Quantitative Methods in Finance 2005

Level, slope and curvature - fact or artefact?

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December 16th, 2005

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Correlation matrices of term structures

Correlation matrices of term structures usually tend to have positive entries, and the following properties:

i) $\rho_{i,j+1} \leq \rho_{ij}$ for $j \geq i$;

ii) $\rho_{i,j-1} \leq \rho_{ij}$ for $j \leq i$;

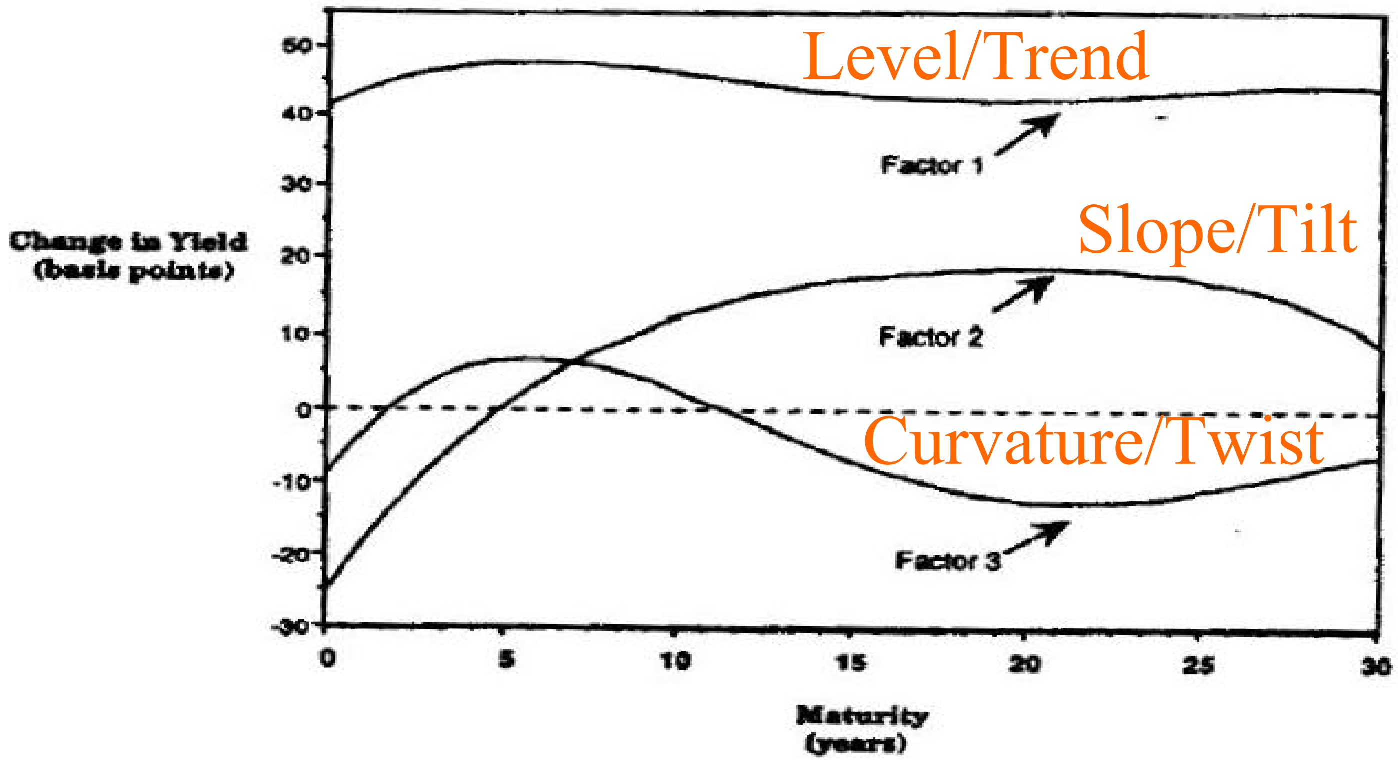
iii) $\rho_{i,i+j} \leq \rho_{i+1,i+j+1}$

i.e. rates that are further apart are less correlated than rates that are closer together. Moreover, rates with longer maturities are more correlated than rates with shorter maturities.

Literature review

Litterman and Scheinkman [1991]

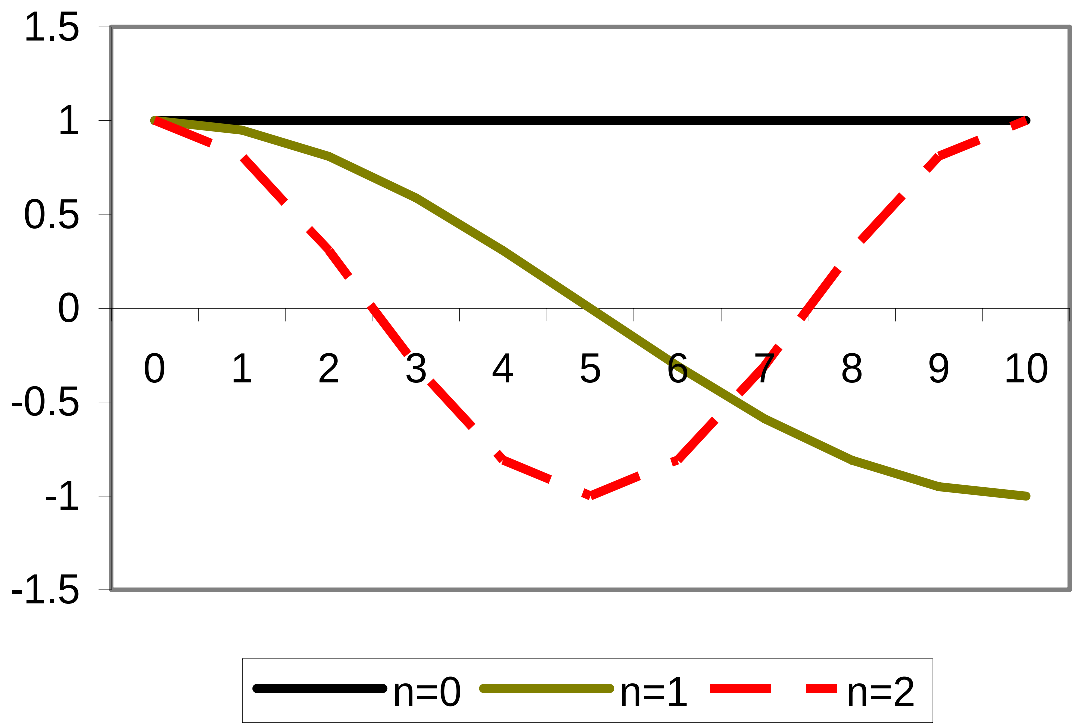
- First paper to apply factor analysis to US zero yields
- 3 factors explain roughly 98% of the variation



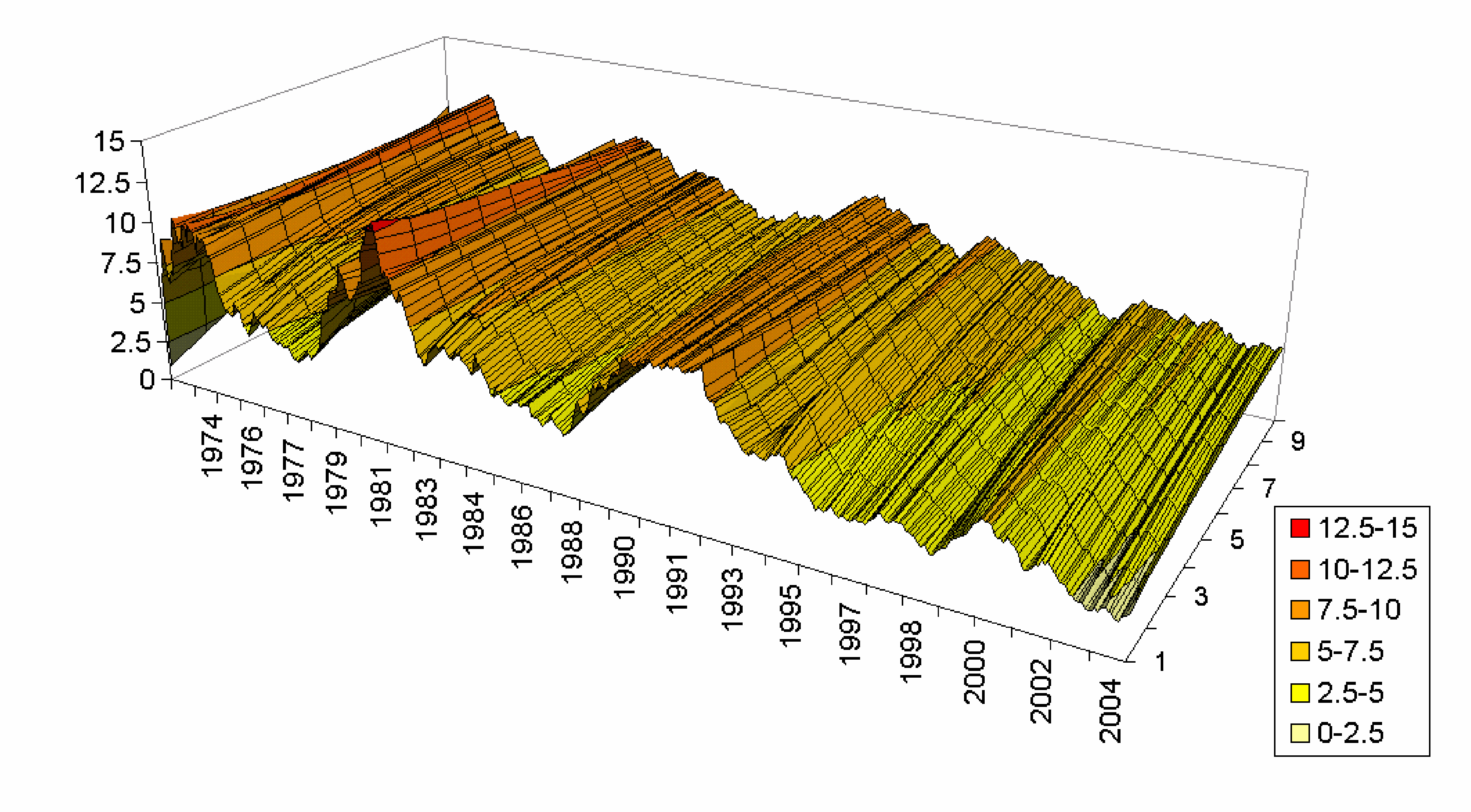
Literature review (2)

Forzani and Tolmasky [2003]

Explicitly solved the eigensystem of the exponentially decaying correlation function $K(x,y) = \rho^{|y-x|}$. The first three eigenfunctions are level, slope and curvature:

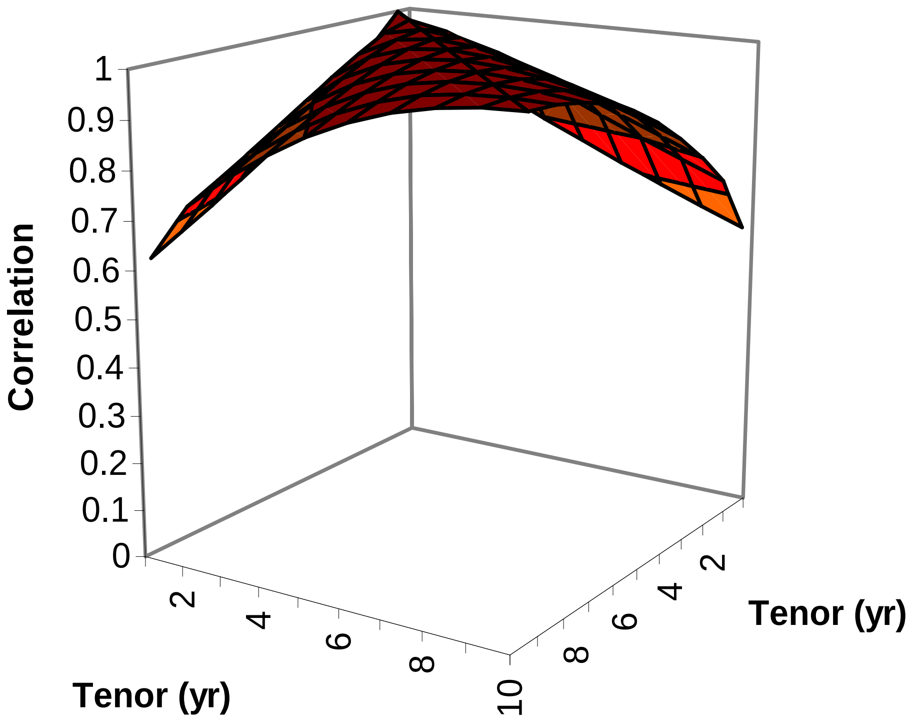


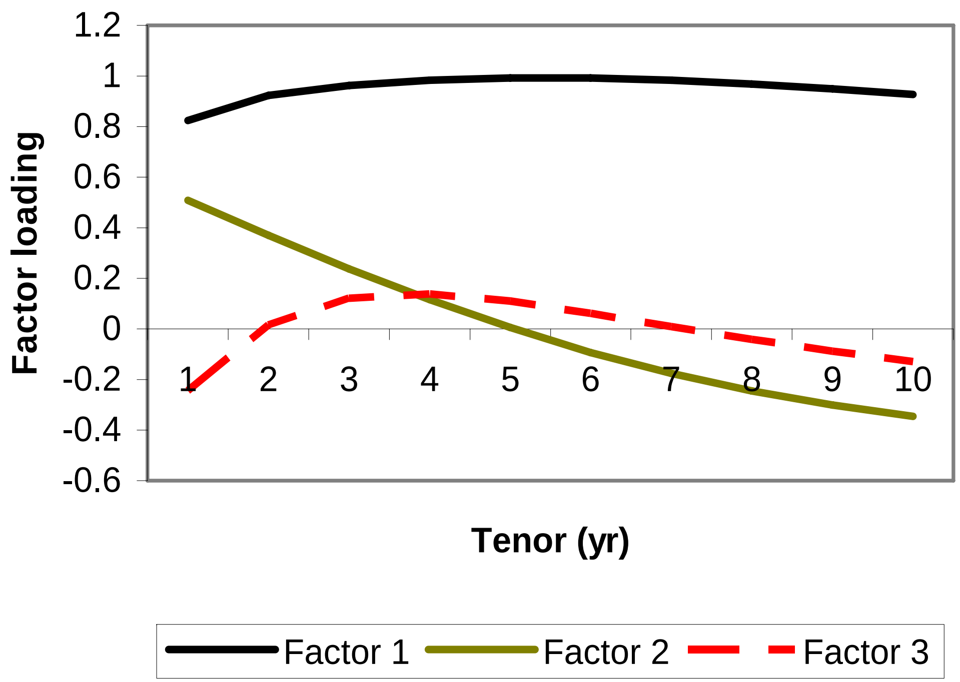
Bundesbank data



Bundesbank data (2)

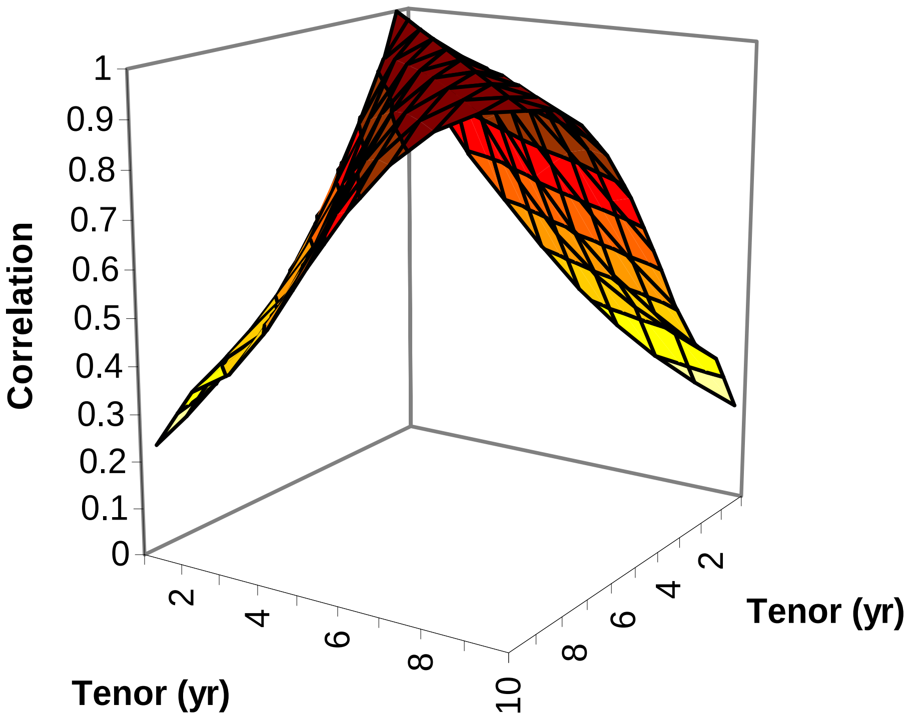
Changes in zero yields, using a Svensson curve

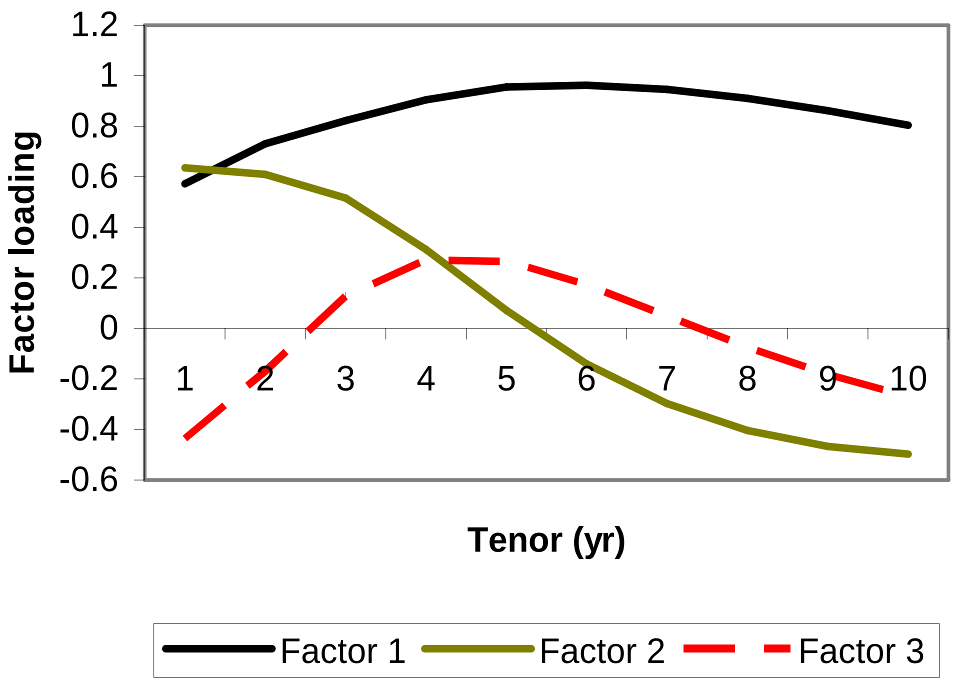




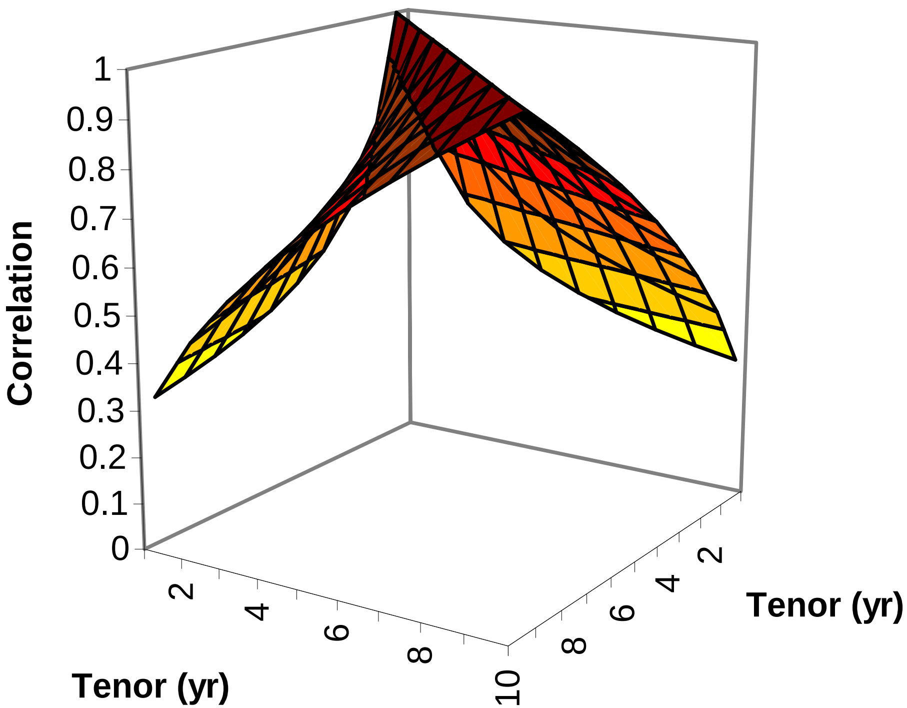
Bundesbank data (3)

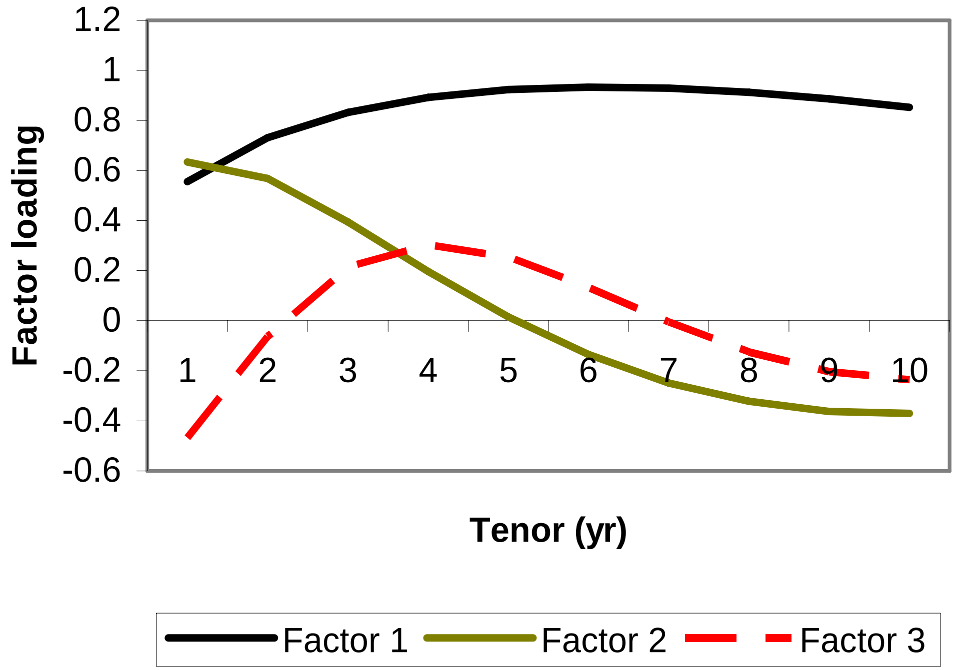
Changes in forward rates, using a Svensson curve





Brownian motion through time





Problem definition

- Given a correlation matrix of a term structure of interest rates (or any other term structure for that matter), do we always have level, slope and curvature (LSC)? Is the pattern an artefact of PCA?
- Alexander: "... *The interpretation of the 1st component as the trend, the 2nd as tilt, and the 3rd as curvature, holds for any highly correlated ordered system, not just a term structure.*"
- If not, can we derive necessary and/or sufficient conditions for LSC to appear? Is there any economic interpretation behind these conditions?

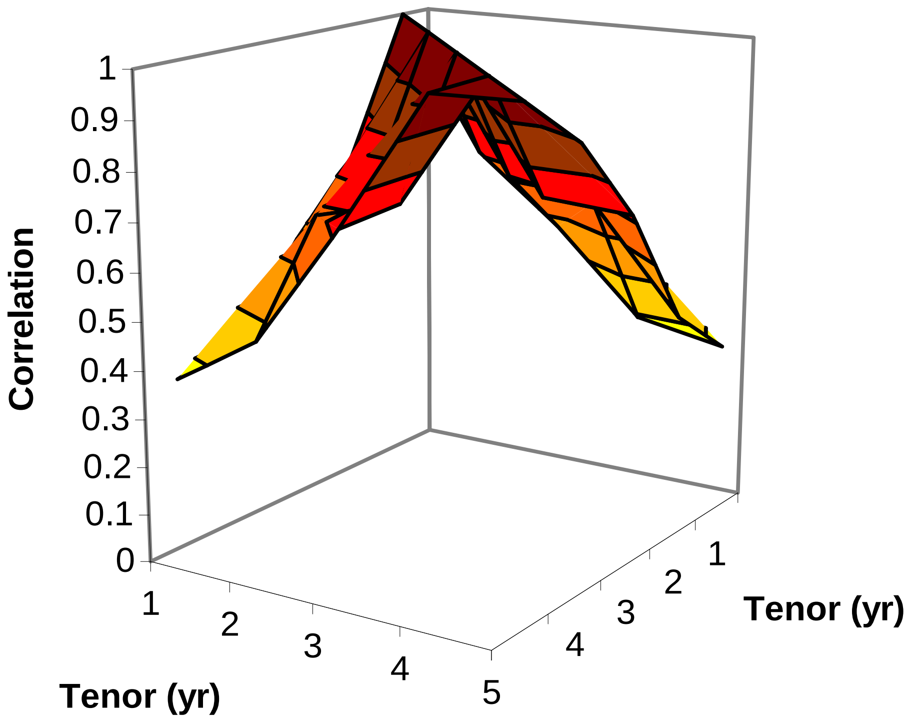
Counterexample

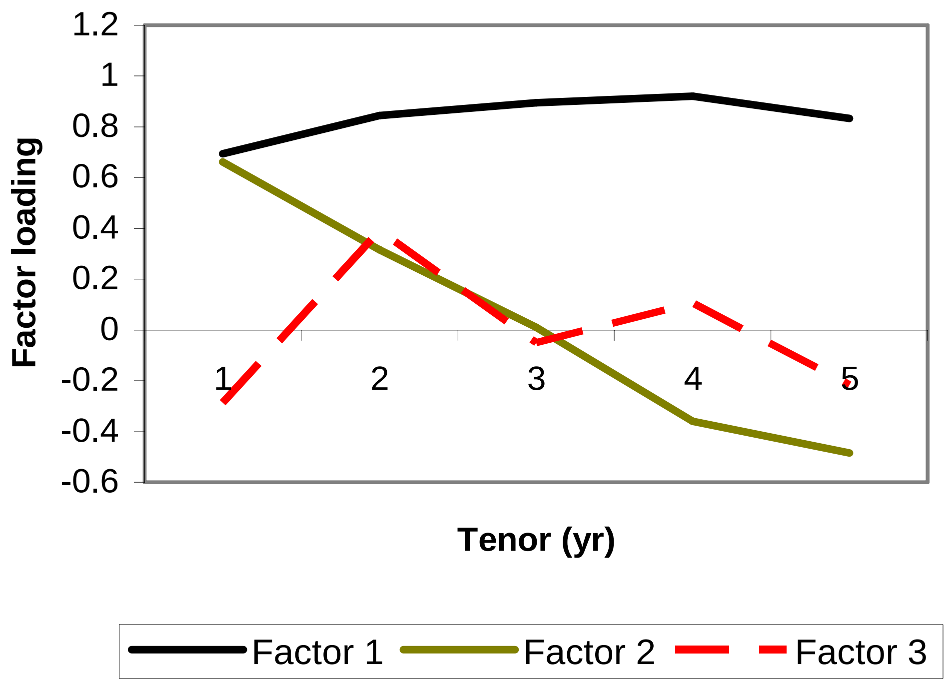
Artificially constructed matrix with typical properties of interest rate correlation matrices: arrows indicate in which direction the entries are increasing in size:

$$\mathbf{R} = \begin{pmatrix} 1 & 0.649 & 0.598 & 0.368 & 0.349 \\ 0.649 & 1 & 0.722 & 0.684 & 0.453 \\ 0.598 & 0.722 & 1 & 0.768 & 0.754 \\ 0.368 & 0.684 & 0.768 & 1 & 0.896 \\ 0.349 & 0.453 & 0.754 & 0.896 & 1 \end{pmatrix}$$

Counterexample (2)

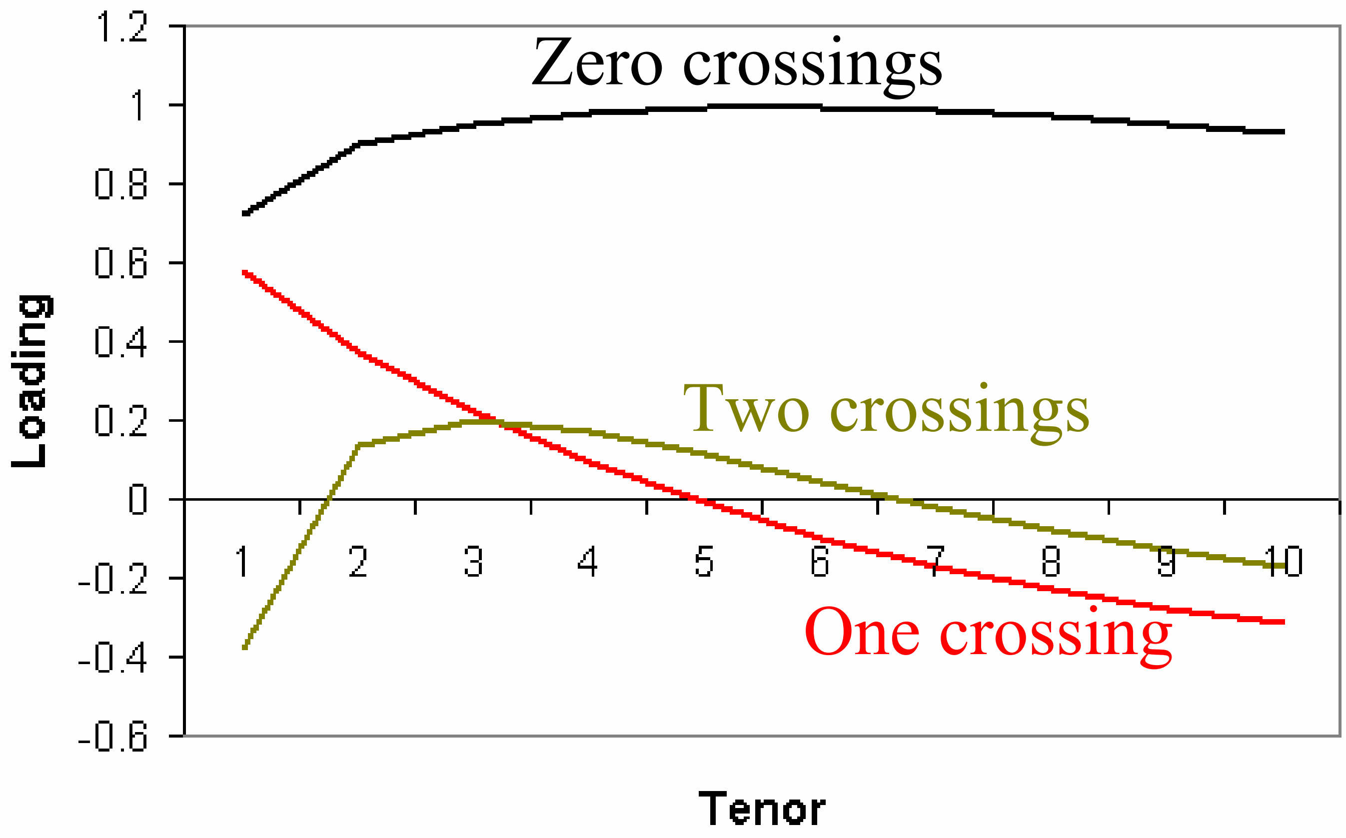
Just level and slope, no curvature:





Problem definition (2)

How do we characterise level, slope and curvature?



Problem definition (3)

Given a vector $x = (x_1, \ldots, x_N)^T$ we define:

- $S^-(x)$: no. of sign changes in x_1 through x_N, with zero terms discarded;
- $S^+(x)$: max. no. of sign changes in x_1 through x_N;

So we could define LSC as:

- Level: $S^-(w_1) = S^+(w_1) = 0$
- Slope: $S^-(w_2) = S^+(w_2) = 1$
- Curvature: $S^-(w_3) = S^+(w_3) = 2$

Sufficient conditions

The level problem is quickly solved:

Perron's theorem

If Σ has strictly positive elements, then Σ has a positive largest eigenvalue with algebraic multiplicity equal to 1. The eigenvector may be chosen such that all components are of one sign.

How about slope and curvature?

Sufficient conditions (2)

Consider the following set:

$$I_p = \left\{ \mathbf{i} = (i_1, \dots, i_p) \in \mathbb{N} \mid 1 \le i_1 < \dots < i_p \le N \right\}$$

For Σ and $\mathbf{i}, \mathbf{j} \in I_p$ consider the following determinant:

$$\Sigma_{[p]}(\mathbf{i}, \mathbf{j}) = \Sigma \begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix} = \det(\Sigma_{i_k j_\ell})_{k,\ell=1}^{p}$$

i.e. determinant of the correlation matrix between the interest rates indexed by **i**, and those indexed by **j**.

Sufficient conditions (3)

We can characterise a matrix as:

- Totally positive of order k (TP_k) if all such determinants of order $\leq$ k are positive;
- STP_k if none of these determinants is zero.

A matrix is oscillatory of order k (O_k) if it:

- is TP_k;
- is non-singular;
- has a strictly positive super- and subdiagonal.

Sufficient conditions (4)

Theorem (after Gantmacher and Kreĭn [1960])

A correlation matrix displays LSC if it is O_3, i.e. if it:

- is TP_3;
- is non-singular;
- has a strictly positive super- and subdiagonal.

Note:

The last two conditions will always be satisfied for valid covariance matrices in a term structure context.

Interpretation of the conditions

Level:

- Solved by Perron's theorem, so the *level* of the correlations causes the *level* factor;
- The higher the level of correlations, the "flatter" the first eigenvector will be.

Interpretation of the conditions (2)

Slope:

The only condition that is relevant here is the TP_2 condition. We will always have positive correlations, so the TP_2 condition reduces to, for $i \le j$ and $k \le \ell$:

$$\left|\begin{pmatrix} \rho_{ik} & \rho_{i\ell} \\ \rho_{jk} & \rho_{j\ell} \end{pmatrix}\right| = \rho_{ik}\rho_{j\ell} - \rho_{i\ell}\rho_{jk} \ge 0 \Leftrightarrow \rho_{ik}\rho_{j\ell} \ge \rho_{i\ell}\rho_{jk}$$

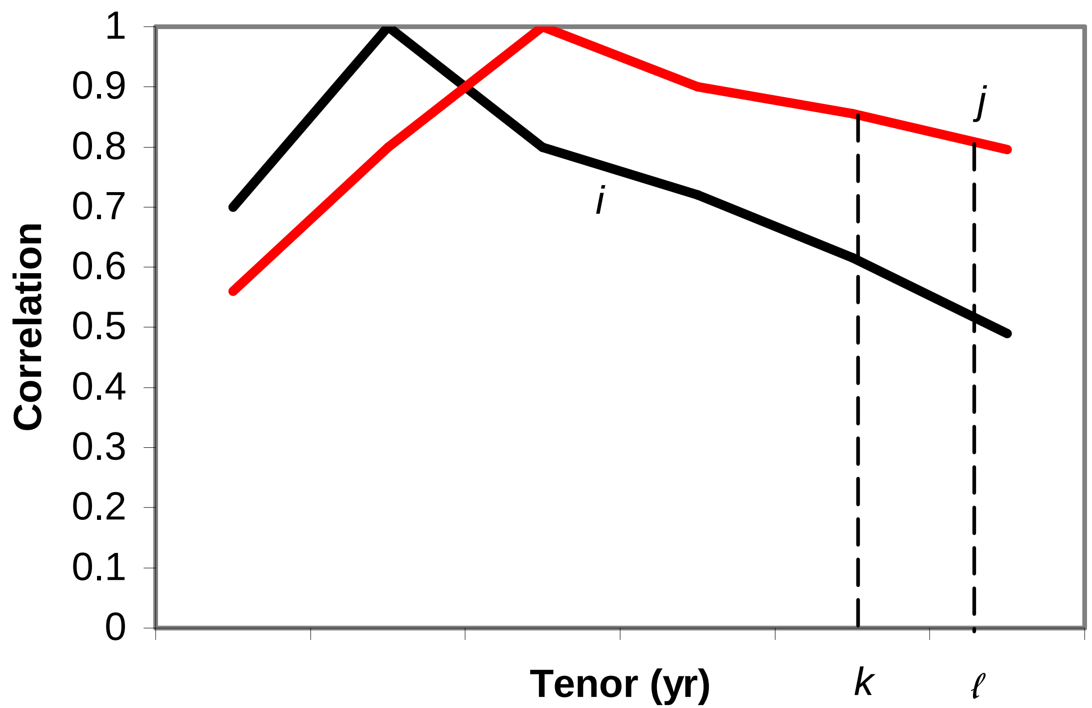
$$\frac{\rho_{ik}}{\rho_{i\ell}} \ge \frac{\rho_{jk}}{\rho_{j\ell}} \Leftrightarrow \frac{\rho_{j\ell} - \rho_{jk}}{\rho_{j\ell}} \ge \frac{\rho_{i\ell} - \rho_{ik}}{\rho_{i\ell}}$$

Slope of the curve

Interpretation of the conditions (3)

Slope (cont'd):

The relative change on the curve for larger tenors should be smaller than for shorter tenors:



Interpretation of the conditions (4)

We refrain from working out the curvature condition here, as it is harder to interpret visually. However, we can conclude:

The sufficient conditions for level, slope and curvature are restrictions on the level, slope and curvature of the correlation matrix.

Lekkos' critique

Lekkos [2000]

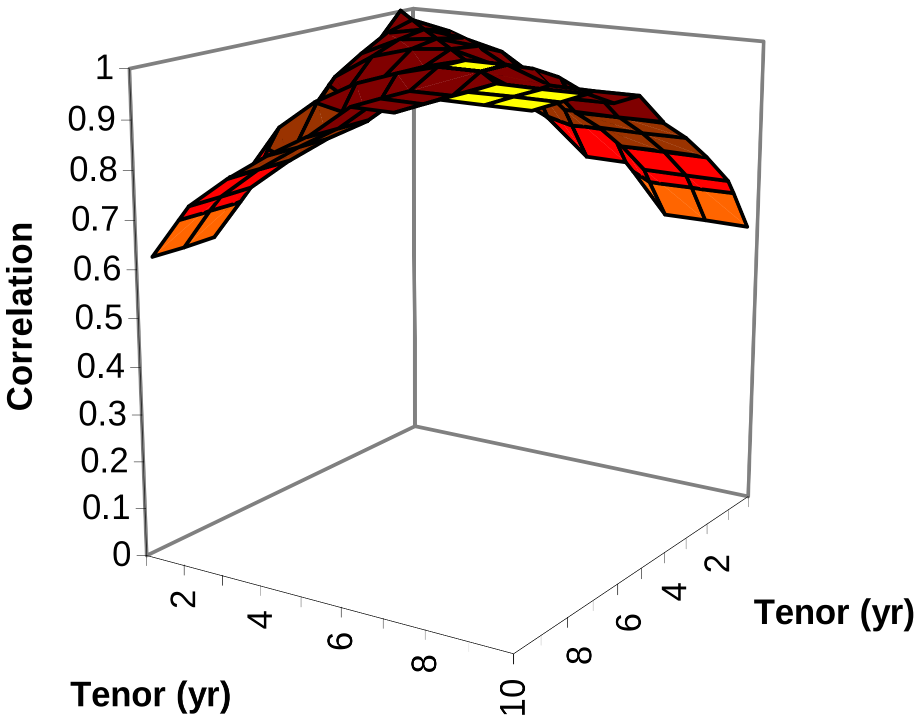
- claims that LSC is caused by the relation between zero yields and forward rates:

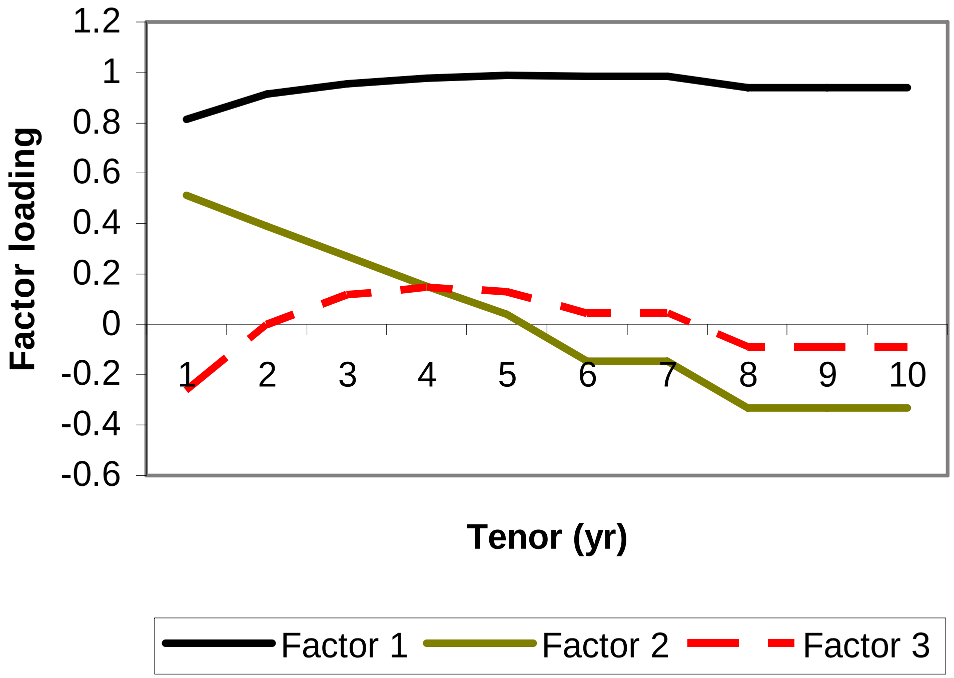
$$R(t,T) = \frac{1}{T-t} \cdot \left(f(t,t,t+\alpha) + \ldots + f(t,T-\alpha,T) \right)$$

- shows (numerically) that if forward rates are uncorrelated, a PCA on zero yields produces LSC;
- doesn't find slope or curvature in forward rates – could be caused by curve construction (bootstrap);
- finds that 5 factors explain 95% of the variation.

Bundesbank data (4)

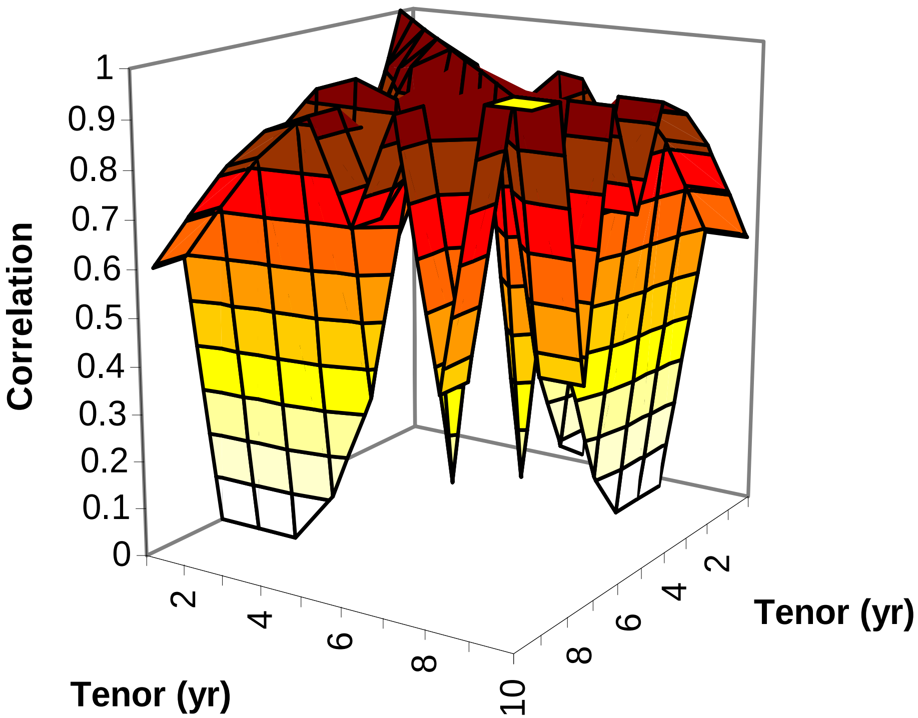
Changes in zero yields, if we bootstrap the curve

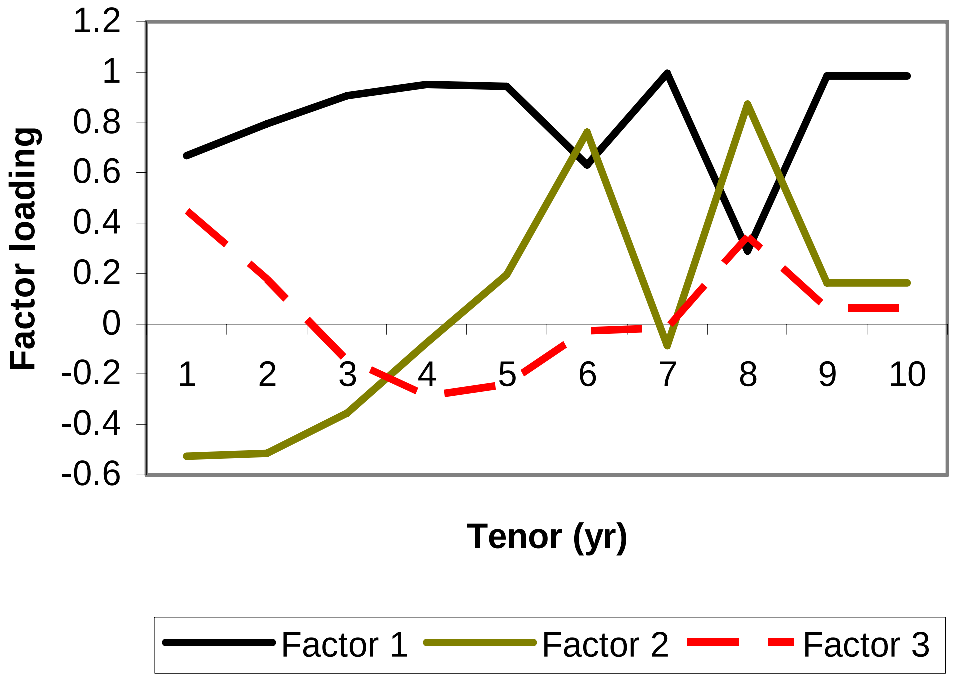




Bundesbank data (5)

Changes in forward rates, if we bootstrap the curve





Lekkos' critique (2)

Back to interest rates:

$$R(t,T) = \frac{1}{T-t} \cdot \big(f(t,t,t+\alpha) + \ldots + f(t,T-\alpha,T)\big)$$

One can easily show that if α-forward rates are uncorrelated, the correlation (or covariance) matrix of the zero yields $(R(t,t+\alpha), \ldots, R(t,T))^T$ will be a Green's or Schoenmakers-Coffey matrix, and will display level, slope and curvature.

This proves Lekkos' numerical finding.

Beyond total positivity

Q: Is total positivity enough?

A: Consider the following matrix, which has all the properties that interest rate correlation matrices typically have, and displays LSC:

$$\mathbf{R} = \begin{pmatrix} 1 & 0.8396 & 0.8297 & 0.8204 \\ 0.8396 & 1 & 0.9695 & 0.901 \\ 0.8297 & 0.9695 & 1 & 0.9785 \\ 0.8204 & 0.901 & 0.9785 & 1 \end{pmatrix} \qquad \begin{vmatrix} 0.8297 & 0.8204 \\ 0.9695 & 0.901 \end{vmatrix} = \mathbf{-0.04782}$$

R, nor any power hereof, is O_3, so that total positivity is not enough. Extensions such as sign regularity are also not useful. So there must be more…

Beyond total positivity (2)

Let us define a quasi-correlation matrix **R** to be a matrix satisfying $\rho_{ii} = 1$ and $-1 \leq \rho_{ij} \leq 1$.

Conjecture

A quasi-correlation matrix **R** with strictly positive entries, satisfying:

i) $\rho_{i,j+1} \leq \rho_{ij}$ for $j \geq i$;

ii) $\rho_{i,j-1} \leq \rho_{ij}$ for $j \leq i$;

iii) $\rho_{i,i+j} \leq \rho_{i+1,i+j+1}$

displays level and slope. □

Beyond total positivity (3)

Results for quasi-correlation matrices:

	Properties (i)-(ii)		Properties (i)-(iii)	
Size	*No slope*	*No curvature*	*No slope*	*No curvature*
3	0%	0%	0%	0%
4	0.04%	19.1%	0%	23.05%
5	0.01%	27.98%	0%	43.81%

For proper correlation matrices:

	Properties (i)-(ii)		Properties (i)-(iii)	
Size	*No slope*	*No curvature*	*No slope*	*No curvature*
3	0%	0%	0%	0%
4	0.13%	14.91%	0%	18.31%
5	0.02%	23.1%	0%	35.38%

Conclusions

- Sufficient condition for LSC is for the matrix to be oscillatory of order 3;
- Formulated an unproven conjecture that order in the correlation matrix causes slope;
- Presence of LSC is probably due to a combination of smoothness, order and flattening out of the correlation surface for larger tenors;
- Empirically shown that forward rates also display LSC, contrary to evidence by Lekkos;
- Shown that the Schoenmakers-Coffey correlation parameterisation displays LSC;
- If forward rates are independent, zero yields display level, slope and curvature.