Bivariate Multi-Fractal (BMF) Model

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Quantitative Methods in Finance Conference
School of Finance and Economics, University of Technology, Sydney
14-17 December 2005

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2. Bivariate Multi-Fractal Model.
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Review of Multifractal Model

Mulitifractal Model of Asset Returns (MMAR).

Mandelbrot et al. (1974, 1997).

Returns $x(t)$ are assumed to follow a compound process:

$$x(t) = B_H[\theta(t)]$$

$B[\cdot]$: Fractional Brownian motion

H: Hurst Exponent

$\theta(t)$: Cumulative distribution function of a multi-fractal measure

$B[\cdot]$ and $\theta(t)$ are independent.

Combinatorial MF:

Review of Multifractal Model

Iterative Version MMAR:

Markov-switiching Multi-Fractal (Calvet & Fisher, 2001):

$$x_t = \sigma \cdot \sqrt{\prod_{i=1}^{k} m_t^{(i)}} \cdot \varepsilon_t$$

Volatility components are renewed by transition probability γ_i

$$Prob\ (\text{new } m_t^{(i)}) = 1 - (1 - \gamma_1)^{(b^{k-1})}$$

⟹ Conceives volatility as a hierarchical, multiplicative process with heterogeneous components.

Review of Multifractal Model

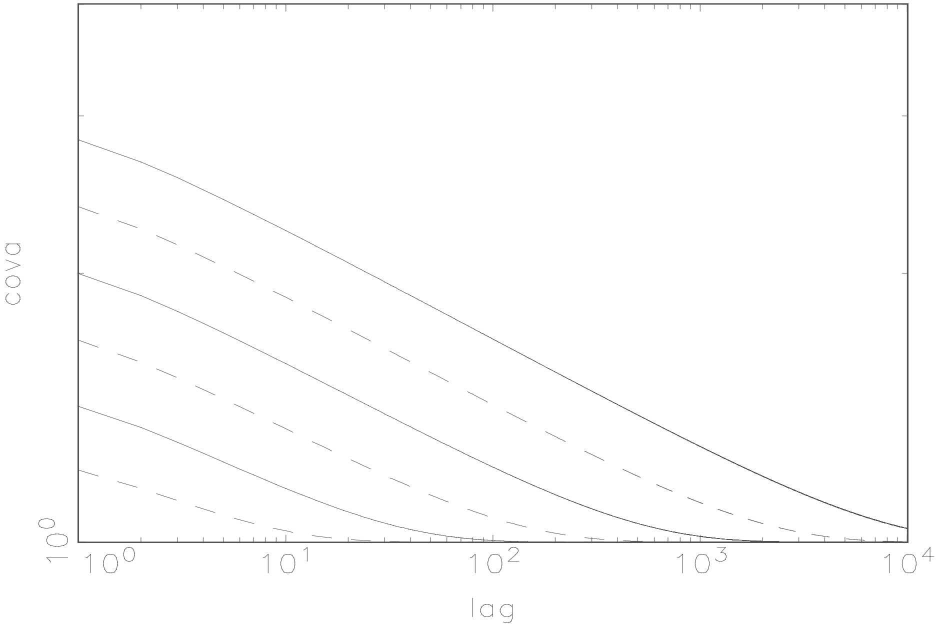
- Outliers.
- Volatility clustering.
- Asymptotic power-law behaviour of autocovariance function.

$$Cov(|x_t|^q, |x_{t+\tau}|^q) \propto \tau^{2d(q)-1}$$

Covariance of squared returns:

MF Binomial model:

$\{1.3,\ 0.7\}$, $k = 4, 6, \ldots, 14$



Bivariate Multifractal Model (BMF)

Two time series having certain amounts of joint cascade levels in both multifracal processes:

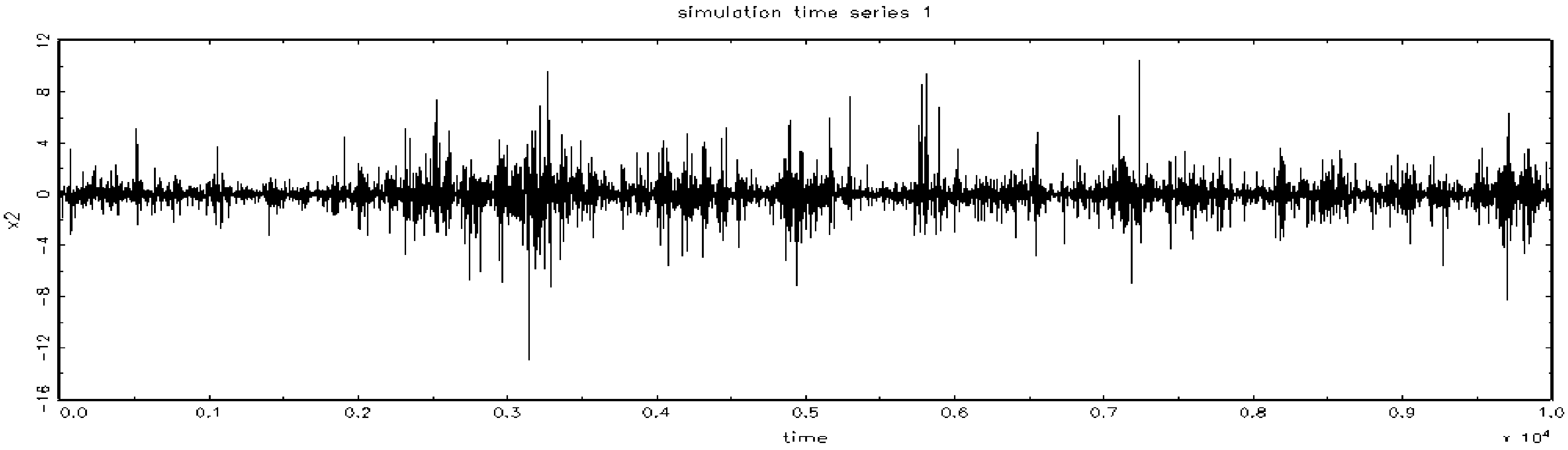
$$x_{t,q} = \sigma_q \cdot (\prod_{i=1}^{k} m_t^{(i)} \prod_{l=k+1}^{n} m_t^{(l)})^{1/2} \cdot u_{t,q} \qquad q = 1;\ 2, \quad u_{t,q} \sim BN(0, \Xi)$$

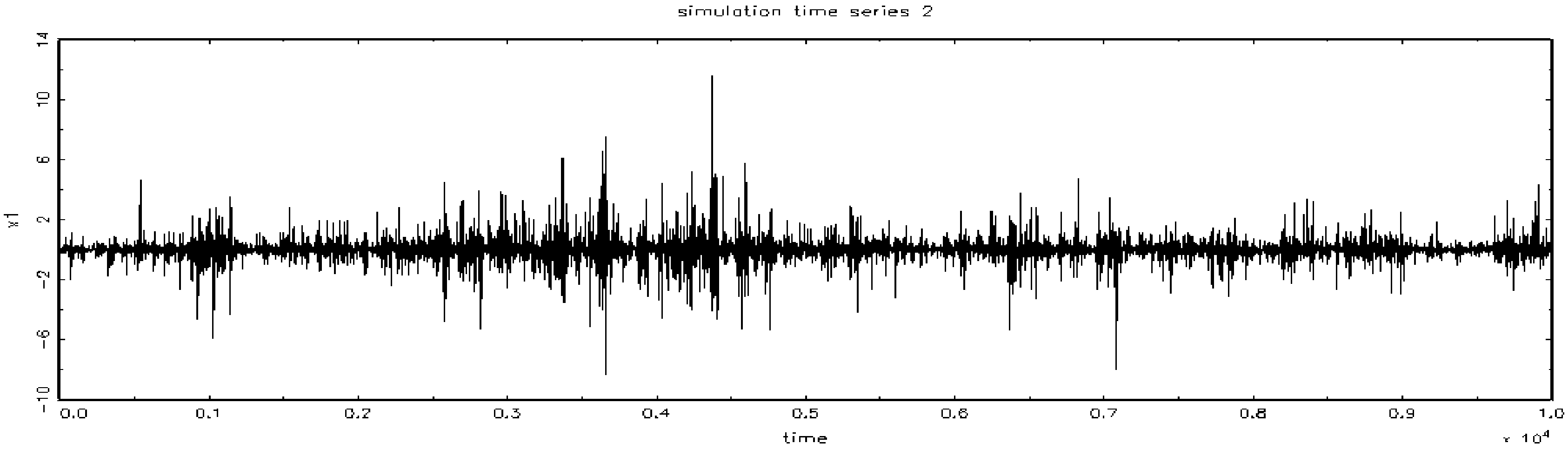
Transition probability:

$$\gamma_j = 2^{-(k'-j)}$$

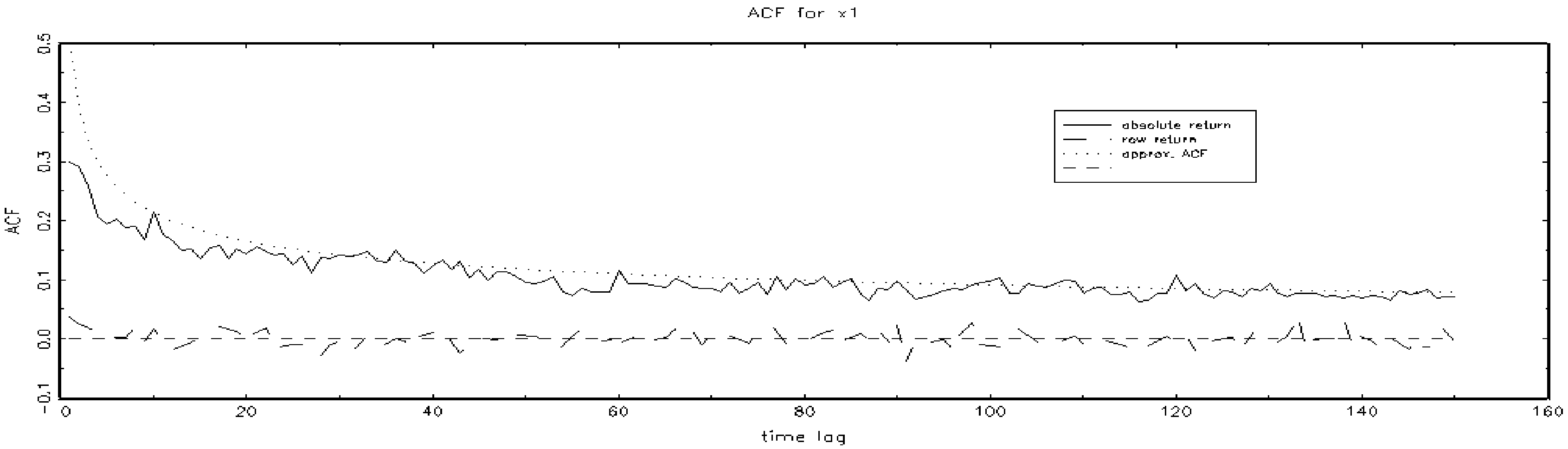
(depending on its rank within the hierarchical structure
or, remains unchanged with probability 1- *r*)

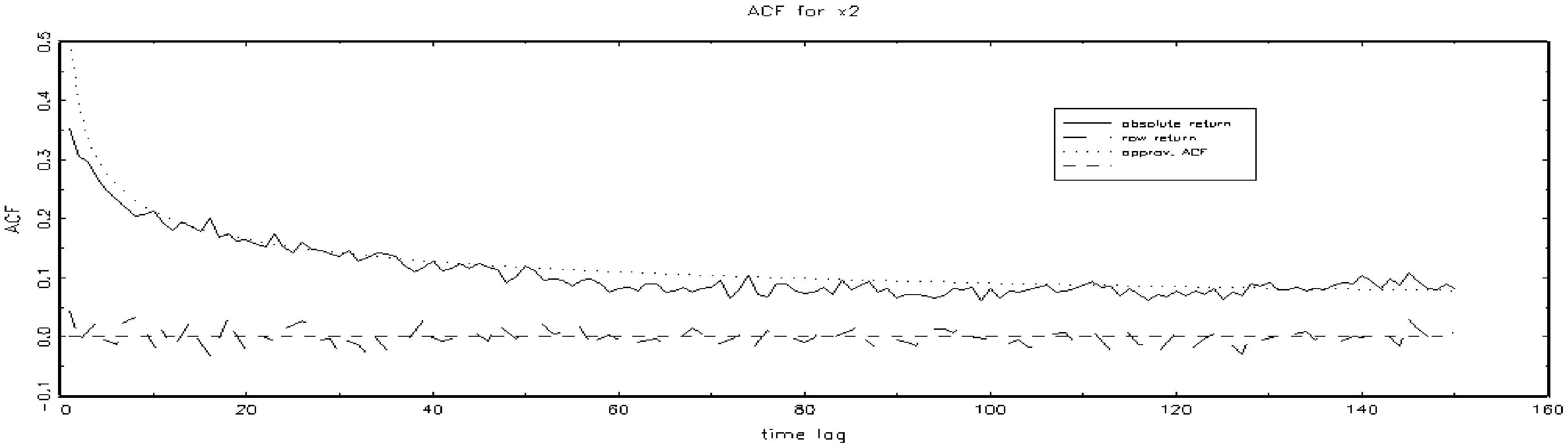
Bivariate Multifractal Model


simulation time series 1


simulation time series 2

Bivariate Multifractal Model





Generalized Method of Moments

GMM (Hansen, 1982)

$$\hat{\beta} = \arg\min_{\beta \in \Theta} \overline{M}(\beta)' W \overline{M}(\beta)$$

Θ : parameter vector

$\overline{M}(\beta)$: vector of differences between sample moments and analytical moments

W : postive definite weighting matrix

Generalized Method of Moments

Applicability of GMM on long memory models (Lux 2003, 2004)

Log increments transfermation: $\varepsilon_t^{(i)} = \ln(m_t^{(i)})$

$$
\begin{aligned}
X_{t,\tau} &= \ln|x_t| - \ln|x_{t-\tau}| \\
&= 0.5\sum_{i=1}^{k}\varepsilon_t^{(i)} + 0.5\sum_{l=1}^{n}\varepsilon_t^{(l)} + \ln|u_t| - (0.5\sum_{i=1}^{k}\varepsilon_{t-\tau}^{(i)} + 0.5\sum_{l=1}^{n}\varepsilon_{t-\tau}^{(l)} + \ln|u_{t-\tau}|) \\
&= 0.5\sum_{i=1}^{k}(\varepsilon_t^{(i)} - \varepsilon_{t-\tau}^{(i)}) + 0.5\sum_{l=k+1}^{n}(\varepsilon_t^{(l)} - \varepsilon_{t-\tau}^{(l)}) + (\ln|u_t| - \ln|u_{t-\tau}|)
\end{aligned}
$$

Generalized Method of Moments

Moment conditions:

Group *1* --- Cross time series

$$Cov(X^q_{t,\tau}, Y^q_{t,\tau})$$

$$Cov(X^q_{t,\tau}, Y^q_{t-\tau,\tau})$$

Group 2 --- Individual time series

$$Cov(X^q_{t,\tau}, X^q_{t-\tau,\tau})$$

$$Cov(Y^q_{t,\tau}, Y^q_{t-\tau,\tau}) \qquad q = 1,2; \qquad \tau = 1,5,10...$$

Generalized Method of Moments

Monte Carlo:

- Number of cascades: $n = 12$.
- Joint cascade levels: $k = 3, \; 6$.
- $\rho = 0.5; \quad \sigma_1 = \sigma_2 = 1.$

  (*1*) Binormial BMF: $\{m_0 ; \, 2\text{-}m_0\}$.

  (*2*) Lognormal BMF: $-\log_2 M \sim N(\lambda, \sigma^2).$

  $$E[M] = 0.5, \rightarrow \sigma_M^2 = 2(\lambda - 1) / \ln 2.$$

- Population: *100,000.*
- Sample sizes: $N_1 = 2{,}000$, $N_2 = 5{,}000$, $N_3 = 10{,}000$.

GMM Binomial

		$\hat{m}_0$			$\hat{\rho}$			$\hat{\sigma}_1$			$\hat{\sigma}_2$		
		Bias	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**
m_0 = 1,20	N1	-0.098	0.133	0.165	-0.002	0.071	0.071	-0.005	0.073	0.073	0.000	0.074	0.074
	N2	-0.081	0.120	0.145	0.001	0.045	0.045	0.000	0.046	0.046	0.000	0.046	0.046
	N3	-0.045	0.102	0.111	-0.001	0.030	0.030	0.000	0.033	0.033	0.003	0.034	0.034
m_0 = 1,30	N1	-0.111	0.149	0.186	0.009	0.072	0.072	-0.009	0.111	0.111	-0.002	0.112	0.112
	N2	-0.057	0.113	0.126	0.002	0.051	0.051	-0.001	0.070	0.070	0.000	0.069	0.069
	N3	-0.015	0.068	0.070	-0.003	0.033	0.033	0.000	0.051	0.051	0.004	0.051	0.051
m_0 = 1,40	N1	-0.083	0.140	0.162	0.012	0.078	0.079	-0.015	0.154	0.155	-0.006	0.153	0.153
	N2	-0.024	0.066	0.070	-0.002	0.052	0.052	-0.003	0.098	0.098	-0.001	0.094	0.094
	N3	-0.002	0.035	0.035	-0.008	0.033	0.034	0.000	0.071	0.071	0.004	0.069	0.069
m_0 = 1,50	N1	-0.042	0.088	0.097	0.002	0.086	0.086	-0.022	0.204	0.205	-0.014	0.197	0.198
	N2	-0.010	0.040	0.041	-0.010	0.055	0.056	-0.006	0.130	0.130	-0.004	0.123	0.123
	N3	0.003	0.024	0.025	-0.016	0.034	0.038	-0.001	0.095	0.095	0.003	0.091	0.091

GMM Binomial

		$\hat{m}_0$			$\hat{\rho}$			$\hat{\sigma}_1$			$\hat{\sigma}_2$		
		Bias	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**
m_0 = 1,20	N1	-0.098	0.133	0.165	-0.002	0.071	0.071	-0.005	0.073	0.073	0.000	0.074	0.074
	N2	-0.081	0.120	0.145	0.001	0.045	0.045	0.000	0.046	0.046	0.000	0.046	0.046
	N3	-0.045	0.102	0.111	-0.001	0.030	0.030	0.000	0.033	0.033	0.003	0.034	0.034
m_0 = 1,30	N1	-0.111	0.149	0.186	0.009	0.072	0.072	-0.009	0.111	0.111	-0.002	0.112	0.112
	N2	-0.057	0.113	0.126	0.002	0.051	0.051	-0.001	0.070	0.070	0.000	0.069	0.069
	N3	-0.015	0.068	0.070	-0.003	0.033	0.033	0.000	0.051	0.051	0.004	0.051	0.051
m_0 = 1,40	N1	-0.083	0.140	0.162	0.012	0.078	0.079	-0.015	0.154	0.155	-0.006	0.153	0.153
	N2	-0.024	0.066	0.070	-0.002	0.052	0.052	-0.003	0.098	0.098	-0.001	0.094	0.094
	N3	-0.002	0.035	0.035	-0.008	0.033	0.034	0.000	0.071	0.071	0.004	0.069	0.069
m_0 = 1,50	N1	-0.042	0.088	0.097	0.002	0.086	0.086	-0.022	0.204	0.205	-0.014	0.197	0.198
	N2	-0.010	0.040	0.041	-0.010	0.055	0.056	-0.006	0.130	0.130	-0.004	0.123	0.123
	N3	0.003	0.024	0.025	-0.016	0.034	0.038	-0.001	0.095	0.095	0.003	0.091	0.091

GMM Lognormal

			$\hat{m}_0$			$\hat{\rho}$			$\hat{\sigma}_1$			$\hat{\sigma}_2$	
		Bias	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**	**Bias**	**SE**	**RMSE**
λ = 1,10	N1	-0.029	0.052	0.060	0.011	0.078	0.079	-0.018	0.110	0.111	-0.004	0.115	0.115
	N2	-0.009	0.036	0.037	-0.001	0.049	0.049	-0.006	0.070	0.071	-0.001	0.072	0.072
	N3	-0.002	0.024	0.024	-0.005	0.039	0.039	-0.004	0.050	0.050	0.004	0.050	0.050
λ = 1,20	N1	-0.027	0.057	0.063	0.002	0.084	0.084	-0.029	0.164	0.166	-0.020	0.180	0.180
	N2	-0.005	0.037	0.038	-0.009	0.054	0.054	-0.017	0.115	0.116	-0.013	0.122	0.123
	N3	0.002	0.026	0.026	-0.015	0.036	0.039	-0.005	0.079	0.079	-0.009	0.085	0.086
λ = 1,30	N1	-0.022	0.063	0.067	-0.017	0.094	0.095	-0.042	0.246	0.250	-0.043	0.265	0.268
	N2	-0.005	0.036	0.037	-0.024	0.056	0.061	-0.026	0.183	0.185	-0.029	0.172	0.174
	N3	0.007	0.028	0.028	-0.031	0.039	0.050	-0.009	0.129	0.129	-0.016	0.121	0.122
λ = 1,40	N1	-0.024	0.068	0.072	-0.032	0.092	0.097	-0.047	0.339	0.341	-0.051	0.351	0.354
	N2	0.002	0.042	0.042	-0.042	0.067	0.079	-0.030	0.217	0.219	-0.038	0.221	0.224
	N3	0.010	0.029	0.031	-0.043	0.040	0.059	-0.023	0.203	0.204	-0.030	0.163	0.165

$$-\log_2 M \sim N(\lambda,\sigma^2), E[M]=0.5, \rightarrow \sigma_M^2 = 2(\lambda-1)/\ln 2$$

Maiximum Likelihood Estimation

- The MF dynamics can be interpreted as a special case of Markov-switching process.

  ⟹ Binomial case: Markov process with 4^k states for volatility

- The likelihood function can be derived by detecting the exact form of each possible component in the transition matrix.

Maiximum Likelihood Estimation

Likelihood function :

$$L(x_{1,} \cdots, x_T; \Theta) = \prod_{t=1}^{T} f(x_t | x_{1,} \cdots, x_{t-1})$$

$$= \prod_{t=1}^{T} [\sum_{i=1}^{4^n} P(M_t = m^i | x_{1,} \cdots, x_{t-1}) \cdot f(x)]$$

$$= \prod_{t=1}^{T} (\Omega_{t-1} \cdot A) f(x)$$

$$\Omega_t^i = P(M_t = m^i | x_1 \cdots, x_t)$$

A : Transition matrix $P(M_{t+1} = m^j | M_t = m^i)$

$f(x)$: Vector of conditional density $\{f(x_t | M_t = m^i)\}_i$

$$\Omega_t = \frac{f(x_t) \otimes (\Omega_{t-1} A)}{\sum f(x_t) \otimes (\Omega_{t-1} A)}$$

ML Binomial ($\underline{n = 5; k = 2}$)

		Bias	**SE**	**RMSE**
$m_0 = 1.4$	N1	-0.001	0.023	0.023
	N2	0.000	0.020	0.019
	N3	0.003	0.019	0.019
$\rho = 0.5$	N1	0.053	0.094	0.106
	N2	0.043	0.092	0.100
	N3	0.040	0.084	0.091
$\sigma_1 = 1$	N1	0.053	0.093	0.106
	N2	0.042	0.091	0.099
	N3	0.040	0.083	0.091
$\sigma_2 = 1$	N1	-0.064	0.112	0.127
	N2	-0.053	0.107	0.118
	N3	-0.041	0.099	0.106

Simulated ML

Berzuini et al. (1996); Chib et al. (2002); Jacquier et al. (1994);
Pitt and Shephard (1999). ----- Particle filter:

$$\Omega_t^i \propto f(x_t \mid M_t = m^i) \sum_{j=1}^{4^n} P(M_t = m^i \mid M_{t-1} = m^j)\Omega_{t-1}^j$$

filtering

$$\approx f(x_t \mid M_t = m^i) \frac{1}{B} \sum_{b=1}^{B} P(M_t = m^i \mid M_{t-1} = m^{(b)})$$

Sampling/Inportance Sampling (SIR):

$$P(q = b) = \frac{f(x_{t+1} \mid M_{t+1} = m_{t+1}^{(b)})}{\sum_{i=1}^{B} f(x_{t+1} \mid M_{t+1} = m_{t+1}^{(i)})}$$

particles

Computational convenient

Simulated ML

One-step ahead desity:

$$f(x_t | x_{t-1}) = \sum_{i=1}^{4^n} f(x_t | M_t = m^i) P(M_t = m^i | x_{t-1})$$

$$\approx \frac{1}{B} \sum_{b=1}^{B} P(x_t | M_t = m^{(b)})$$

Approximated Likelihood:

$$\hat{L}(x_{1,} \cdots, x_T; \Theta) = \prod_{t=1}^{T} f(x_t | x_{1,} \cdots, x_{t-1})$$

Value-at-Risk

Value-at-Risk (VaR):

- Specified target horizon
- Statistical confidence level
- The worst loss

Looking forward *h*-period return at time *t*: $\tilde{x}_{t,t+h} = \sum_{i=1}^{h} x_{t+i}$

VaR at the *h*-period horizon:

$$\Pr(\tilde{x}_{t,t+h} \leq VaR_{t,t+h}^{\alpha} | I_t) = \alpha$$

Value-at-Risk

Empirical daily data:

- Stock Exchange Index (*Jan. 1969 - Aug. 1999*)

  Dow Jones Composite 65 Average Index

  Nikkei 225 Average Index

- Foreign Exchange rates: (*Mar.1973 - Nov.2004*)

  British Pound to US Dollar
  Australian Dollar to US Dollar.

- Bonds: (*Jun.1976 - Oct.2004*).

  U.S. 1 Year and 2 Year Treasury Constant Maturity Rate

Returns: $x_t = \ln(p_t) - \ln(p_{t-1})$.
Failure Rate: Number of Obsrvations above VaR.

Failure rate for multi-period VaR forecast (GMM)

		One day horizon			Two days horizon			Five days horizon		
		DOW	*NIK*	*EW*	*DOW*	*NIK*	*EW*	*DOW*	*NIK*	*EW*
Stocks	p = 10%	0.1074	0.0896	0.0974	0.1025	0.0912	0.1062	0.1052	0.0872	0.1110
	p = 5%	**0.0625+**	0.0422	0.0488	**0.0627+**	0.0451	0.0584	0.0604	0.0453	**0.0645+**
	p = 1%	0.0180	0.0081	0.0130	0.0193	0.0100	0.0165	0.0186	0.0087	0.0191
		BP	*AUD*	*EW*	*BP*	*AUD*	*EW*	*BP*	*AUD*	*EW*
Fxs	p = 10%	0.1059	0.9660	0.1104	0.1088	0.1035	0.1108	0.1051	0.0926	0.1051
	p = 5%	0.0542	0.0487	0.0512	0.0547	0.0501	0.0557	0.0563	0.0419	0.0469
	p = 1%	0.0106	0.0077	0.0085	0.0097	0.0092	0.0100	0.0118	0.0068	0.0069
		T1	*T2*	*EW*	*T1*	*T2*	*EW*	*T1*	*T2*	*EW*
Bonds	p = 10%	0.1010	0.1104	0.1083	0.1043	0.1096	0.1070	0.0902	0.1063	0.1041
	p = 5%	0.0577	**0.0665+**	**0.0642+**	0.0530	**0.0659+**	**0.0662+**	0.0541	**0.0633+**	0.0535
	p = 1%	0.0156	0.0187	0.0191	0.0146	**0.0263+**	**0.0237+**	0.0175	**0.0229+**	**0.0227+**

- EW denotes Equal-Weight portfolio
- + and * denote too risky and conservative VaR respectively

Failure rate for multi-period VaR forecast (ML)

		One day horizon			Two days horizon			Five days horizon		
		DOW	*NIK*	*EW*	*DOW*	*NIK*	*EW*	*DOW*	*NIK*	*EW*
Stocks	p = 10%	0.1175	0.0959	0.0904	0.1140	0.0968	0.0967	0.1125	0.0898	0.1000
	p = 5%	**0.0690+**	0.0439	0.0435	**0.0677+**	0.0465	0.0495	**0.0650+**	0.0475	0.0563
	p = 1%	0.0173	0.0063	0.0082	0.0180	0.0098	0.0120	0.0191	0.0088	0.0162
		BP	*AUD*	*EW*	*BP*	*AUD*	*EW*	*BP*	*AUD*	*EW*
Fxs	p = 10%	0.1176	0.1119	0.0903	0.1168	0.1143	0.0875	0.1095	0.0951	0.0892
	p = 5%	0.0577	0.0516	0.0403	0.0560	0.0517	**0.0302***	0.0563	0.0431	**0.0244***
	p = 1%	0.0084	0.0054	**0.0031***	0.0082	0.0070	**0.0027***	0.0106	0.0056	**0.0031***
		T1	*T2*	*EW*	*T1*	*T2*	*EW*	*T1*	*T2*	*EW*
Bonds	p = 10%	0.1001	0.1092	0.1060	0.1020	0.1085	0.1041	0.0872	0.1019	0.0982
	p = 5%	0.0572	**0.0662+**	0.0609	0.0521	**0.0648+**	**0.0653+**	0.0527	0.0611	0.0528
	p = 1%	0.0156	0.0187	0.0191	0.0146	**0.0263+**	**0.0237+**	0.0175	**0.0229+**	**0.0227+**

- EW denotes Equal-Weight portfolio.
- + and * denote too risky and conservative VaR respectively

Conclusion (1)

MMF:

- Generalized Method of Moments Estimation.
  - → No upbound for cascade levels.
  - → Continuous distribution applicable.
- Maximum Likelihood Estimation.
  - → Cascade levels bound: $n = 5$ for Bivariate Mf and $n = 3$ for Trivariate MF.
  - → Only for discrete distribution.

	Computation time (GAUSS)
GMM (Table *1*)	*2* Days
ML (Table *4*)	*5* Weeks

Conclusion (2)

Extension --- High Dimension MF:

$$x_{t,q} = \sigma_q \cdot (\prod_{i=1}^{k} m_t^{[i]} \prod_{h(\cdot)=k+1}^{n} m_t^{[h(\cdot)]})^{1/2} \cdot u_{t,q}$$

σ_q and $u_{q,t}$ are q dimentional vectors

Estimation: GMM

Maximum Likelihood $(q^2)^n \times (q^2)^n$

SML