A Free Boundary Problem Related to Environmental Management Systems

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Introduction

For social and economic reasons corporations and government agencies attach high priority to the development and implementation of environmental policies. The study of real options has led to the application of option pricing techniques from financial engineering to the valuation of projects which include embedded choices of action.

Introduction

In this vein there have recently been papers relating to the application of such ideas to environmental questions. Papers discussing environmental problems include those by Cortazar, Schwartz and Salinas (1998), Fisher (2000) and Scheinkman and Zariphopoulou (2001). The present work extends that of Pindyck (2002): *Optimal timing problems in environmental economics.* J. Econ. Dynamics and Control, 2002 (26), 1677 – 1697.

Introduction

In this paper we consider the optimal time to implement a reduction in the level of pollution. The dynamics we choose for S, the level of pollution, are of log-normal form, rather than the deterministic or mean reverting dynamics of Pindyck (2002). This seems reasonable as the level of pollution may continue to grow in a stochastic manner. We also consider a second process Θ which models the financial, social or political impact (or ‘cost’) of the pollution.

Fixed switching times

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $W = \{W_t\,,\ t \geq 0\}$ be a standard Brownian motion on Ω.

Suppose $S = \{S_t\,,\ t \geq 0\}$ is a stochastic process which measures the level of some environmental degradation or pollution. For example, S might represent the amount of oil residue on some site or the level of degradation of grassland.

Fixed switching times

The evolution of the process S can be controlled by the choice of some control variable u which we take to be real valued. In fact, u will describe the rate of increase of S. Initially u has a value $u_0 > 0$. At some time we can reduce u to a value $u_1 < u_0$, so reducing the rate of increase of S. The results of this paper concern the optimal time to reduce u_0 to u_1.

Fixed switching times

We also consider another process $\Theta = \{\Theta_t\,,\ t \geq 0\}$ which models the financial, social or political impact (or ‘cost’) of pollution. For example, Θ might be reduced by the arrival of new clean-up technology, or Θ might be increased by heightened political concerns.

We assume that initially Θ takes the value Θ_0 and at a later fixed time T it takes one of the values $\underline{\Theta}$ or $\overline{\Theta}$:

$$\underline{\Theta} < \Theta_0 < \overline{\Theta}$$

with probability $1/2$.

Fixed switching times

Suppose the rate of cost of the pollution is $\Theta_t \, S_t$ and the cost of changing from u_0 to u_1 is K.

Further, we suppose S has log-normal dynamics

$$dS_t = \big(\beta \, u_t - \delta\big) \, S_t \, dt + \sigma \, S_t \, dW_t \, .$$

Here σ is the diffusion parameter, δ describes the rate at which, on average, the pollution level decays, and β is an absorption parameter.

Fixed switching times

Let τ be the time of changing from u_0 to u_1, then the total cost is

$$\Phi = \mathbf{E}\left[\int_0^\infty e^{-\rho t}\,\Theta_t\,S_t\,dt + e^{-\rho\tau}\,K\right],$$

where ρ is a discount rate.

We assume $\rho + \delta - \beta\,u_0 > 0$.

Fixed switching times

If the control value is never changed to u_1, then $\tau = \infty$ and

$$\begin{aligned}
\Phi &= \Phi_\infty = \Theta_0 \int_0^T \mathbf{E}\Big[S_t \,\Big|\, S_0\Big]\, e^{-\rho t}\, dt \\
&\qquad + \left(\frac{\overline{\Theta} + \underline{\Theta}}{2} \right) \int_T^\infty \mathbf{E}\Big[S_t \,\Big|\, S_0\Big]\, e^{-\rho t}\, dt \\
&= \frac{\Theta_0\, S_0}{\rho + \delta - \beta\, u_0} \left(1 - e^{(\beta\, u_0 - \delta - \rho) T} \right) \\
&\qquad + \left(\frac{\overline{\Theta} + \underline{\Theta}}{2} \right) \frac{S_0\, e^{(\beta\, u_0 - \delta - \rho) T}}{\rho + \delta - \beta\, u_0} \,.
\end{aligned}$$

Fixed switching times

If the control value is changed to u_1 at time 0, then

$$\begin{aligned}\Phi &= \Phi_0 = \frac{\Theta_0\, S_0}{\rho + \delta - \beta\, u_1}\left(1 - e^{(\beta\, u_1 - \delta - \rho)T}\right) \\ &\qquad + \left(\frac{\overline{\Theta} + \underline{\Theta}}{2}\right)\frac{S_0\, e^{(\beta\, u_1 - \delta - \rho)T}}{\rho + \delta - \beta\, u_1} - K\,.\end{aligned}$$

Fixed switching times

If the control value is changed to u_1 at time T only when Θ switches to $\overline{\Theta}$, then

$$\begin{aligned}\Phi &= \Phi_T = \frac{\Theta_0\, S_0}{\rho+\delta-\beta\, u_0}\left(1-e^{(\beta\, u_0-\delta-\rho)T}\right) \\ &\quad +\frac{\overline{\Theta}}{2}\,\frac{S_0\, e^{(\beta\, u_1-\delta-\rho)T}}{\rho+\delta-\beta\, u_1}+\frac{\underline{\Theta}}{2}\,\frac{S_0\, e^{(\beta\, u_0-\delta-\rho)T}}{\rho+\delta-\beta\, u_0}-\frac{e^{-\rho T}}{2}\,K\,.\end{aligned}$$

Fixed switching times

The relative benefit of changing from u_0 to u_1 at either $0\,,\ T$ or ∞ can be determined by comparing $\Phi_0\,,\ \Phi_T$ and Φ_∞. For example, if $\Theta_0 = 1/2\left(\overline{\Theta} + \underline{\Theta}\right)$, then

$$\Phi_\infty = \frac{\Theta_0\, S_0}{\rho + \delta - \beta\, u_0}\,, \qquad \Phi_0 = \frac{\Theta_0\, S_0}{\rho + \delta - \beta\, u_1} - K\,,$$

$$\begin{aligned}\Phi_T \quad = \quad & \frac{\Theta_0\, S_0}{\rho + \delta - \beta\, u_0} + \frac{\overline{\Theta}}{2}\,\frac{S_0\, e^{(\beta\, u_1 - \delta - \rho)T}}{\rho + \delta - \beta\, u_1} \\ & - \frac{\overline{\Theta}}{2}\,\frac{S_0\, e^{(\beta\, u_0 - \delta - \rho)T}}{\rho + \delta - \beta\, u_0} - \frac{e^{-\rho T}}{2}\, K\,.\end{aligned}$$

Fixed switching times

If we consider, for example,

$$
\begin{aligned}
\Phi_T - \Phi_0 \quad = \quad & \left(1 - \frac{e^{-\rho T}}{2}\right) K \\
& + \Theta_0 \, S_0 \left(\frac{1}{\rho + \delta - \beta \, u_0} - \frac{1}{\rho + \delta - \beta \, u_1} \right) \\
& + \frac{\overline{\Theta}}{2} \, \frac{S_0 \, e^{(\beta \, u_1 - \delta - \rho) T}}{\rho + \delta - \beta \, u_1} - \frac{\overline{\Theta}}{2} \, \frac{S_0 \, e^{(\beta \, u_0 - \delta - \rho) T}}{\rho + \delta - \beta \, u_0} \, ,
\end{aligned}
$$

the first term on the right represents the present value of the cost savings from delaying the implementation.

Fixed switching times

As

$$\mathbf{E}\Big[S_t \,\Big|\, S_0\Big] = S_0 \, e^{(\beta\, u_i - \delta)\, t}$$

with $i = 0$ or 1, depending on whether the control policy is adopted, the remaining terms represent the difference in expected discounted cost.

A continuous time environmental model

We now suppose the cost of the pollution Θ is described by log-normal dynamics in continuous time.

As before we suppose S has log-normal dynamics

$$dS_t = \big(\beta\, u_t - \delta\big)\, S_t\, dt + \sigma\, S_t\, dW_t\,. \tag{1}$$

Note that if $u_t = 0$ then

$$\mathbf{E}\big[S_t\big] = e^{-\delta t}\, S_0\,,$$

so then δ describes the rate at which, on average, the pollution level declines over time.

A continuous time environmental model

Here $u = \{u_t\,,\ t \geq 0\}$ represents a control policy and regulates the rate at which the pollution level is increasing.

We suppose that initially u takes some initial value $u_0 > 0$. It remains at this level until a time τ, which is to be chosen. At time τ, u takes a new lower value $0 \leq u_1 < u_0$. Switching u from u_0 to u_1 represents a decrease in the rate of pollution. We suppose that the cost of implementing this policy is K.

A continuous time environmental model

Consider a second independent Brownian motion $B = \left\{ B_t \,,\ t \geq 0 \right\}$ defined on $\left(\Omega, \mathcal{F}, \mathbf{P} \right)$.

We introduce a process $\Theta = \left\{ \Theta_t \,,\ t \geq 0 \right\}$ which represents a cost or weighting of the pollution process S.

For example, Θ might represent the introduction of new technology which reduces the impact of S, or a change in political opinion which increases the impact of S.

A continuous time environmental model

We suppose the dynamics of Θ are given by

$$d\Theta_t = \mu\,\Theta_t\,dt + \gamma\,\Theta_t\,dB_t\,,$$

so that

$$\Theta_t = \Theta_0\,\exp\left\{\left(\mu - \gamma^2/2\right)t + \gamma\,B_t\right\}. \tag{2}$$

We suppose in general the flow of social cost of the pollution is given by a function $b(S_t, \Theta_t)$. For simplicity we suppose

$$b(S_t, \Theta_t) = S_t\,\Theta_t\,. \tag{3}$$

A continuous time environmental model

Before the time τ the process S has the dynamics

$$dS_t = \Big(\beta\, u_0 - \delta\Big)\, S_t\, dt + \sigma\, S_t\, dW_t\ ,$$

so

$$S_t = S_0\ \exp\left\{\Big(\beta\, u_0 - \delta - \sigma^2/2\Big)\, t + \sigma\, W_t\right\} \tag{4}$$

for $0 \le t \le \tau$.

A continuous time environmental model

After time τ the dynamics are

$$dS_t = \big(\beta\, u_1 - \delta\big)\, S_t\, dt + \sigma\, S_t\, dW_t\ ,$$

so

$$S_t = S_\tau\ \exp\left\{\left(\beta\, u_1 - \delta - \sigma^2/2\right)(t-\tau) + \sigma\,\big(W_t - W_\tau\big)\right\} \qquad (5)$$

for $t \geq \tau$.

A continuous time environmental model

The objective is to minimize the value function

$$\begin{aligned} v(S_t, \Theta_t) &= \mathbf{E}\left[\int_t^{\infty} e^{-\rho(s-t)}\, b(S_s, \Theta_s)\, ds + e^{-\rho(\tau - t)}\, K \,\middle|\, \mathcal{F}_t \right] \qquad (6) \\ &= \mathbf{E}\left[\int_t^{\infty} e^{-\rho(s-t)}\, S_s\, \Theta_s\, ds + e^{-\rho(\tau - t)}\, K \,\middle|\, S_t, \Theta_t \right] \end{aligned}$$

over stopping times $\tau \geq t$.

A continuous time environmental model

As the time horizon is infinity, the value function is independent of t. The problem is one of *impulse control*. The quadrant $\{\Theta \geq 0,\ S \geq 0\}$ is divided into two regions by an optimal curve $\Theta^*(S)$. If $(\Theta,\ S)$ is such that $\Theta \geq \Theta^*(S)$, then the optimal policy is to implement the reduction from u_0 to u_1 immediately. That is, knowing the values Θ_t and S_t at time t, if $\Theta_t \geq \Theta^*(S_t)$, then we should choose $\tau = t$. If $\Theta_t < \Theta^*(S_t)$, then we should allow our processes Θ and S to continue to run.

A continuous time environmental model

If $\Theta_t \geq \Theta^*(S_t)$ we say we are in the *adoption region* and from equations (6), (5) and (2) we have

$$\begin{aligned} v(S_t, \Theta_t) &= v^A(S_t, \Theta_t) \qquad (7) \\ &= \mathbf{E}\left[\int_t^\infty e^{-\rho(s-t)}\, S_s\, \Theta_s\, ds + K \,\middle|\, S_t, \Theta_t \right] \\ &= S_t\, \Theta_t \int_t^\infty \exp\left\{ -\big(\rho - \beta\, u_1 + \delta - \mu\big)\,(s-t) \right\} ds + K\,. \end{aligned}$$

A continuous time environmental model

If $\rho - \beta\, u_1 + \delta - \mu > 0$ then we have

$$v(S_t, \Theta_t) = v^A(S_t, \Theta_t) = \frac{S_t\, \Theta_t}{\rho - \beta\, u_1 + \delta - \mu} + K\,. \qquad (8)$$

If $\Theta_t < \Theta^*(S_t)$ we say we are in the *continuation region* and then dynamic programming shows that $v(S_t, \Theta_t) = v^C(S_t, \Theta_t)$ satisfies the equation

$$\frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 v^C}{\partial S^2} \quad + \quad \frac{1}{2}\,\gamma^2\, \Theta^2\, \frac{\partial^2 v^C}{\partial \Theta^2} + (\beta\, u_0 - \delta)\, S\, \frac{\partial v^C}{\partial S} \qquad (9)$$
$$+\mu\, \Theta\, \frac{\partial v^C}{\partial \Theta} + \Theta\, S - \rho\, v^C = 0\,.$$

A continuous time environmental model

On the boundary curve $\Theta^*(S)$ we must have

$$v^C(S, \Theta^*(S)) = v^A(S, \Theta^*(S))\,. \tag{10}$$

Further, the first order pasting conditions should be satisfied on $\Theta^*(S)$:

$$\frac{\partial v^C(S, \Theta^*(S))}{\partial S} = \frac{\partial v^A(S, \Theta^*(S))}{\partial S} \tag{11}$$

and

$$\frac{\partial v^C(S, \Theta^*(S))}{\partial \Theta} = \frac{\partial v^A(S, \Theta^*(S))}{\partial \Theta}\,. \tag{12}$$

A continuous time environmental model

A particular solution of equation (9) is

$$\frac{\Theta\, S}{\rho - \mu - \beta u_0 + \delta}\,.$$

Therefore we look for a solution of (9) of the form

$$v^C(S,\Theta) = \frac{\Theta\, S}{\rho - \mu - \beta u_0 + \delta} + \kappa\, \Theta^a\, S^b\,. \tag{13}$$

Substituting (13) in (9) we have that a and b should satisfy

$$\frac{1}{2}\sigma^2 b^2 + \frac{1}{2}\gamma^2 a^2 + b\big(\beta u_0 - \delta\big) + \mu a - \rho = 0\,. \tag{14}$$

A continuous time environmental model

Substituting (8) and (13) in (10), (11) and (12) we obtain

$$\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_0+\delta}+\kappa\left(\Theta^*(S)\right)^a S^b=\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_1+\delta}+K\,, \tag{15}$$

$$\frac{\Theta^*(S)}{\rho-\mu-\beta u_0+\delta}+\kappa\,b\left(\Theta^*(S)\right)^a S^{b-1}=\frac{\Theta^*(S)}{\rho-\mu-\beta u_1+\delta}\,, \tag{16}$$

$$\frac{S}{\rho-\mu-\beta u_0+\delta}+\kappa\,a\left(\Theta^*(S)\right)^{a-1} S^b=\frac{S}{\rho-\mu-\beta u_1+\delta}\,. \tag{17}$$

A continuous time environmental model

Hence

$$\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_0+\delta}+\kappa\,b\left(\Theta^*(S)\right)^a S^b=\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_1+\delta} \qquad (18)$$

and

$$\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_0+\delta}+\kappa\,a\left(\Theta^*(S)\right)^a S^b=\frac{\Theta^*(S)\,S}{\rho-\mu-\beta u_1+\delta}\,,$$

which implies

$$\kappa\,(a-b)=0\,,$$

so $a=b$.

A continuous time environmental model

Thus equation (14) becomes

$$a^2\left(\sigma^2+\gamma^2\right)+2a\left(\beta u_0-\delta+\mu\right)-2\rho=0\,.$$

The positive root of this equation for a is

$$a=\frac{-\left(\beta u_0-\delta+\mu\right)+\sqrt{\left(\beta u_0-\delta+\mu\right)^2+2\left(\sigma^2+\gamma^2\right)\rho}}{\left(\sigma^2+\gamma^2\right)}\,.$$

A continuous time environmental model

Then

$$v^C(S,\Theta) = \frac{\Theta\, S}{\rho - \mu - \beta u_0 + \delta} + \kappa\, \Theta^a\, S^a$$

and subtracting equation (18) from equation (15) we have

$$\kappa\,(1-a)\,\big(\Theta^*(S)\big)^a\, S^a = K\,.$$

A continuous time environmental model

Hence

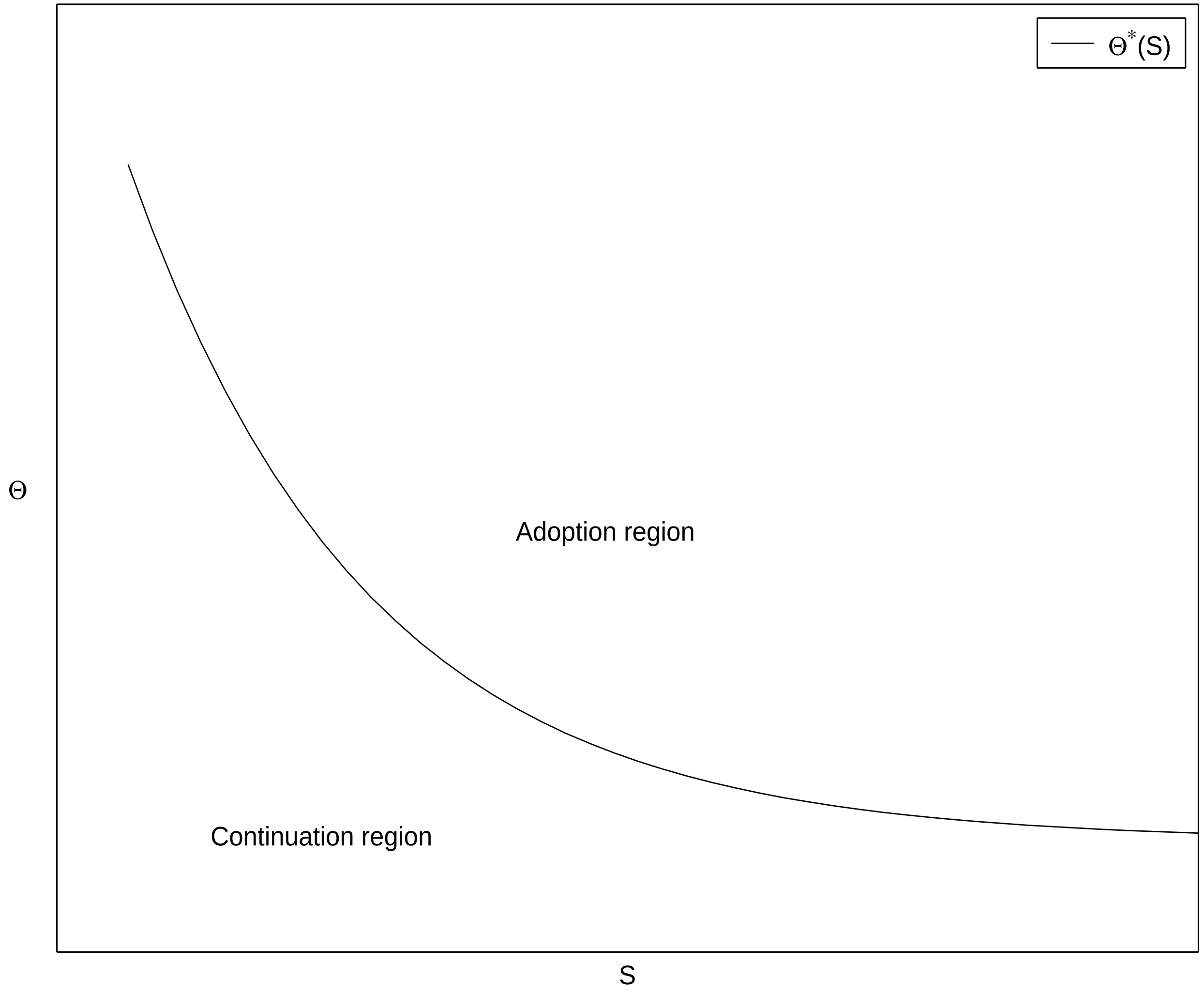
$$\kappa = \frac{K}{(1-a)\left(\Theta^{*}(S)\right)^{a} S^{a}}$$

and therefore equation (16) implies

$$\Theta^{*}(S) = \frac{K\,a}{S\,(a-1)}\;\frac{(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}{\beta\,(u_0-u_1)}\,. \tag{19}$$

This is the equation for the *free boundary* which determines the optimal policy.

A continuous time environmental model



A verification result

In this section we re-derive, using a different approach, the free boundary and we show why it provides the optimal switching curve. We shall write Θ^* for $\Theta^*(S)$. Recall equation (15)

$$\frac{\Theta^*\,S}{\rho-\mu-\beta u_0+\delta}+\kappa\left(\Theta^*\right)^a S^b=\frac{\Theta^*\,S}{\rho-\mu-\beta u_1+\delta}+K\,.$$

This gives

$$\kappa=\frac{1}{\left(\Theta^*\right)^a S^b}\left[\frac{\Theta^*\,S\,\beta\,(u_1-u_0)}{(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}+K\right]. \tag{20}$$

A verification result

Substituting in (13) we have

$$
\begin{aligned}
v^C(S,\Theta) &= \frac{\Theta\, S}{\rho - \mu - \beta u_0 + \delta} \\
&+ \frac{\Theta^a}{\left(\Theta^*\right)^a}\left[\frac{\Theta^*\, S\, \beta\, (u_1 - u_0)}{(\rho - \mu - \beta u_0 + \delta)\,(\rho - \mu - \beta u_1 + \delta)} + K\right]
\end{aligned}
\tag{21}
$$

Now $\Theta^* = \Theta^*(S)$ should be chosen so that $v^C(S,\Theta)$ is minimized. Consequently, differentiating (21) with respect to Θ^* we have

A verification result

$$\begin{aligned}&\frac{\partial v^C}{\partial \Theta^*} \\ &= -a\left(\frac{\Theta}{\Theta^*}\right)^a \frac{1}{\Theta^*}\left[\frac{\Theta^*\, S\,\beta\,(u_1-u_0)}{(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}+K\right] \\ &+\left(\frac{\Theta}{\Theta^*}\right)^a \frac{S\,\beta\,(u_1-u_0)}{(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}\end{aligned} \tag{22}$$

and $\partial v^C/\partial\Theta^* = 0$ when as in (19)

$$\Theta^* = \Theta^*(S) = \frac{K\,a\,(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}{S\,(a-1)\,\beta\,(u_0-u_1)}\,. \tag{23}$$

A verification result

To show that $\Theta^*(S)$ does indeed give the minimum value we now check that

$$\frac{\partial^2 v^C}{\partial(\Theta^*)^2} > 0$$

at $\Theta^*(S)$. From (22) we have

$$(\Theta^*)^{a+1}\,\frac{\partial v^C}{\partial\Theta^*} = \frac{\Theta^a\,\Theta^*\,S\,(a-1)\,\beta\,(u_0-u_1)}{(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)} + K\,.$$

A verification result

Differentiating again in $\Theta^*(S)$ we obtain

$$
\begin{aligned}
(a+1)(\Theta^*)^a \frac{\partial v^C}{\partial \Theta^*} &+ (\Theta^*)^{a+1} \frac{\partial^2 v^C}{\partial (\Theta^*)^2} \\
&= \frac{\Theta^a \, S \, (a-1) \, \beta \, (u_0 - u_1)}{(\rho - \mu - \beta u_0 + \delta) \, (\rho - \mu - \beta u_1 + \delta)} \, .
\end{aligned}
$$

Evaluating both sides at the optimal $\Theta^* = \Theta^*(S)$, where $\partial v^C / \partial \Theta^* = 0$, we have

A verification result

$$\left(\Theta^*(S)\right)^{a+1} \frac{\partial^2 v^C\left(S,\, \Theta^*(S)\right)}{\partial\left(\Theta^*(S)\right)^2} = \frac{\Theta^a\, S\,(a-1)\,\beta\,(u_0 - u_1)}{(\rho - \mu - \beta u_0 + \delta)\,(\rho - \mu - \beta u_1 + \delta)}\,.$$

Consequently,

$$\frac{\partial^2 v^C\left(S,\, \Theta^*(S)\right)}{\partial\left(\Theta^*(S)\right)^2} > 0$$

A verification result

and $\Theta^*(S)$ gives the unique local minimum for values of $\Theta^* \in (0,\infty)$. This enables us to state the following result.

Theorem 1 *For the pollution process S with dynamics given by (1) and the weight process Θ given by (2) the optimal time to lower the pollution rate from u_0 to u_1, is the first time at which $\Theta \geq \Theta^*(S)$.*

A verification result

We also note from the form of the equation (19) that for any $\Theta \in (0,\infty)$ there is an $S^*(\Theta)$ such that $\Theta = \Theta^*\big(S^*(\Theta)\big)$. In fact,

$$S^*(\Theta) = \frac{K\,a\,(\rho-\mu-\beta u_0+\delta)\,(\rho-\mu-\beta u_1+\delta)}{\Theta\,S\,(a-1)\,\beta\,(u_0-u_1)}\,.$$

A verification result

Then, for a fixed value Θ if

$$S < S^*(\Theta)$$

the processes should be allowed to continue. If

$$S \geq S^*(\Theta)$$

the pollution rate should be switched from u_0 to u_1.

Conclusion

Modifying the model of Pindyck we have discussed an environmental management problem described by two random processes: the pollution level S and the impact or social cost of pollution Θ. The effect of reducing the rate of pollution is investigated. When S and Θ are described by log-normal dynamics, the optimal time to reduce the rate of pollution is the solution of a free boundary problem. We solve for the free boundary in two ways and show that this gives the optimal policy.

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