RATE OF CONVERGENCE OF THE EULER SCHEME FOR THE PRICE, DELTAS AND GAMMAS OF A EUROPEAN OPTION

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Structure of the talk

- Part 1: Introduction and objectives
- Part 2: Mathematical results
- Part 3: Rate of convergence of the Euler scheme for the deltas and gammas of a european option

Objective

(1) Estimate the law of X_1^x, i.e. estimate $\mathbb{E}[f(X_1^x)]$, where

$$X_t^x = x + \int_0^t b(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dB_s.$$

(2) Estimate $\partial_x^\alpha \mathbb{E}[f(X_1^x)]$.

Time discretization + Monte Carlo (1)

Euler scheme

$t_k^n = k/n$, $X_0^{n,x} = x$ and, for all $k \in \{0, \ldots, n-1\}$ and $t \in [t_k^n, t_{k+1}^n]$,

$$X_t^{n,x} = X_{t_k^n}^{n,x} + b\left(X_{t_k^n}^{n,x}\right)(t - t_k^n) + \sigma\left(X_{t_k^n}^{n,x}\right)\left(B_t - B_{t_k^n}\right).$$

Monte Carlo

$$\mathbb{E}\left[f\left(X_1^x\right)\right] \approx \frac{1}{N}\sum_{i=1}^{N} f(X_{i,1}^{n,x}).$$

What is already well known?

$$\mathbb{E}\left[f\left(X_1^{n,x}\right)\right] - \mathbb{E}\left[f\left(X_1^{x}\right)\right] = C_1 f(x)/n + O\left(1/n^2\right) \tag{1}$$

(1) when f is smooth with polynomially growing derivatives (Talay-Tubaro, '90),

(2) under a uniform hypoellipticity condition for X, when f is only measurable and bounded (Bally-Talay, '96).

Time discretization + Monte Carlo (2)

(1) $\frac{1}{N}\sum_{i=1}^{N} f(X_{i,1}^{n,x})$: In a time of order nN, one gets an error of order $1/\sqrt{N} + 1/n$. Given a tolerance $\varepsilon \ll 1$, in order to minimize the time of calculus, one should then choose $N = O\left(n^2\right)$ and gets a result in a time of order $1/\varepsilon^3$.

(2) $\frac{1}{N}\sum_{i=1}^{N}(2f(X_{i,1}^{2n,x}) - f(X_{i,1}^{n,x}))$: **Romberg**'s extrapolation technique: in a time of order nN, one gets an estimate of $\mathbb{E}\left[f\left(X_1^x\right)\right]$ whose accuracy is of order $1/\sqrt{N}+1/n^2$, since (1) implies that $\mathbb{E}[2f(X_1^{2n,x}) - f(X_1^{n,x})] = \mathbb{E}[f(X_1^x)] + O(1/n^2)$. Given a tolerance $\varepsilon \ll 1$, one should now choose $N = O\left(n^4\right)$ and gets a result in a time of order $1/\varepsilon^{5/2}$.

Objective: time discretization

Take f to be a tempered distribution. $\mathbb{E}\left[f\left(X_1^x\right)\right]$? Expansion in powers of $1/n$?

Example: $f = \delta_y$, the Dirac mass at point $y \in \mathbb{R}^d$.

$$\mathbb{E}\left[\delta_y\left(X_1^x\right)\right] = \int_{\mathbb{R}^d} \delta_y(y')p(1,x,y')\,dy' = p(1,x,y)$$

Find a function π and a “bounded” sequence of functions $(\pi_n, n \geq 1)$ such that

$$p_n(t,x,y) - p(t,x,y) = \pi(t,x,y)/n + \pi_n(t,x,y)/n^2$$

Part 2: Mathematical results

Theorem. Under (B) and (C),

(i) for all $t \in (0,1]$ and $x \in \mathbb{R}^d$, X_t^x has a density $p(t,x,\cdot)$ and $p \in \mathcal{G}(\mathbb{R}^d)$,

(ii) for all $t \in (0,1]$, $x \in \mathbb{R}^d$ and $n \geq 1$, $X_t^{n,x}$ has a density $p_n(t,x,\cdot)$ and $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$,

(iii) there exists $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and a bounded sequence $(\pi_n, n \geq 1)$ in $\mathcal{G}_4(\mathbb{R}^d)$ such that for all $n \geq 1$,

$$p_n - p = \pi/n + \pi_n/n^2. \tag{2}$$

Moreover,

$$\pi(t,x,y) = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} p(s,x,z) L_2^*(p(t-s,\cdot,y))(z)\, dzds,$$

where the differential operator L_2^* is explicitely given in terms of the functions a and b by

$$-L_2^* = \sum_{i=1}^d \left(b\cdot\nabla b_i + \frac{1}{2}\mathrm{tr}\left(a\nabla^2 b_i\right)\right)\partial_i$$

$$+\sum_{i,j=1}^d \left(\frac{1}{2} b\cdot\nabla a_{i,j} + a_j\cdot\nabla b_i + \frac{1}{4}\mathrm{tr}\left(a\nabla^2 a_{i,j}\right)\right)\partial_{ij} + \frac{1}{2}\sum_{i,j,k=1}^d a_k\cdot\nabla a_{i,j}\partial_{ijk}.$$

Functional sets

- $C^\infty_{\text{pol}}(\mathbb{R}^d)$: the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with polynomially growing derivatives of any order, i.e. such that for all $\alpha \in \mathbb{N}^d$, there exists $c \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha f(x)| \leq c\,(1 + \|x\|^q)\,,$$

- $C^\infty_b(\mathbb{R}^d)$: the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives of any order, i.e. such that $\partial^\alpha f \in L^\infty(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

Assumptions

(A) For all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_{\text{pol}}(\mathbb{R}^d)$ and have bounded first derivatives.

(B) For all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_b(\mathbb{R}^d)$.

(C) There exists $\eta > 0$ such that for all $x, \xi \in \mathbb{R}^d$, $\xi^* a(x) \xi \geq \eta \|\xi\|^2$.

Functional sets (continued)

- $\mathcal{G}_l(\mathbb{R}^d)$: the set of all measurable functions $\pi : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that
  - for all $t \in (0,1]$, $\pi(t,\cdot,\cdot)$ is infinitely differentiable,
  - for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp\left(-c_2 \|x-y\|^2 / t\right). \quad (3)$$

- $\mathcal{G}(\mathbb{R}^d)$: defined in the same way as $\mathcal{G}_l(\mathbb{R}^d)$ with (3) replaced by the following two conditions:

$$\begin{aligned}\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| &\leq c_1 t^{-(|\alpha|+|\beta|+d)/2} \exp\left(-c_2 \left\|x-y\right\|^2/t\right),\\ \left|\partial_x^\alpha \left(\pi\left(t,x,x+y\sqrt{t}\right)\right)\right| &\leq c_1 t^{-d/2} \exp\left(-c_2 \left\|y\right\|^2\right).\end{aligned}$$

Spatial derivatives of the density

Corollary. Under (B) and (C), for all $\alpha, \beta \in \mathbb{N}^d$, there exists $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\partial_x^\alpha \partial_y^\beta p_n(t,x,y) - \partial_x^\alpha \partial_y^\beta p(t,x,y) = \frac{1}{n} \partial_x^\alpha \partial_y^\beta \pi(t,x,y) + r_n(t,x,y)$$

and

$$|r_n(t,x,y)| \leq c_1 n^{-2} t^{-(|\alpha| + |\beta| + d + 4)/2} \exp\left(-c_2 \|x-y\|^2 / t\right).$$

Polynomially growing (non-smooth) f's

Corollary. Assume (B) and (C). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that there exists $c' \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$, $|f(x)| \leq c'(1 + \|x\|^q)$. Then there exists $c \geq 0$ such that for all $n \geq 1$, $t \in (0, 1]$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n} \int_{\mathbb{R}^d} f(y)\pi(t, x, y)\, dy + r_n(t, x) \qquad (4)$$

and

$$|r_n(t, x)| \leq cn^{-2}t^{-2}\left(1 + \|x\|^q\right).$$

Exponentially growing (non-smooth) f's

$\mathcal{E}_\mu$: the set of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that there exists $c_1, c_2 \geq 0$ such that for all $y \in \mathbb{R}^d$,

$$|f(y)| \leq c_1 \exp\left(c_2 \|y\|^\mu\right),$$

Corollary. Under (B) and (C), for all $\mu \in (0,2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$, $f(X_t^x)$ and $f(X_t^{n,x})$ are integrable and

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n}\int_{\mathbb{R}^d} f(y)\pi(t,x,y)\,dy + r_n(t,x) \qquad (5)$$

$$|r_n(t,x)| \leq c_1 n^{-2} t^{-2} \exp\left(c_2 \|x\|^\mu\right).$$

Rate of convergence of $\partial_x^\alpha \mathbb{E}[f(X_t^{n,x})]$

Corollary. Under (B) and (C), for all $\alpha \in \mathbb{N}^d$, $\mu \in (0,2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$,

$$\partial_x^\alpha \mathbb{E}[f(X_t^{n,x})] - \partial_x^\alpha \mathbb{E}[f(X_t^x)] = \frac{1}{n} \int_{\mathbb{R}^d} f(y) \partial_x^\alpha \pi(t,x,y)\, dy + r_n(t,x) \quad (6)$$

with

$$|r_n(t,x)| \leq c_1 n^{-2} t^{-(|\alpha|+4)/2} \exp\left(c_2 \|x\|^\mu\right).$$

Euler scheme and tempered distributions

Theorem. Under (B) and (C), for all $S \in \mathcal{S}'(\mathbb{R}^d)$, there exists $c \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \langle S, p_n(t,x,\cdot)\rangle - \langle S, p(t,x,\cdot)\rangle &= \frac{1}{n}\langle S, \pi(t,x,\cdot)\rangle + r'_n(t,x), \\ \langle S, p_n(t,\cdot,y)\rangle - \langle S, p(t,\cdot,y)\rangle &= \frac{1}{n}\langle S, \pi(t,\cdot,y)\rangle + r''_n(t,y), \end{aligned}$$

and

$$|r'_n(t,x)| + |r''_n(t,x)| \leq c n^{-2} t^{-(d+4+\#S)/2} \left(1 + \|x\|^{\#S}\right).$$

Conclusion: (1) is valid for f's being only tempered distributions provided we define $\mathbb{E}\left[S(Y)\right] = \langle S, p_Y \rangle$.

Part 3: Rate of convergence of the Euler scheme for the deltas and gammas of a european option with general payoff

Setting and objective

$S^v = (S^{v,1}, \ldots, S^{v,d})$: a basket of assets satisfying

$$\frac{dS_t^{v,i}}{S_t^{v,i}} = \mu_i(S_t^v)\,dt + \sum_{j=1}^{r} \sigma_{i,j}(S_t^v)\,dB_t^j, \qquad S_0^{v,i} = v^i > 0,$$

with $\mu, \sigma \in C_b^\infty(\mathbb{R}^d)$ and σ satisfying (C).

Purpose

Given a measurable and polynomially growing function ϕ, we try to estimate the price Price $= \mathbb{E}[\phi(S_t^v)]$, the deltas Delta$_i = \partial_v^{e_i}\mathbb{E}[\phi(S_t^v)]$ and the gammas Gamma$_{i,j} = \partial_v^{e_i+e_j}\mathbb{E}[\phi(S_t^v)]$ of the european option of maturity t and payoff ϕ ($(e_1, \ldots, e_d)$ is the canonical base of $\mathbb{R}^d$).

Change of variables

We set $x = \ln v$ (i.e. $x^i = \ln v^i$) and $X_t^{x,i} = \ln(S_t^{v,i})$

X is the solution of the initial SDE with $b = \mu - \|\sigma\|^2/2 \in C_b^\infty(\mathbb{R}^d)$, where $\|\sigma\|_i^2(x) = \sum_{j=1}^r \sigma_{i,j}^2(x)$.

If we set $\exp(x) = (\exp(x^1), \ldots, \exp(x^d))$ and $f(x) = \phi(\exp(x))$, we define a function $f \in \mathcal{E}_1$ and Price $= \mathbb{E}[f(X_t^x)]$.

Results

$$\mathsf{Price}^n - \mathsf{Price} = C_t^{\mathsf{Price}}\phi(v)/n + O\left(n^{-2}t^{-2}\exp\left(c_2\left\|\ln v\right\|\right)\right),$$

where Price^n stands for the approximated price $\mathbb{E}[f(X_t^{n,x})]$ and

$$C_t^{\mathsf{Price}}\phi(v) = \int_{(\mathbb{R}_+^*)^d} \phi(u)\frac{\pi(t, \ln v, \ln u)}{u_1 \cdots u_d}\, du.$$

If we set $\mathsf{Delta}_i^n = \partial_v^{e_i}\mathbb{E}[f(X_t^{n,\ln v})]$ and $\mathsf{Gamma}_{i,j}^n = \partial_v^{e_i+e_j}\mathbb{E}[f(X_t^{n,\ln v})]$, (6) shows that

$$\begin{aligned}
\mathsf{Delta}^n - \mathsf{Delta} &= C_t^{\mathsf{Delta}}\phi(v)/n + O\left(n^{-2}t^{-5/2}\exp\left(c_2\,\|\ln v\|\right)\right),\\
\mathsf{Gamma}^n - \mathsf{Gamma} &= C_t^{\mathsf{Gamma}}\phi(v)/n + O\left(n^{-2}t^{-3}\exp\left(c_2\,\|\ln v\|\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
C_t^{\mathsf{Delta}}\phi(v)_i &= \frac{1}{v_i}\int_{(\mathbb{R}_+^*)^d}\phi(u)\frac{\partial_2^{e_i}\pi(t,\ln v,\ln u)}{u_1\cdots u_d}\,du,\\
C_t^{\mathsf{Gamma}}\phi(v)_{i,j} &= \frac{1}{v_i v_j}\int_{(\mathbb{R}_+^*)^d}\phi(u)\frac{\partial_2^{e_i+e_j}\pi(t,\ln v,\ln u) - \mathbf{1}_{\{i=j\}}\partial_2^{e_i}\pi(t,\ln v,\ln u)}{u_1\cdots u_d}\,du.
\end{aligned}$$

Explosion at maturity

	principal part	remainder
price	$t^{-1/2}$	t^{-2}
delta	t^{-1}	$t^{-5/2}$
gamma	$t^{-3/2}$	t^{-3}

Article

- to appear in *Stochastic Processes and Their Applications*
- available online at `http://cermics.enpc.fr/~guyon/home.html`