Credit Risk under Incomplete Information

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1. Credit Risk and Incomplete Information

In credit risk the information set used in constructing a model frequently differs from the information available to investors. As shown by the following examples, this can have far-reaching economic implications.

1. Firm-value models and credit spreads.

- In firm-value models the default time τ is modelled as $\tau = \inf\{t \geq 0, V_t < B\}$, V the asset-value process and B a threshold related to liabilities. Typically, V is a diffusion, making τ predictable and leading to unrealistically low short-term credit spreads.
- If investors have only noisy information about asset value, τ admits an intensity wrt the investors' filtration, and short-term credit spreads show more reasonable behaviour; see for instance [Duffie and Lando, 2001].

Credit Risk and Information ctd

2. Information-based default contagion.

- In standard portfolio credit risk models survival probabilities and default intensities of firms are assumed to depend on some common factor variable/process $\boldsymbol{X}$.

- If (parts of) $\boldsymbol{X}$ are not specified explicitly, we may have default contagion: new default information leads to updating of conditional distribution of $\boldsymbol{X}$ and hence to changing default intensities. Information-based default contagion occurs for instance in the popular factor copula models; see [Schönbucher, 2004], [Giesecke and Goldberg, 2004] or Chapter 9 of [McNeil et al., 2005].

Portfolio Credit Risk and Nonlinear Filtering

In this talk we show how nonlinear filtering techniques can be used for constructing "information-consistent" portfolio credit risk models with unobservable state variables.

We use a two-step procedure for model building.

Step 1. Construct a portfolio credit risk model where default intensities are driven by a common Markovian state variable process $\boldsymbol{X}$ on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. This is the full-information-model, $(\mathcal{F}_t)$ is the full-information filtration.

Step 2. Denote by $\mathcal{F}_t^I \subset \mathcal{F}_t$ information actually available to investors (default history, prices of credit derivatives etc.) Construct incomplete-information model by projecting default intensities and price dynamics from the full-information model on $(\mathcal{F}_t^I)$ via nonlinear filtering.

Applications

- Construction of pricing models for credit derivatives via martingale methods; in that case P represents martingale measure.
- Statistical analysis of credit risk models; in that case P is historical probability measure.

Overview

1. Credit Risk and Incomplete Information
2. A special case with conditionally independent defaults
3. A general full-information model with default contagion
4. Nonlinear filtering (W. Runggaldier)

General Notation

- We consider m firms with default times τ_j and default indicator process $\mathbf{Y} = (Y_{t,1}, \ldots, Y_{t,m})$ with $Y_{t,j} = 1_{\{\tau_j \le t\}}$.
- The ordered default times are denoted by $0 = T_0 < T_1 < \ldots < T_m$. $\xi_n \in \{1, \ldots, m\}$ gives identity of the firm defaulting at time T_n.

2. Conditionally Independent Defaults

We explain the approach in special case of conditionally independent doubly-stochastic default-times.

- The factor process $\boldsymbol{X}$ is a d-dimensional jump-diffusion of the form

$$d\boldsymbol{X}_t = b(\boldsymbol{X}_t)dt + \sigma(\boldsymbol{X}_t)d\boldsymbol{W}_t + d\boldsymbol{Z}_t,$$

  $\boldsymbol{Z}$ a $\mathbb{R}^d$-valued compound Poisson process with compensator measure $F_{\boldsymbol{Z}}(d\boldsymbol{x})ds$ (for instance an affine model as in [Duffie and Singleton, 2003]).

- Default intensity of firm j at time t is equal to $\lambda_j(\boldsymbol{X}_t)$.

- Default times are conditionally independent doubly-stochastic random times.

Credit Derivatives under Full Information

Since $\boldsymbol{X}$ is Markov, in the full-information model prices of credit derivatives at time t are functions of $(t, \boldsymbol{X}_t, \boldsymbol{Y}_t)$.

Example 1. Consider a defaultable zero-coupon bond $p_j(\cdot, T)$ on firm j with zero recovery rate and maturity T. Suppose that P represents equivalent martingale measure and that default-free interest rate is given by $r(\boldsymbol{X}_t)$. We get

$$\begin{aligned} p_j(t,T) &= E\left(e^{-\int_t^T r(\boldsymbol{X}_s)ds}(1 - Y_{T,j}) \mid \mathcal{F}_T\right) \\ &= E_{(\boldsymbol{X}_t,\boldsymbol{Y}_t)}\left(e^{-\int_0^{T-t} r(\boldsymbol{X}_s)ds}(1 - Y_{T-t,j})\right) \\ &\overset{\text{cond. indep.}}{=} (1 - Y_{t,j}) E_{\boldsymbol{X}_t}\left(e^{-\int_0^{T-t} R(\boldsymbol{X}_s)\, ds}\right) \end{aligned}$$

where $R(\boldsymbol{X}_t) := r(\boldsymbol{X}_t) + \lambda_j(\boldsymbol{X}_t)$. Last equality uses properties of conditionally independent doubly-stochastic default times.

Credit Derivatives under Full Information ctd.

Example 2. Consider a recovery claim which pays the amount $\delta(\boldsymbol{X}_t)$ at the default time τ_j if $\tau_j < T$. We get

$$
\begin{aligned}
H_t &= (1 - Y_{t,j}) E\left(e^{-\int_t^{\tau_j} r(\boldsymbol{X}_s)ds}\delta(\boldsymbol{X}_{\tau_j})Y_{T,j} \mid \mathcal{F}_t\right) \\
&= (1 - Y_{t,j}) E_{(\boldsymbol{X}_t, \boldsymbol{Y}_t)}\left(e^{-\int_0^{\tau_j} r(\boldsymbol{X}_s)ds}\,\delta(\boldsymbol{X}_{\tau_j})Y_{T-t,j}\right) \\
&\overset{\text{cond.indep.}}{=} (1 - Y_{t,j}) E_{\boldsymbol{X}_t}\Big(\int_0^{T-t} \lambda_j(\boldsymbol{X}_s)\delta(\boldsymbol{X}_s)e^{-\int_0^s R(\boldsymbol{X}_u)\,du}\Big)\,;
\end{aligned}
$$

Examples 1 and 2 cover corporate bonds and CDSs; similar argument applies to portfolio credit derivatives such as CDO tranches.

Computation of prices can be done using Feynman-Kac and PDE-techniques or via simulation with $(\boldsymbol{X}_t, \boldsymbol{Y}_t)$ as state variable.

Incomplete Information and Investor Filtration

Recall that information actually available to investors is $\mathcal{F}_t^I \subset \mathcal{F}_t$.

Assumptions:

- Incomplete information. $\boldsymbol{X}$ is not $(\mathcal{F}_t^I)$-adapted.
- $\mathcal{F}_t^I$ contains default history. Formally, $\mathcal{H}_t \subset \mathcal{F}_t^I$, where $\mathcal{H}_t = \sigma(\{Y_{s,i} : s \leq t, 1 \leq i \leq m\}) = \sigma(\{(T_n, \xi_n) : T_n \leq t\})$.
- Default-free interest rate (r_t) is $(\mathcal{F}_t^I)$-adapted.
- In this talk we generally assume $\mathcal{F}_t^I = \mathcal{H}_t$.
  - ⋆ In line with literature on dynamic properties of factor copula models and information-based default contagion.
  - ⋆ Extension to richer information sets containing (noisy) derivative prices possible.

Credit Derivatives under Incomplete Information

We use martingale modelling for constructing an arbitrage- free pricing model for credit derivatives wrt investor filtration $(\mathcal{F}_t^I)$. Here P denotes the martingale measure used for pricing.

Consider a credit derivative with payoff H and maturity T (for instance $H = 1 - Y_{T,i}$), and assume that its 'full-information price' $E(\exp(-\int_t^T r_s ds)\, H \mid \mathcal{F}_t)$ is given by a function $h(t, \boldsymbol{X}_t, \boldsymbol{Y}_t)$.

Under the martingale modelling approach the price of H in the incomplete- information model is given by

$$H_t = E\Big(e^{-\int_t^T r_s ds} H \mid \mathcal{F}_t^I\Big).$$

Credit Derivatives and Nonlinear Filtering

We get by iterated conditional expectations

$$\begin{aligned} H_t &= E\Big(E\big(e^{-\int_t^T r_s ds} H \mid \mathcal{F}_t\big) \mid \mathcal{F}_t^I\Big) \\ &= E(h(t, \boldsymbol{X}_t, \boldsymbol{Y}_t) \mid \mathcal{F}_t^I)\,. \end{aligned} \tag{1}$$

Since $\boldsymbol{Y}_t$ is known at t, in order to compute (1) we have to compute $E(f(X_t) \mid \mathcal{F}_t^I)$ for "reasonable" f. This is a typical nonlinear filtering problem.

Calibration via Filtering. Suppose that price process of certain credit derivatives are $(\mathcal{F}_t^I)$-adapted, i.e. part of the investor information. If observed price processes are consistent with the full-information model, the approach yields an incomplete-information model which is automatically calibrated to observed prices; see [Gombani et al., 2004].

Nonlinear Filtering and Estimation

Estimation of default probabilities and model parameters also leads to nonlinear filtering problem; now P represents physical measure.

- Default probability. In analogy with zero-coupon bond pricing we have $P(\tau_i > T \mid \mathcal{F}_t) = g_i(t, \boldsymbol{X}_t, \boldsymbol{Y}_t)$ where $g_i(t, \boldsymbol{x}, \boldsymbol{y}) = E_{\boldsymbol{x},\boldsymbol{y}}(1 - Y_{T-t,i})$. Hence

$$P(\tau_i > T \mid \mathcal{F}_t^I) = E\left(P(\tau_i > T \mid \mathcal{F}_t) \mid \mathcal{F}_t^I\right) = E(g_i(t, \boldsymbol{X}_t, \boldsymbol{Y}_t) \mid \mathcal{F}_t^I) .$$

- Parameter estimation. Certain components of $\boldsymbol{X}$ might represent unknown model parameters. Bayesian parameter estimation then amounts to computing the conditional distribution of these parameters given $\mathcal{F}_t^I$. This is related to Bayesian estimation approaches for credit risk models proposed in [McNeil and Wendin, 2003].

3. Full-Information Model with Default Contagion

The nonlinear filtering approach works in a fairly general full information model allowing for default contagion and systemic risk. There it is assumed that $\boldsymbol{X}$ and $\boldsymbol{Y}$ solve the following SDE.

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t b(\boldsymbol{X}_{s-}, \boldsymbol{Y}_{s-})ds + \int_0^t \sigma(\boldsymbol{X}_{s-}, \boldsymbol{Y}_{s-})d\boldsymbol{W}_s + \int_0^t \int_E K^{\boldsymbol{X}}(\boldsymbol{X}_{s-}, \boldsymbol{Y}_{s-}, u)\mathcal{N}(ds, du), \quad (2)$$

$$Y_{t,j} = Y_{0,j} + \int_0^t \int_E (1 - Y_{s-,j})K_j^{\boldsymbol{Y}}(\boldsymbol{X}_{s-}, \boldsymbol{Y}_{s-}, u)\mathcal{N}(ds, du). \quad (3)$$

Here $\mathcal{N}(ds, du)$ denotes a $(P, (\mathcal{F}_t))$-standard Poisson random measure on $\mathbb{R}_+ \times E$, E some Euclidian space; the compensator measure is $F_{\mathcal{N}}(du)ds$; $\boldsymbol{W}$ and $\mathcal{N}$ are independent; $K_j^{\boldsymbol{Y}}$ takes values in $\{0, 1\}$.

Full-Information Model ctd

The following subsets of E are closely linked to jumps of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

$$D_i^{\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{y}) := \{u \in E : K_i^{\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{y},u) \neq 0\}\,, \quad 1 \leq i \leq d\,, \tag{4}$$

$$D_j^{\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y}) := \{u \in E : K_j^{\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y},u) \neq 0\}\,, \quad 1 \leq j \leq m\,. \tag{5}$$

Assumptions: A1. Existence and uniqueness of the system (2), (3).

A2. (finite jump-intensities) $E\left(\int_0^T F_{\mathcal{N}}(D_i^{\boldsymbol{X}}(\boldsymbol{X}_s,\boldsymbol{Y}_s))\,ds\right) < \infty$ and $E\left(\int_0^T F_{\mathcal{N}}(D_j^{\boldsymbol{Y}}(\boldsymbol{X}_s,\boldsymbol{Y}_s))ds\right) < \infty$.

A3. (no joint defaults) $F_{\mathcal{N}}(D_{j_1}^{\boldsymbol{Y}} \cap D_{j_2}^{\boldsymbol{Y}}) = 0$, $j_1 \neq j_2$.

Further Model-Features

- The model allows for common jumps of $\boldsymbol{X}$ and $\boldsymbol{Y}$. More precisely, X_i and Y_j have common jumps if $F_{\mathcal{N}}(D_j^{\boldsymbol{Y}} \cap D_i^{\boldsymbol{X}}) > 0$. In this way phenomena like 'systemic risk', 'default contagion' or 'flight to quality' can be modelled in full-information model.

- Default intensities. By definition of the compensator of a Poisson random measure,

$$Y_{t,j} - \int_0^t (1 - Y_{s-,j}) F_{\mathcal{N}}(D_j^{\boldsymbol{Y}}(\boldsymbol{X}_{s-}, \boldsymbol{Y}_{s-}))\, ds$$

is a martingale, so that $\lambda_j(\boldsymbol{X}_{t-}, \boldsymbol{Y}_{t-}) := F_{\mathcal{N}}(D_j^{\boldsymbol{Y}}(\boldsymbol{X}_{t-}, \boldsymbol{Y}_{t-}))$ is the $(\mathcal{F}_t)$-default intensity of firm j.

Extended [Davis and Lo, 2001]-Model

Informal model description.

- X is modelled as a finite-state Markov chain with state space $S^X = \{0, \ldots, K\} \subset \mathbb{R}$.
- Default intensity of firm j is given by $\lambda_j(X_t)$.
- Default contagion. At a default time T_n, X jumps upward by one unit with probability p_{ξ_n} and remains constant with probability $1-p_{\xi_n}$. (In original Davis-Lo model occurrence of contagious effect is deterministic).
- If $X_t > 0$, X jumps to $X_t - 1$ with intensity $\gamma(X_t)$, independent of the default history.

Extended Davis-Lo Model ctd.

Formal model description. To model these features we put $E = \mathbb{R}$, $F_{\mathcal{N}} = \nu$, ν Lebesgue-measure, and let

$$K^X(x, \boldsymbol{y}, u) = 1_{\{[0,K-1]\}}(x) \Big(1_{\{ [0,(1-y_1)p_1\lambda_1(x)] \}}(u) + \cdots + 1_{\{ [\bar{\lambda}_{m-1}(x,\boldsymbol{y}),\bar{\lambda}_{m-1}(x,\boldsymbol{y})+(1-y_m)p_m\lambda_m(x)] \}}(u) \Big) \tag{6}$$

$$- 1_{\{[1,k]\}}(x) 1_{\{[-\gamma(x),0]\}}(u) \tag{7}$$

$$K_j^{\boldsymbol{Y}}(x, \boldsymbol{y}, u) = 1_{\{[\bar{\lambda}_{j-1}(x,\boldsymbol{y}),\bar{\lambda}_j(x,\boldsymbol{y})]\}}(u), j = 1, \ldots, m \,, \tag{8}$$

where $\bar{\lambda}_j(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{j} (1 - y_i) \lambda_i(\boldsymbol{x}, \boldsymbol{y})$, $0 \leq j \leq m$, and $\bar{\lambda}(\boldsymbol{x}, \boldsymbol{y}) := \bar{\lambda}_m(\boldsymbol{x}, \boldsymbol{y})$.

Conditionally Independent Defaults

To obtain a model with conditionally independent doubly-stochastic default times, where $\boldsymbol{X}$ solves the SDE

$$d\boldsymbol{X}_t = b(\boldsymbol{X}_t)dt + \sigma(\boldsymbol{X}_t)d\boldsymbol{W}_t + d\boldsymbol{Z}_t,$$

$\boldsymbol{Z}$ a $\mathbb{R}^d$-valued compound Poisson process with compensator measure $F_{\boldsymbol{Z}}(d\boldsymbol{x})ds$, we can take $E = \mathbb{R}^d \times \mathbb{R}$, $F_{\mathcal{N}} = \mathcal{F}_{\boldsymbol{Z}} \times \nu$, and

$$K_j^{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{u}) = 1_{\left\{\left[\sum_{i=1}^{j-1}\lambda_j(\boldsymbol{x}), \sum_{i=1}^{j}\lambda_j(\boldsymbol{x})\right]\right\}}(u_{d+1}), \quad 1 \le j \le m, \text{ and}$$

$$K_i^{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{u}) = u_i 1_{\{[-1,0)\}}(u_{d+1}), \quad 1 \le i \le d.$$

Note that $F_{\mathcal{N}}(D_i^{\boldsymbol{X}} \cap D_j^{\boldsymbol{Y}}) = 0$.

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