Quadratic Models for Portfolio Credit Risk with Shot-noise Effects

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Credit Risk Models

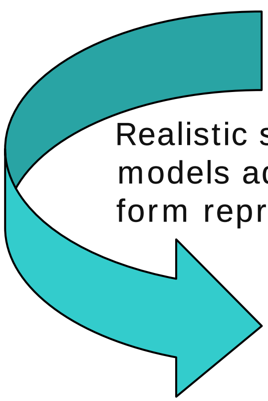
- **Approaches**

Structural approach

- Based on capital structure of the firm (economic intuition)
- Easy to study equity/credit link within firm

but

- Rare close form solutions (must rely on simulations)
- Unrealistic short term spreads
- Hard to make realistic models
- Hard to justify a multiple-firm model
- Hard to implement in practice



Reduced form approach

- Based on arbitrage theory with jump processes
- Allow for realistic assumptions on the default process
- Easily extends to a multi-firm setup
- Close-form solutions (in non-trivial cases)
- Easier to implement in practice

but

- Difficult to study equity/credit link

➢Our Goals

- Use the **general quadratic setting** to obtain closed-form solutions

  This class of TS include as special cases ATS and Gaussian-QTS.

  It is as far as we can go in terms of exponential polynomials.

- Incorporate realistic features using **shot-noise processes**:

  Clustering of defaults within firms

  Correlation of defaults across firms

- Deal with **portfolio** credit risk derivatives

➢Previous Literature

Closed-form solutions	deterministic intensity of default
	Affine intensity models
Realistic features	Hard to replicate.
	Even in affine models reasonable correlation across defaults imply unreasonable intensities.
Other Problems	Some affine intensity models do not guarantee positiveness of intensity.

The Model

- The default intensity is $$\mu_t = \eta_t + J_t$$

$$\eta(t, Z_t) = Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t)$$

Predictable component

Finite state variable driven by Wiener Process.

$$dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW_t$$

$$J_t = \sum_{\tilde{\tau}_i \leq t} Y_i h(t - \tilde{\tau}_i)$$

Unpredictable component

Jump process where $\tilde{\tau}_i, i = 1, 2, \ldots$ are the jump times of a standard Poisson process.

The function h can be, for instance,

$$h(t) = ae^{-bt}$$

BIG Advantages:

- The intensity is positive in a natural way
- The processes η_t and J_t are **independent**.

Predictable Component

- **General Quadratic Term Structures (GQTS)** are a generalization of affine term structures and Gaussian-quadratic term structures.
- From the previous literature - Gaspar(2004) – we know that

If

$$\eta(t, Z_t) = Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t)$$

with

$$dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW_t$$

$$\alpha(t, z) = d(t) + E(t)z$$

$$\sigma(t, z)\sigma^\top(t, z) = k_0(t) + \sum_{i=1}^{m} k_i(t) z_i + \sum_{i,j=1}^{m} z_i\, g_{ij}(t) z_j$$

Possible due to the *a priori* classification of factors in GQTS

I :

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \eta_u\, du\right)\Big|\mathcal{F}_t^W\right] = \exp\left[\mathcal{A}(t,T) + \mathcal{B}^\top(t,T) Z_t + Z_t^\top \mathcal{C}(t,T) Z_t\right]$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ solve the basic ODE system.

Unpredictable Component

$$J_t = \sum_{\tilde{\tau}_i \le t} Y_i h(t - \tilde{\tau}_i)$$

Advantage 1 :

II : It is treatable and it fits the GQTS framework !

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T J_u\,du\right)\Big|\mathcal{F}_t^J\right] = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + (T-t)l\big(D(T-t)-1\big)\right]$$

where

$$\int_0^1 \varphi_Y\Big(H(xu)\Big)\,du =: D(x)$$

$$H(x) = \int_0^x h(u)\,du$$

$$\tilde{J}(t,T) = \sum_{\tilde{\tau}_i \le t} Y_i H(T - \tilde{\tau}_i) \qquad \tilde{J}(t,t) = \tilde{J}_t$$

Advantage 2 :

Clustering of defaults!

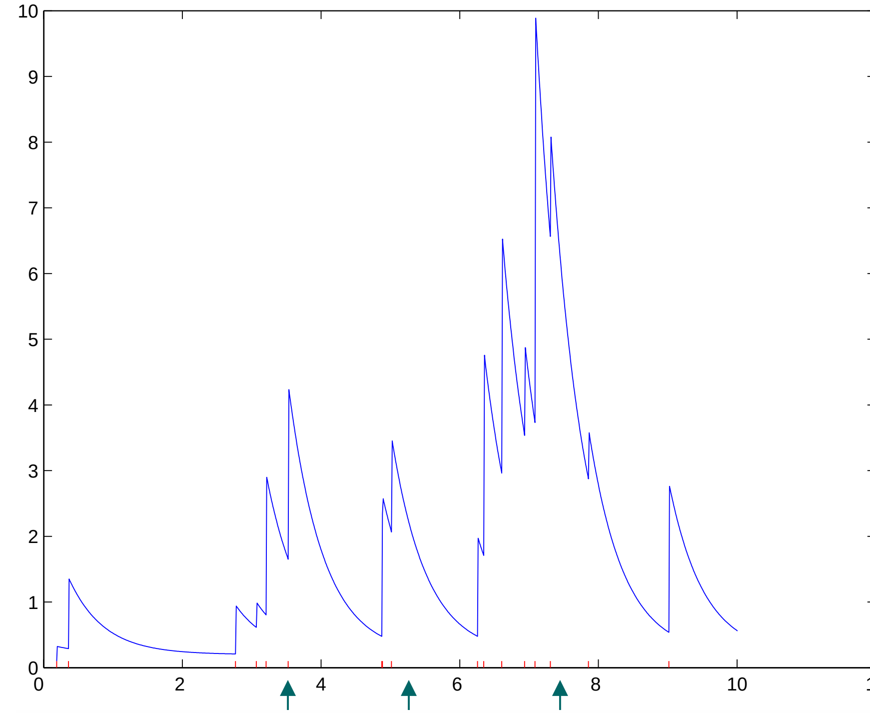


Figure 2: Possible realization of the process J with $h(x) = e^{-0.5x}$ and χ_2^2-distributed Y_i.

Building Blocks

- **Survival Probabilities**

$$
\begin{aligned}
\mathbb{Q}\left[\tau > T \middle| \mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_u\, du} \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_t^T \eta_u + J_u\, du) \middle| \mathcal{F}_t\right] \\
&= \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_t^T \eta_u\, du) \middle| \mathcal{F}_t^W\right]}_{\text{I}} \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_t^T J_u\, du) \middle| \mathcal{F}_t^J\right]}_{\text{II}}
\end{aligned}
$$

$$
\boxed{\mathbb{Q}_S(t,T) = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + \mathcal{A}(t,T) + xl\big(D(x)-1\big) + \mathcal{B}^\top(t,T)Z_t + Z_t^\top \mathcal{C}(t,T) Z_t\right]}
$$

➢Zero-coupon bond prices

Assuming that the default-free short rate is also quadratic

$$r(t, Z_t) = Z_t^\top Q(t) Z_t + g^\top(t) Z_t + f(t)$$

$$\bar{p}_0(t,T) = \exp\left[\tilde{J}_t - \tilde{J}(t,T) + \bar{A}(t,T) + xl\big(D(x)-1\big) + \bar{B}^\top(t,T)Z_t + Z_t^\top \bar{C}(t,T)Z_t\right]$$

where $\bar{A}, \bar{B}, \bar{C}$ solve the basic ODE system as before but with different parameters.

➢Default-digital payoffs

$$e(t,T) = \bar{p}_o(t,T) \cdot \Big\{ \bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top \bar{c}(t,T) Z_t + J(t,T) - l \cdot \left[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\right] \Big\}$$

$$e^*(t, T_{n-1}, T_n) = \bar{p}_o(t, T_{n-1}) e^{\alpha(t,T_{n-1},T_n) + \beta^\top(t,T_{n-1},T_n)Z_t + Z_t^\top \gamma(t,T_{n-1},T_n) Z_t} - \bar{p}_o(t, T_n)$$

where $\bar{a}, \bar{b}, \bar{c}$ and α, β, γ solve the ODE systems that depend on solutions of basic ODE systems.

Recovery / Credit Derivatives

Idea :

Use previously computed key ingredients to price in closed-form more complex credit products.

In the paper we price in closed-form:

- Defaultable bonds with **different recovery assumptions**
- **credit derivatives**:
  - Default Digital Payoffs
  - Credit Default Swaps
  - Options on defaultable bonds (up to numerical evaluation of the Laplace transform)

Extension to portfolios

- $k = 1, \cdots, \bar{K}$
- Each firm may default only once and its default time is denoted by T^k
- $N_t := \sum \mathbf{1}_{\{T^k \leq t\}}$

We order the default times $T^1, \ldots, T^{\bar{K}}$ and denote the outcome by $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{\bar{K}}$

Basic Assumption :

Consider independent processes μ^i of the form quadratic plus jump

The default intensity of each defaultable firm $k \in \mathbf{k}$ is modeled as

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c,$$

- Jump part: Clustering/Contagion
- Quadratic part: Business Cycle

The higher ϵ_i the bigger is the dependence of the common default risk driven by μ^c.

Building Blocks for Portfolio Credit Risk

$$S^i_\eta(\theta,t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right] = e^{\mathcal{A}^i(\theta,t,T)+\mathcal{B}^{i\top}(\theta,t,T)Z_t+Z_t^\top\mathcal{C}^i(\theta,t,T)Z_t}$$

$$S^i_J(\theta,t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right] = e^{\theta(\tilde{J}_t-\tilde{J}(t,T))+l^i x[D^i(\theta,x)-1]}$$

$$\bar{S}^c_\eta(\theta,t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right] = e^{\bar{A}^c(\theta,t,T)+\bar{B}^{c\top}(\theta,t,T)Z_t+Z^\top\bar{C}^c(\theta,t,T)Z_t}$$

$$\begin{aligned}
\Gamma^i_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^i_T e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right]\\
&= S^i_\eta(\theta,t,T)\cdot\left(a^i(\theta,t,T)+b^{i\top}(\theta,t,T)Z_t+Z_t^\top c^i(\theta,t,T)Z_t\right)\\
\Gamma^i_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta J^i_T e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right]\\
&= S^i_J(\theta,t,T)\cdot\left[\theta J^i(t,T)-l^i\cdot\left(D^i(\theta,x)(1-x)-1+x\varphi^i(\theta H^i(x))\right)\right]\\
\bar{\Gamma}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^c_T e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right]\\
&= \bar{S}^c_\eta(\theta,t,T)\cdot\left(\bar{a}^c(\theta,t,T)+\bar{b}^{c\top}(\theta,t,T)Z_t+Z_t^\top\bar{c}^c(\theta,t,T)Z_t\right)
\end{aligned}$$

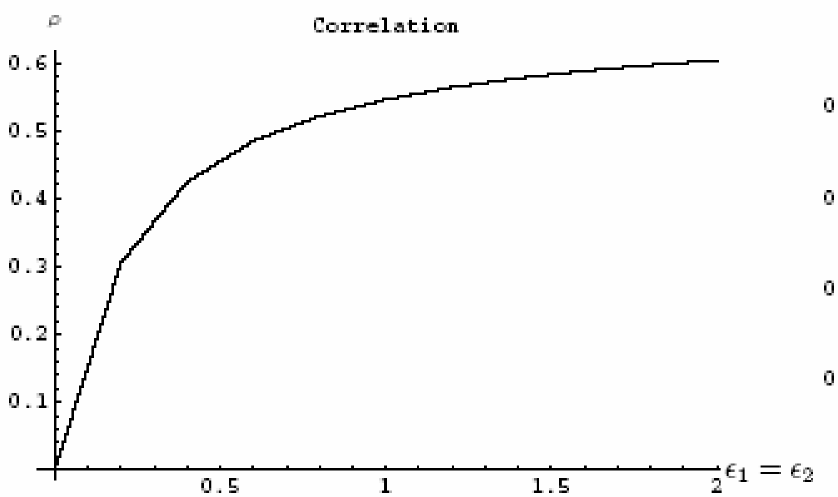
Two Firms Case

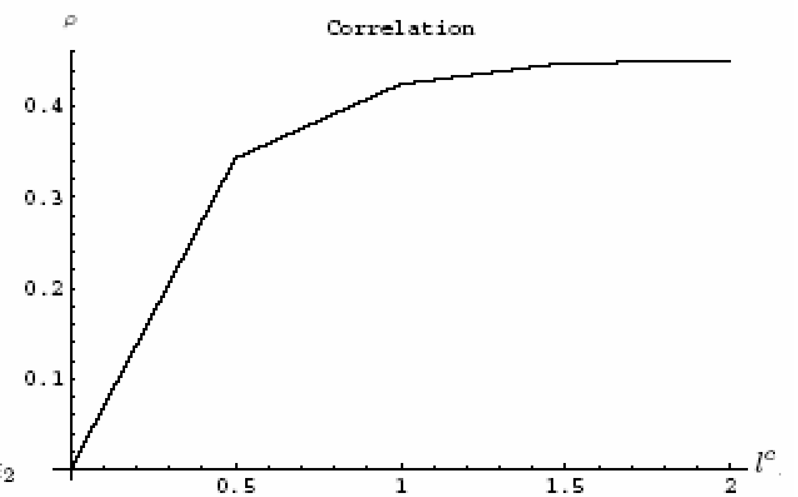
- Default Correlation

$$\rho^{i,j}(t,T) = \frac{\mathbb{Q}_D^{i,j}(t,T) - \mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T)}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}$$

$$\mathbb{Q}_D^{i,j}(t,T) = 1 - \mathbb{Q}_S^i(t,T) - \mathbb{Q}_S^j(t,T) + S^i(t,T)S^j(t,T)S^c(\epsilon^i + \epsilon^j, t, T)$$

$$\mathbb{Q}_S^k(t,T) = S^k(t,T) \cdot S^c(\epsilon^k, t, T) \qquad \mathbb{Q}_D^k(t,T) = 1 - \mathbb{Q}_S^k(t,T)$$





Copula models

Our model

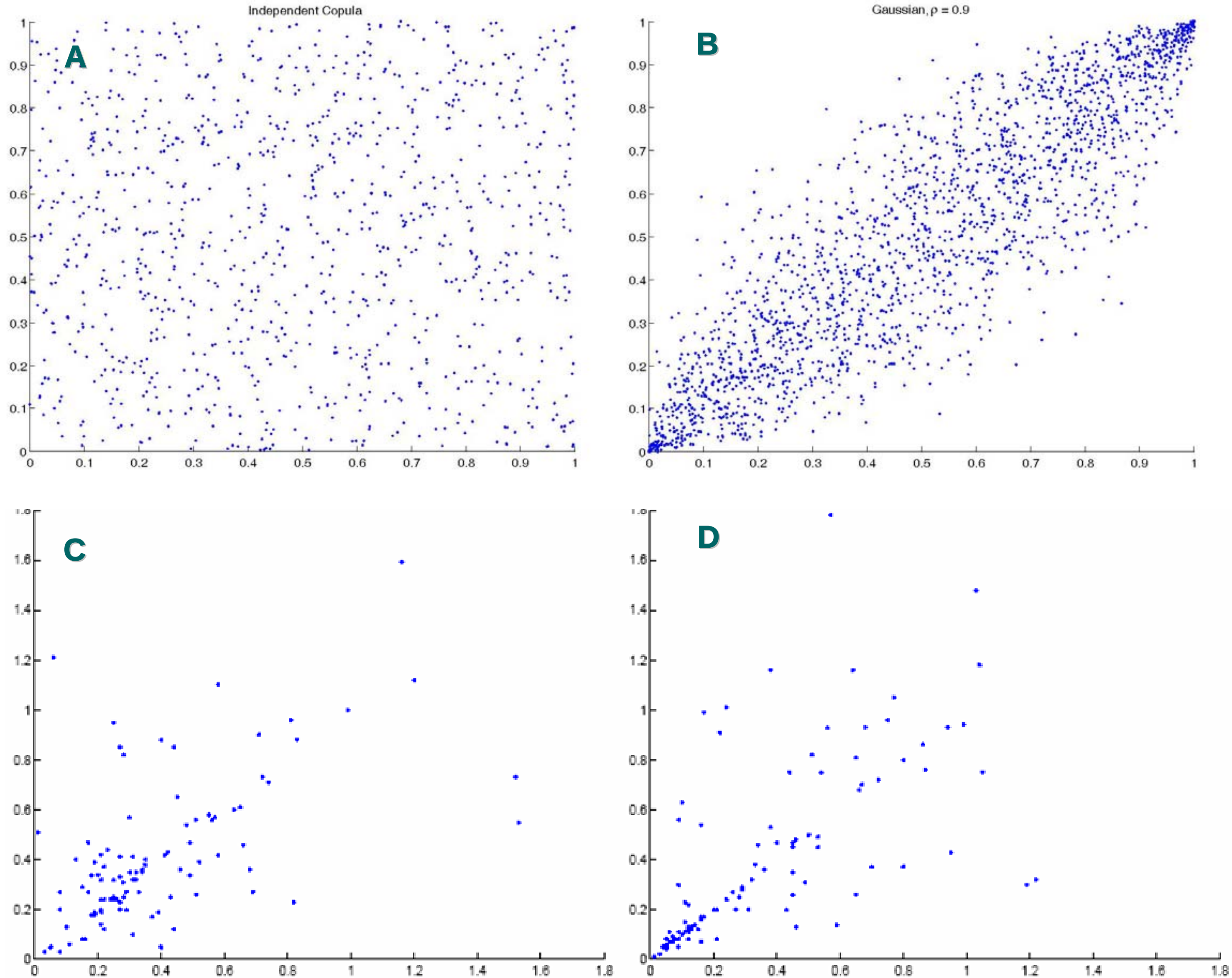
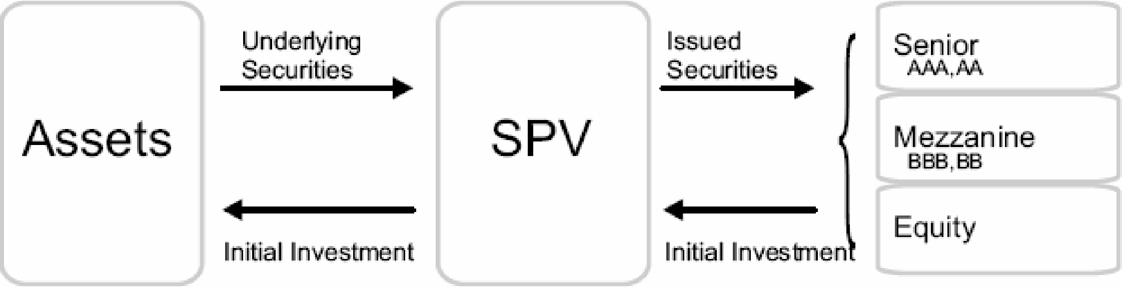


Figure 5: The left picture has $\epsilon_i = 0.1$, the right $\epsilon_i = 0.5$.

Collateralized Debt Obligations

- **CDO**



- There exist at least three tranches, in descendent order of seniority: an *equity tranche* (or first loss), a *mezzanine tranche* and a *senior tranche*. It is, however, common to find CDOs with several mezzanine tranches with different seniorities.

- If during the life of the CDO one (of several) of the names default, all recovery payments are reinvested in default-free securities.

- At the maturity of the CDO, the portfolio is liquidated and the proceeds distributed to the tranches, according to their seniority ranking.

The loss process

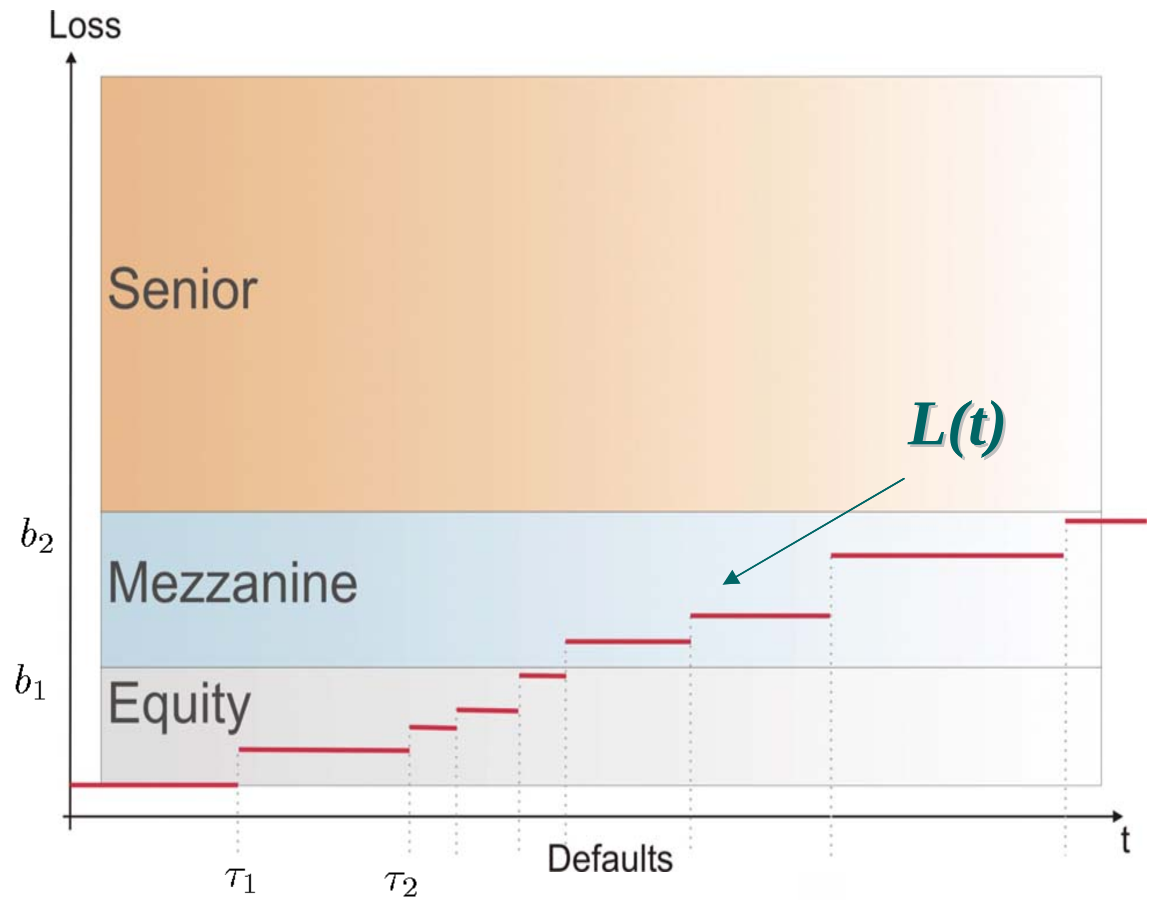
- Initial Value of tranches

$$V^i(0) = b^i - b^{i-1}$$

- Total Loss Process

$$L(t) := \sum_{j=1}^{N_t} \xi_j$$

loss at default time τ_j



- loss in tranche i

$$L^i(t) = \begin{cases} 0 & \text{if } L(t) < b^{i-1} \\ L(t) - b^{i-1} & \text{if } b^{i-1} \leq L(t) < b^i \\ b^i - b^{i-1} & \text{if } L(t) \geq b^i \end{cases}$$

Coupons and tranches value

- coupon dates $t_1, \cdots, t_{N^*}$ → maturity
- intermediate payments $\left(1 - \frac{L^i(t_j)}{b^i - b^{i-1}}\right) c^i$
- maturity payment $\left(1 - \frac{L^i(t_{N^*})}{b^i - b^{i-1}}\right) c^i + \left(b^i - b^{i-1} - L^i(t_{N^*})\right)$

Value of each tranche

$$V^i(t) = \mathbb{E}_t^Q \left[\sum_{j=1}^{N^*} e^{-\int_t^{t_j} r(u)du} \left(1 - \frac{L^i(t_j)}{b^i - b^{i-1}}\right) c^i + e^{-\int_t^{t_{N^*}} r(u)du} \left(b^i - b^{i-1} - L^i(t_{N^*})\right) \right]$$

everything boils down to compute $\mathbb{E}_t^Q \left[e^{-\int_t^{t_j} r(u)du} L^i(t_j) \right]$

We get closed-form expressions again!

Conclusions

- Current credit risk models are either unrealistic, or hard to implement in practice. Serious drawbacks for realistic modeling of CDOs.
- Using the intuitions from GQTS and shot-noise processes, we propose a class of credit risk model which has the models proposed previously as special cases and we derive closed formulas for all credit risk building blocks.
- The model can be easily extended to the several firm setup and to include firm specific and systematic risks.
- Even with a simple model we are able to generate reasonable default correlation levels

Extra Slides

Main References

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Duffie, D. and D. Lando (2001). Term structures of credit spreads with incomplete accounting information. *Econometrica 69*, 633–664.

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Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research 2*, 99–120.

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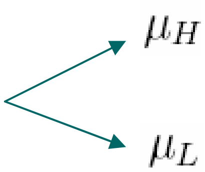
Unpredictable Component



➢ **Motivation** : **Incomplete Information**

Agents in the market have incomplete information about the quality of firms and adapt their expectations.

$$\mathbb{P}(\mu = \mu_H | \tau > t) = \frac{\mathbb{P}(\mu = \mu_H, \tau > t)}{\mathbb{P}(\tau > t)} = \frac{pe^{-\mu_H t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}$$

$$\mathbb{E}(\mu | \tau > t) = \frac{\mu_H pe^{-\mu_H t} + \mu_L (1-p)e^{-\mu_L t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}$$

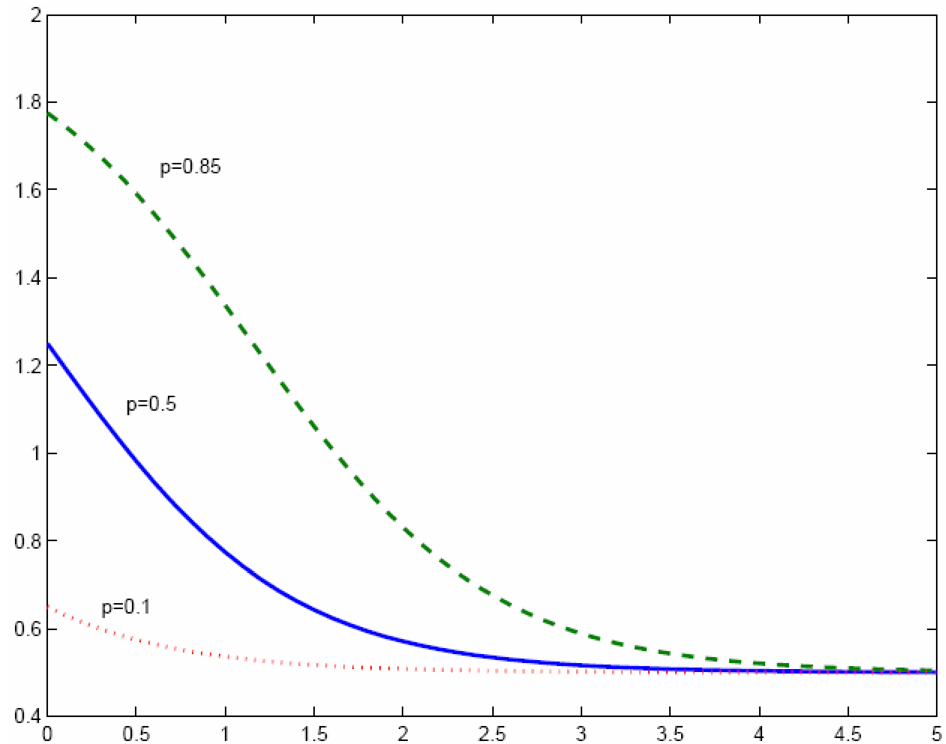


Figure 1: The values of μ_H and μ_L are 2.5 and 0.5, respectively.

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ solve the system of ODES

$$\begin{cases} \dfrac{\partial \mathcal{A}}{\partial t} + d^*(t)\mathcal{B} + \dfrac{1}{2}\mathcal{B}^* k_0(t)\mathcal{B} + \operatorname{tr}\{\mathcal{C}k_0(t)\} & = & \mathsf{f}(t) \\ \mathcal{A}(T,T) & = & 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \mathcal{B}}{\partial t} + E^*(t)\mathcal{B} + 2\mathcal{C}d(t) + \dfrac{1}{2}\tilde{\mathcal{B}}^* K(t)\mathcal{B} + 2\mathcal{C}k_0(t)\mathcal{B} & = & \mathsf{g}(t) \\ \mathcal{B}(T,T) & = & 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \mathcal{C}}{\partial t} + \mathcal{C}E(t) + E^*(t)\mathcal{C} + 2\mathcal{C}k_0(t)\mathcal{C} + \dfrac{1}{2}\tilde{\mathcal{B}}^* G(t)\tilde{\mathcal{B}} & = & \mathsf{Q}(t) \\ \mathcal{C}(T,T) & = & 0 \end{cases}$$

$$\tilde{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \quad K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_{11}(t) & \cdots & g_{1m}(t) \\ \vdots & \ddots & \vdots \\ g_{m1}(t) & \cdots & g_{mm}(t) \end{pmatrix}$$

Illustration

The previous pictures are based on the following simple three-factor model

$$Z = \begin{pmatrix} Z^1 \\ Z^2 \\ r \end{pmatrix}$$

$$\begin{aligned} dZ_t^1 &= \left[\beta_1(t) - \alpha_1 Z_t^1\right] dt + \sigma_1 dW_t^1 \\ dZ_t^2 &= \left[\beta_2(t) - \alpha_2 Z_t^2\right] dt + \sigma_2 dW_t^2 \\ dr_t &= \alpha_r \left[\beta_r - r_t\right] dt + \sigma_r \sqrt{r_t} dW_t^r \end{aligned}$$

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c, \quad \mu_t^k = \eta_t^k = \left(Z_t^k\right)^2, \quad k = 1,2 \; \epsilon^1, \epsilon^2 \in \mathbb{R}$$

$$\mu^c = J^c + \delta r, \quad J_t^c = \sum_{\tilde{\tau}_i < t} Y_i h^c(t - \tilde{\tau}_i), \quad Y_i \sim \chi^2(2) \quad h^c(x) = e^{-bx}, \; b \in \mathbb{R}_+$$

$\tilde{\tau}_i$ are the jumps of a Poisson process with intensity l^c.

We get :

- $$S^k(\theta,t,T) = \sqrt{\frac{2\gamma_k e^{(\gamma_k+\alpha_k)x}}{(\gamma_k+\alpha_k)(e^{2\gamma_k x}-1)+2\gamma_k}} \times \exp\left\{\frac{\theta\left[1-e^{2\gamma_k x}\right]}{(\gamma_k+\alpha_k)\left[e^{2\gamma_k x}-1\right]+2\gamma_k}\left(Z_t^k\right)^2\right\}\times$$

$$\times \exp\left\{-\int_t^T\left(\beta_k(s)\mathcal{B}^k(\theta,s,T)+\frac{1}{2}\sigma_k^2\left(\mathcal{B}^k(\theta,s,T)\right)^2\right)ds+\mathcal{B}^k(\theta,t,T)Z_t^k\right\}$$

where

$$\mathcal{B}^k(t,T) = \left[2(\alpha_k+\gamma_k)(e^{2\gamma_k x}-1)+4\gamma_k\right]^{1/2\sigma_k^2}\int_t^T\left(\beta_k(s)\frac{4\theta e^{\left(\alpha_k+\frac{\theta}{\gamma_k-\alpha_k}\right)(s-t)}\left(1-e^{2\gamma_k(T-s)}\right)}{\left[2(\alpha_k+\gamma_k)(e^{2\gamma_k(T-s)}-1)+4\gamma_k\right]^{1+\frac{1}{2\sigma_k^2}}}\right)ds$$

- $$\begin{aligned} S_\eta^c(\theta,t,T) &= \exp\left\{\mathcal{A}^c(\theta,t,T)+\mathcal{B}^{c\top}(\theta,t,T)Z_t+Z_t^\top\mathcal{C}^c(\theta,t,T)Z_t\right\} \\ \bar{S}_\eta^c(\theta,t,T) &= \exp\left\{\bar{A}^c(\theta,t,T)+\bar{B}^{c\top}(\theta,t,T)Z_t+Z_t^\top\bar{C}^c(\theta,t,T)Z_t\right\}\end{aligned}$$

where

$$\begin{aligned}\bar{A}^c(\theta,t,T) &= A((1+\theta\delta),t,T)\\ \mathcal{A}^c(\theta,t,T) &= A(\theta\delta,t,T)\\ \bar{B}^c(\theta,t,T) &= B((1+\theta\delta),t,T)\\ \mathcal{B}^c(\theta,t,T) &= B(\theta\delta,t,T)\\ \bar{C}^c(\theta,t,T) &= \mathbf{0}\\ \mathcal{C}^c(\theta,t,T) &= \mathbf{0}\end{aligned}$$

$$A(\Delta,t,T) = \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left[\frac{2\hat{\gamma}_r e^{(\hat{\gamma}_r+\alpha_r)\frac{T-t}{2}}}{(\alpha_r+\hat{\gamma}_r)[e^{\hat{\gamma}_r(T-t)}-1]+2\hat{\gamma}_r}\right]$$

$$B(\Delta,t,T) = \begin{bmatrix}0\\0\\ \dfrac{2\Delta\left[e^{\hat{\gamma}_r(T-t)}-1\right]}{(\hat{\gamma}_r+\alpha_r)\left[e^{\hat{\gamma}_r(T-t)}-1\right]+2\hat{\gamma}_r}\end{bmatrix},$$

- $$S_J^c(\theta,t,T) = \left[1+\frac{2\theta}{b}\left(1-e^{-b(T-t)}\right)\right]^{\frac{l^c}{b+2\theta}}\exp\left\{\frac{1}{b}\left[e^{-bx}-1\right]J_t+l^c(T-t)\left[\frac{b}{b+2\theta}-1\right]\right\}$$

- $$\Gamma^k(\theta,t,T) = \Gamma^k_\eta(\theta,t,T) = S^k_\eta(\theta,t,T)\exp\left(a^k(\theta,t,T) + b^k(\theta,t,T)Z^k_t + c^k(\theta,t,T)\left(Z^k_t\right)^2\right)$$

where

$$a^k(\theta,t,T) = -\int_t^T \beta_k(s) + \sigma_k^2\mathcal{B}^k(s,T)b + \sigma_k^2 c^k(s,T)ds$$

$$b^k(\theta,t,T) = -2\int_t^T e^{\int_t^s \alpha_k - 2\sigma_k^2\mathcal{C}^k(u,T)du}\left(\beta_k(s) - \sigma^2\mathcal{B}^k(s,T)\right)c^k(s,T)ds$$

$$c^k(\theta,t,T) = \frac{\theta\left[(\gamma_k+\alpha_k)(e^{\gamma_k x}-1)+2\gamma_k\right]}{\gamma_k e^{(\gamma_k+3\alpha_k)x}}$$

- $$\Gamma^c_\eta(\theta,t,T) = S^c_\eta(\theta,t,T)\exp\{a(\theta\delta,t,T) + b(\theta\delta,t,T)r_t\}$$

$$\bar{\Gamma}^c_\eta(\theta,t,T) = \bar{S}^c_\eta(\theta,t,T)\exp\{a((1+\theta\delta),t,T) + b((1+\theta\delta),t,T)r_t\}$$

where

$$a(\Delta,t,T) = \int_t^T \alpha_r\beta_r b(\Delta,s,T)ds$$

$$b(\Delta,t,T) = \theta e^{-\int_t^T \alpha_r - \frac{1}{2}\sigma_r^2 B(s,T)ds} = \frac{\theta\left[(\alpha_r+\hat{\gamma}_r)(e^{\hat{\gamma}_r x}-1)+2\hat{\gamma}_r\right]}{2\hat{\gamma}_r e^{(3\alpha_r+\hat{\gamma}_r)\frac{x}{2}}}$$

- $$\Gamma^c_J(\theta,t,T) = S^c_J(\theta,t,T)\left\{\theta J^c(t,T) - l^c\left[\frac{1}{2+b\theta}\left(b+\frac{1}{x}\ln\left(1+\frac{2\theta}{c}(1-e^{-bx})\right)\right)(1-x) - 1\right] + x\frac{1}{1+\frac{2\theta}{b}(1-e^{-bx})}\right\}$$