium approach*: [Föllmer-Schweizer, '93];
- [Brennan-Schwartz, '89], [Frey-Sremme, '97], [Platen-Schweizer, '98];
- [Sircar-Papanicolaou, '98], [Frey, '98], [Schönbucher-Wilmott, 2000], [Mancino-Ogawa, '03].

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$\Delta = u_S(t, S)$ $\Leftarrow$ self-financing replicating strategy

Itô formula

ITÔ FORMULA

$\Downarrow$

Price dynamics

ITÔ FORMULA

$\Downarrow$

$$dS_t = \tilde{\mu}(t, S_t, u_{xx})dt + \tilde{\sigma}(t, S_t, u_{xx})dW_t$$

$$\tilde{\sigma}(t, S_t, u_{xx}) = -\frac{\rho S_t}{\gamma + S_t \alpha u_{xx}}$$

$$\tilde{\mu}(t, S_t, u_{xx}) = \frac{\tilde{\sigma}(t, S_t, u_{xx})}{(\gamma + S_t \alpha u_{xx})^2}\left[m\rho(\gamma + S_t \alpha u_{xx}) - \frac{1}{2}\rho\gamma\right].$$

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The *existence* and *uniqueness* result in a suitable functions space is given in [Mancino-Ogawa, '03] for $r = 0$, $\gamma < 0$, $\alpha > 0$, $\rho \neq 0$ and for $\lambda := \frac{\alpha}{-\gamma}$ smaller than a particular $\overline{\lambda}$ depending on f.

Preliminary numerical results

◇ *Semi-implicit Finite Difference Method* on uniform mesh, with *one-sided treatment of the boundary conditions*.

◇ We evaluate the capability of the above model of addressing some phenomena observed in financial markets, such as

- the effect of the hedging strategies on the *price process*, on the *greeks* and on the implicit *probability density function*;

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- the *volatility effect*;

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- the effect of the hedging strategies on the *price process*, on the *greeks* and on the implicit *probability density function*;
- the *volatility effect*;
- the *smile-skewness effect*.

$$F(\Delta) = \\ = \alpha\Delta + \beta$$

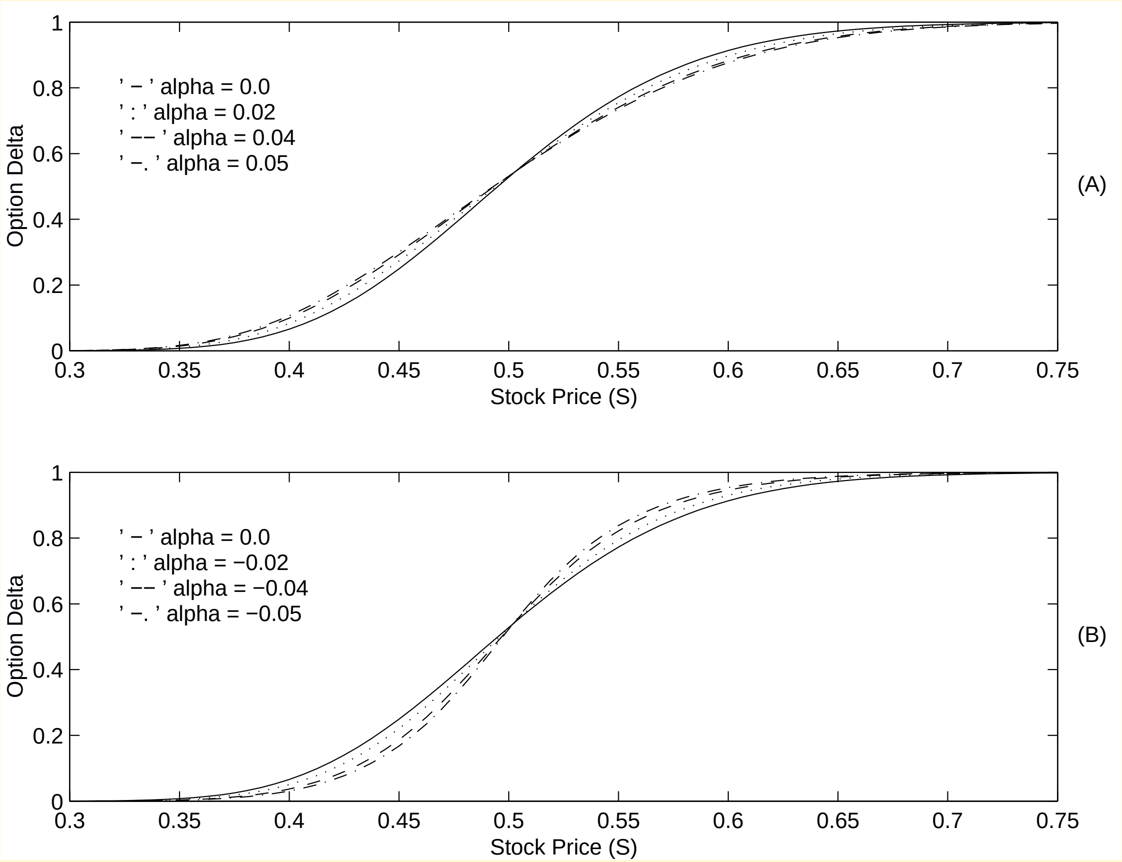

Figure 1: Option delta. Parameter values: $\rho = 0.2$, $\gamma = -1$, $r = 0$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.02,\ 0.04,\ 0.05$ (Panel A), $\alpha = 0,\ -0.02,\ -0.04,\ -0.05$ (Panel B).

Probability density function (PDF)

○

- Date-T payoff S_T
- Continuously compounded τ-period return $u_\tau \equiv \log(S_T/S_t)$

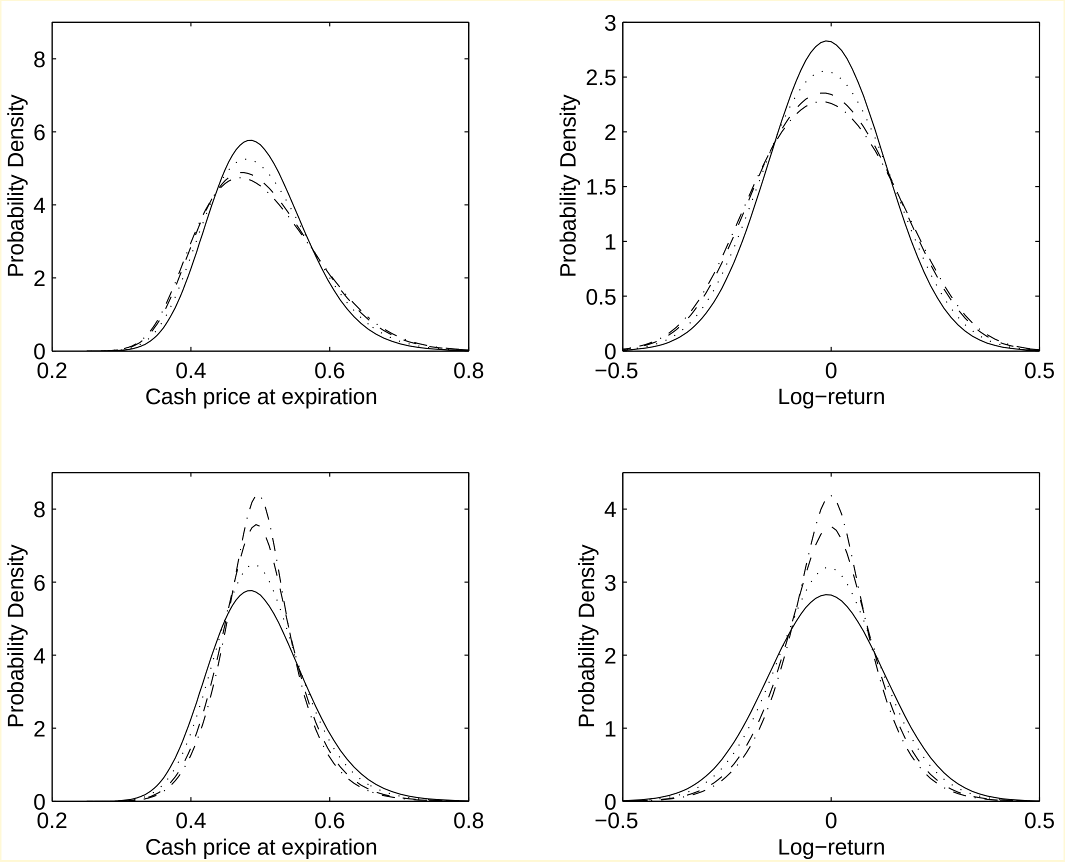

Figure 2: Risk-neutral PDF of S_T and of u_τ. Parameter values: $\rho = 0.2$, $\gamma = -1$, $r = 0$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.02,\ 0.04,\ 0.05$ (on the top), $\alpha = 0,\ -0.02,\ -0.04,\ -0.05$ (on the bottom). $x = S_t$ is taken to be equal to K.

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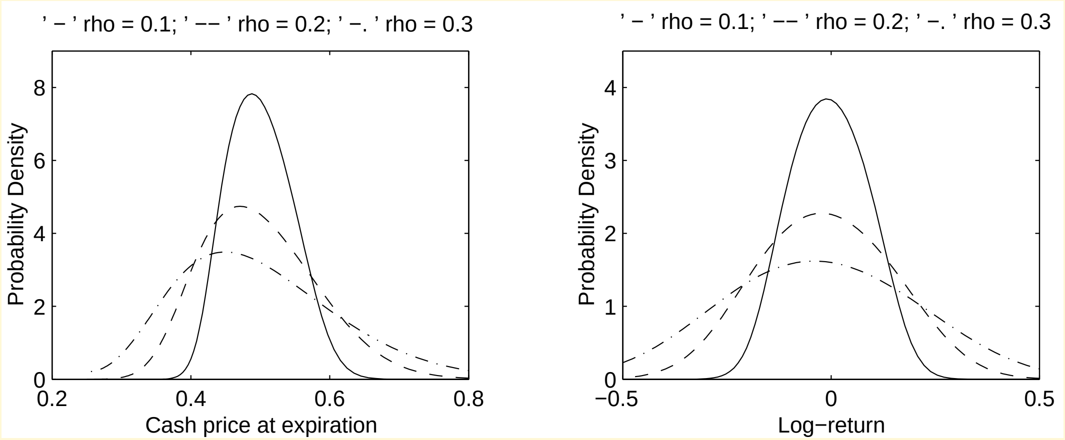

Figure 2: Risk-neutral PDF for S_T and for the continuously compounded τ-period return u_τ. Parameter values: $\alpha = 0.05$, $\gamma = -1$, $r = 0$, $K = 0.5$, $T = 0.5$, $t = 0$, $\rho = 0.1,\ 0.2,\ 0.3$. $x = S_t$ is taken to be equal to K. Discretization parameters: $k = 0.005$ and $h = 0.0125$.

Modified volatility

$\tilde{\sigma}(x, u_{xx}) = \frac{\beta x}{1 - \lambda x u_{xx}}$

$\beta = \frac{\rho}{-\gamma}$, $\lambda = \frac{\alpha}{-\gamma}$

Feedback impact:

$F_x(\Delta) = F'(\Delta) u_{xx} =$

$= \alpha u_{xx}$

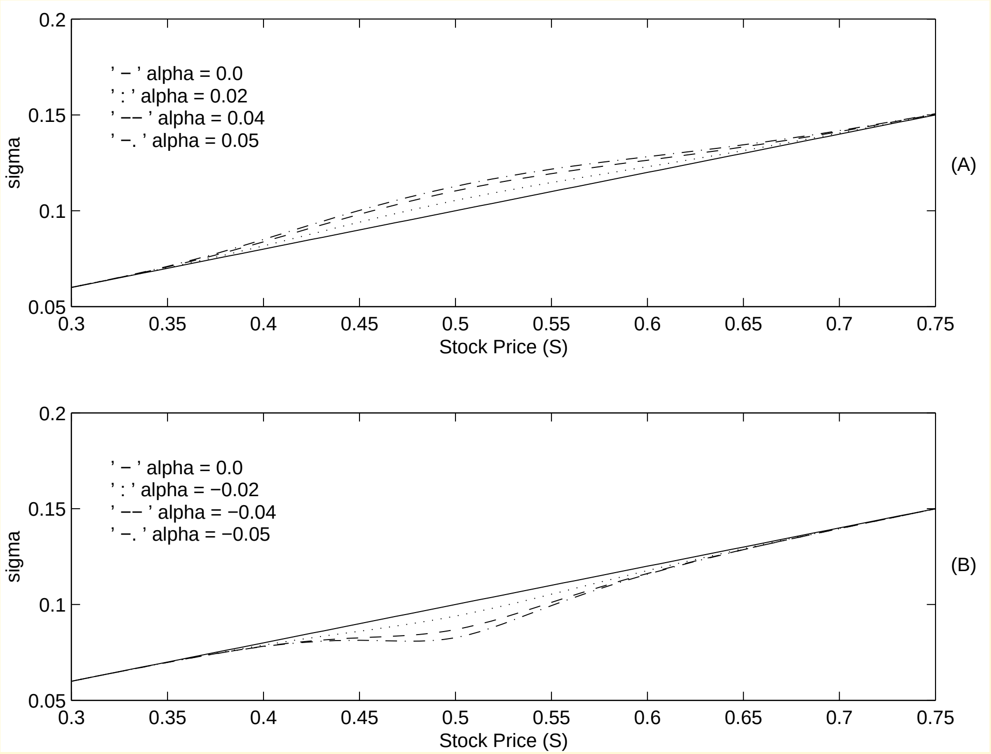

Figure 3: Modified volatility. Parameter values: $\rho = 0.2$, $\gamma = -1$, $r = 0$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.02,\ 0.04,\ 0.05$ (Panel A), $\alpha = 0,\ -0.02,\ -0.04,\ -0.05$ (Panel B).

Implied volatility

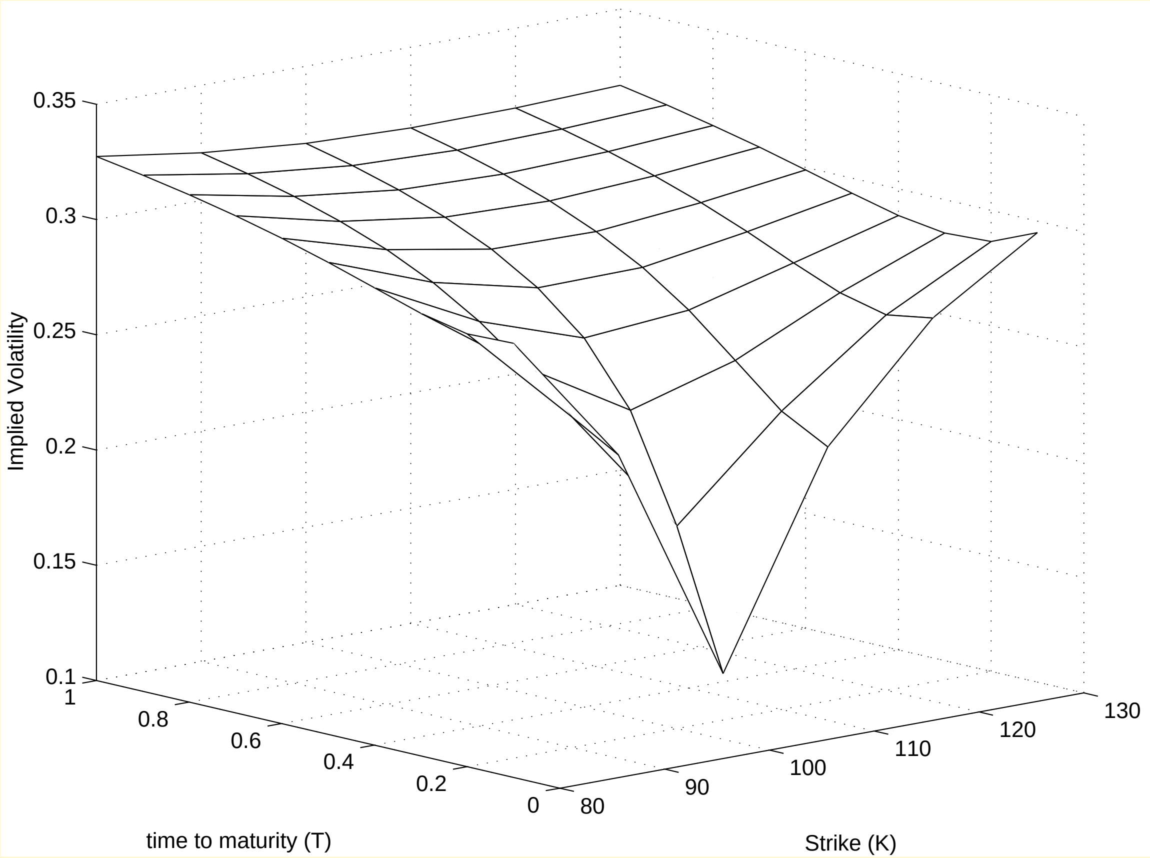

Figure 4: Implied volatility. Parameter values: $\rho = 0.2$, $\gamma = 0.5$, $r = 0$, $\alpha = 0.05$, $S_t = 100$, $t = 0$; $T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$; $K = 80, 90, 100, 110, 120, 130$.

Calibration to market data

◇ Given the "spot" riskless interest rate r, we have

$$u = u(t, S_t; r, K, T; \beta, \lambda),$$

where $\beta = \frac{\rho}{-\gamma}$ and $\lambda = \frac{\alpha}{-\gamma}$.

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◇ DETERMINATION OF PARAMETERS: given prices C_i for traded options with strikes K_i and maturity T_i, $1 \leq i \leq m$, parameters β^*, λ^* are estimated as

$(\beta^*, \lambda^*) =$
$\arg\min_{\beta,\lambda} \sum_{i=1}^{m} [u(t, S_t; r, K_i, T_i; \beta, \lambda) - C_i]^2$

i.e. by minimizing the squared distance of the theoretical prices from the observed option prices.
(Nonlinear Least Squares method)

Alternative models

- *Black-Scholes* model (BS):

$$\boxed{dS_t = \mu S_t dt + \sigma S_t dW_t;}$$

- *Jump-Diffusion* model (JD):

$$\boxed{dS_t = (\mu - \lambda_j k_j) S_t dt + \sigma_j S_t dW_t + (J_t - 1) S_t dq,}$$

where $k_j = E[J_t - 1]$; if the logarithm of J_t is normally distributed, with standard deviation δ_j

$C_{JD} = \sum_{n=0}^{\infty} \frac{e^{-\lambda' \tau}(\lambda' \tau)^n}{n!} C_{BS}(S, \tau; \sigma_n, r_n);$

- *Heston* model (H):

$$\boxed{\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t &= [\theta_v - k_v V_t] dt + \sigma_v \sqrt{V_t} dW_t^2, \end{aligned}}$$

where W^1 and W^2 are two Wiener processes with correlation ρ.

Data description

- S&P 500 call option prices for the sample period January 4, 1993 to December 31, 1993 (252 days).

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Variable	Mean	Std. Dev.	Min	Max
Call price C ($)	18.29	17.32	0.0249	68.60
Implied volatility (%)	11.01	2.98	5.07	36.83
τ (days)	79.96	66.56	1	350
K (index points)	445.52	27.48	375	550
F (index points)	454.87	10.21	429.18	474.21
r (%)	3.06	0.08	2.85	3.21

Table 1: Summary statistics for the data set. F denotes the S&P 500 futures value implied from the call and put prices and "Std. Dev." the sample standard deviation of the variable.

Data description

- S&P 500 call option prices for the sample period January 4, 1993 to December 31, 1993 (252 days).
- We divide the option data into several categories according to either moneyness S_t/K or term to maturity τ.
  - *at-the-money* (ATM) if $S_t/K \in (0.98, 1.02]$,
  - *out-of-the-money* (OTM) if $S_t/K \leq 0.98$,
  - *in-the-money* (ITM) if $S_t/K > 1.02$.
  - *short-term* if $\tau < 60$ days,
  - *medium-term* if $60 \leq \tau \leq 180$ days,
  - *long-term* if $\tau > 180$ days.

In-sample performance

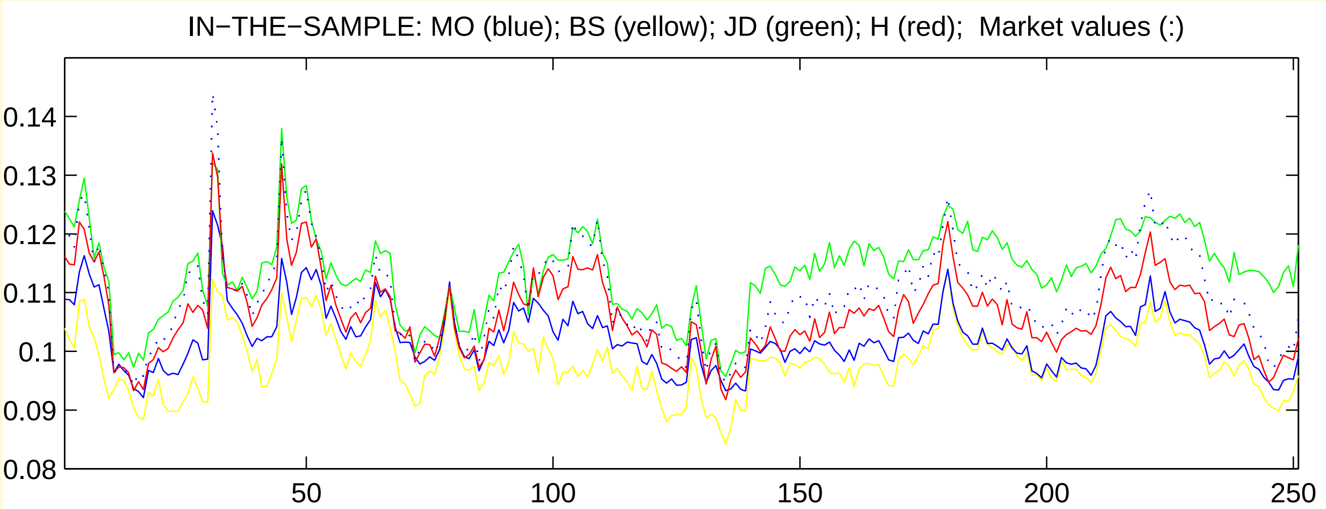

Figure 5: Daily average implied volatility series for the four models during the period January 4, 1993 to December 31, 1993.

Model	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Imp. Vol.	SSE	APE	PPE	MISP
BS	9.82%	-	-	-	-	9.82%	29.08	0.56	0.39	0.92
	(0.03)					(0.03)				
MO	12.12%	-0.02	-	-	-	10.21%	23.35	0.51	0.49	0.93
	(0.06)	(0.06)				(0.03)				
JD	7.12%	0.77	4.48%	-0.07	-	11.34%	2.88	0.16	0.08	0.45
	(0.03)	(1.25)	(0.10)	(0.07)		(0.05)				
H	8.80%	6.17	46.62%	8.62%	-0.63	10.64%	0.66	0.08	0.06	0.22
	(0.09)	(12.65)	(0.74)	(0.14)	(0.51)	(0.04)				
Sample	-	-	-	-	-	10.96%				

Table 1: Average implied volatility and relevant structural parameters for the whole data set. The parameters are $\Phi = (\sigma)$ for the BS model, $\Phi = (\beta, \lambda)$ for the MO model, $\Phi = (\sigma_j, \lambda_j, \delta_j, k_j)$ for the JD model and $\Phi = (\sqrt{V}, k_v, \sigma_v, \theta_v, \rho)$ for the H model. APE, PPE and MISP are respectively the daily-averaged absolute and percentage pricing errors and the daily-averaged mispricing index.

In-sample performance

Model	Class	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Imp. Vol.	SSE	APE	PPE
BS	OTM	8.67%	-	-	-	-	8.67%	2.33	0.24	0.49
	ATM	9.59%	-	-	-	-	9.59%	3.13	0.37	0.11
	ITM	11.49%	-	-	-	-	11.49%	4.53	0.30	0.01
MO	OTM	9.30%	-0.01	-	-	-	8.72%	1.77	0.22	0.59
	ATM	11.78%	-0.02	-	-	-	9.46%	1.29	0.26	0.08
	ITM	17.09%	-0.05	-	-	-	12.28%	2.26	0.18	0.007
JD	OTM	6.76%	0.44	5.04%	-0.12	-	8.20%	0.12	0.06	0.14
	ATM	7.15%	0.47	9.15%	-0.10	-	9.31%	0.27	0.11	0.03
	ITM	8.73%	0.44	12.06%	-0.04	-	13.64%	0.55	0.10	0.004
H	OTM	8.70%	3.76	27.88%	5.65%	-0.73	8.20%	0.06	0.04	0.10
	ATM	8.70%	4.84	41.41%	7.26%	-0.65	9.39%	0.19	0.08	0.03
	ITM	9.74%	5.86	77.13%	9.61%	-0.21	12.86%	0.17	0.06	0.002
Sample	OTM	-	-	-	-	-	8.25%	-	-	-
	ATM	-	-	-	-	-	9.42%	-	-	-
	ITM	-	-	-	-	-	13.17%	-	-	-

Table 2: Average implied volatility and relevant structural parameters for each moneyness-based category.

In-sample performance

Model	Class	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Imp. Vol.	SSE	APE	PPE
BS	Short	9.25%	-	-	-	-	9.25%	3.79	0.32	0.26
	Medium	9.78%	-	-	-	-	9.78%	14.96	0.68	0.39
	Long	10.35%	-	-	-	-	10.35%	19.06	1.13	0.89
MO	Short	12.39%	-0.02	-	-	-	10.36%	3.19	0.30	0.41
	Medium	12.74%	-0.02	-	-	-	10.22%	12.37	0.62	0.53
	Long	12.17%	-0.01	-	-	-	10.54%	16.34	1.05	0.95
JD	Short	7.14%	0.97	2.75%	-0.06	-	11.45%	0.27	0.08	0.06
	Medium	6.72%	0.88	3.69%	-0.07	-	10.54%	0.51	0.09	0.05
	Long	6.28%	0.75	5.23%	-0.08	-	10.49%	0.57	0.10	0.08
H	Short	9.01%	8.49	57.10%	10.07%	-0.56	10.93%	0.18	0.07	0.06
	Medium	8.02%	1.78	34.81%	5.19%	-0.63	10.47%	0.19	0.06	0.05
	Long	8.87%	1.64	20.81%	2.79%	-0.71	10.46%	0.15	0.04	0.04
Sample	Short	-	-	-	-	-	11.44%	-	-	-
	Medium	-	-	-	-	-	10.50%	-	-	-
	Long	-	-	-	-	-	10.44%	-	-	-

Table 3: Average implied volatility and relevant structural parameters for each maturity-based category.

In-sample performance

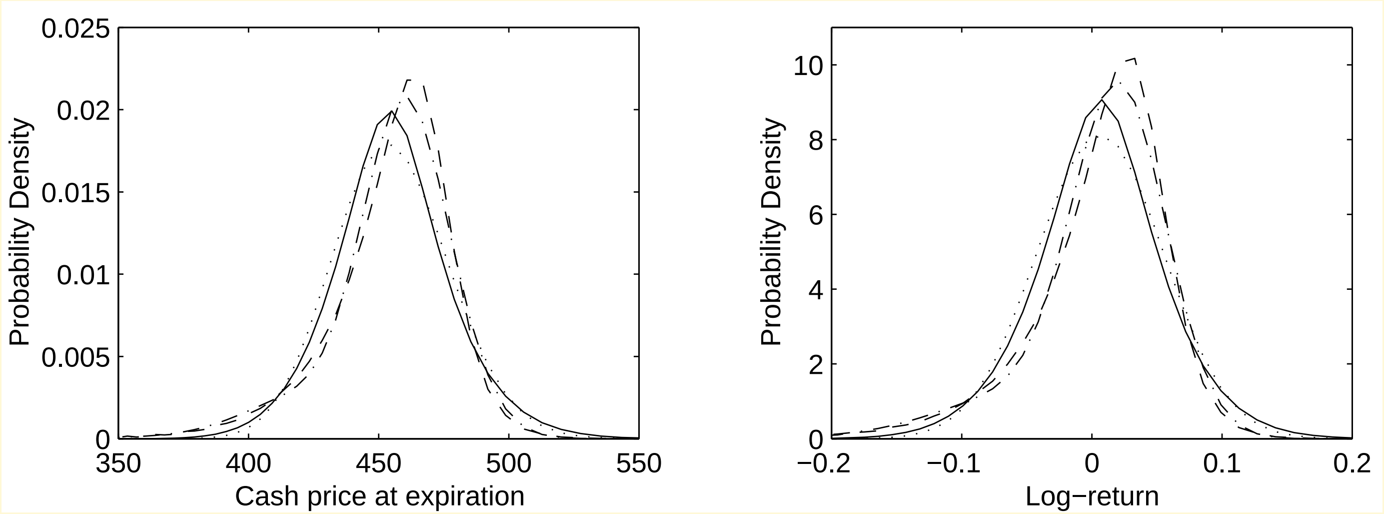

Figure 6: Risk-neutral PDF of S_T and of the continuously compounded τ-period return u_τ ($\tau = 0.25$). '-' MO model; ':' BS model; '- -' H model; '-.' JD model.

Out-of-sample performance

	Model	All Options	Moneyness-Based			Maturity-Based		
			OTM	ATM	ITM	Short	Medium	Long
APE	BS	0.57	0.26	0.41	0.32	0.33	0.69	1.15
	MO	0.52	0.25	0.31	0.20	0.31	0.63	1.08
	JD	0.20	0.11	0.20	0.14	0.12	0.17	0.19
	H	0.14	0.11	0.20	0.11	0.13	0.15	0.18
PPE	BS	0.40	0.53	0.12	0.01	0.27	0.39	0.90
	MO	0.50	0.64	0.09	0.008	0.42	0.54	1.02
	JD	0.10	0.20	0.05	0.006	0.08	0.07	0.10
	H	0.08	0.19	0.05	0.005	0.10	0.07	0.08
MISP	BS	0.90	0.74	0.59	-0.09	0.84	0.88	0.82
	MO	0.91	0.82	0.45	-0.29	0.91	0.90	0.84
	JD	0.40	0.13	-0.003	0.32	0.19	0.39	0.43
	H	0.17	0.02	-0.03	0.03	0.39	0.45	0.29

Table 4: Out-of-sample mispricing index (MISP), absolute (APE) and percentage (PPE) pricing errors between the market price and the model price for each call in a given moneyness-maturity category.

Concluding remarks

- Mancino-Ogawa (MO) model really represents an improvement with respect to BS.
- However, its in-sample performance remains worse than for JD and H, as a consequence of its limited capability of reproducing skewed probability distributions.
- Nevertheless, the JD and the H models are less stable through time (out-of-sample analysis) and the H model is somehow misspecified in terms of internal consistency.
- Therefore, nonlinear feedback due to hedging strategies can contribute to explain the strong pricing biases of the BS formula. Interesting and more reliable future developments of nonlinear feedback models should take into account the presence of a second source of randomness, such as a stochastic volatility.