Quantitative Methods in Finance.

On Finite Dimensional Realizations of a Two Country Interest Rate Models.

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December 14, 2005

Pricing on international markets

- The global international markets development brings the need for creating tractable pricing models on the international integrated market.

- The approach employed by most of the previous studies:
  - Exogenously specify a finite set of local and global **factors** and specify their dynamics
  - Specify the desired **functional form** of the domestic and foreign bond prices (e.g. affine term structures)
  - Find necessary and sufficient conditions on the drift and volatility of the factors, as well as short rate, for the term structures to possess the specified form.

Our approach:

- Specify the domestic forward rate dynamics for every T under the domestic martingale measure (**HJM** Framework).
- Specify the foreign forward rate dynamics for every T under the foreign martingale measure (**HJM** Framework).
- Specify the exchange rate dynamics.
- The entire domestic and forward rate curves are are chosen as the infinite dimensional state variables.

Main Problems:

- Find necessary and sufficient conditions on the forward rate volatilities for the forward rate model to be realized by a finite dimensional state space model.

or

when **both** forward rate curves evolve on a given finite dimensional submanifold in a forward curve space?

- Determine the dimension of the state space and the dynamics of the state variables

Basic definitions

- $p(t,x)$ and $\tilde{p}(t,x)$ - time t prices of the domestic and foreign zero coupon bond maturing at $T = t+x$

- $r(t,x)$ - domestic forward interest rate contracted at time t with maturity $t+x$

$$r(t,x) = -\frac{\partial \log p(t,x)}{\partial x}$$

- $\tilde{r}(t,x)$ - foreign forward interst rate contracted at time t with maturity $t+x$

$$\tilde{r}(t,x) = -\frac{\partial \log \tilde{p}(t,x)}{\partial x}$$

- Q^d and Q^f - domestic and foreign martingale measures

The International Market

- The domestic forward rate dynamics under Q^d (Heath-Jarrow-Morton-Musiela equation)

$$\begin{aligned} dr_t(x) &= \left\{ \frac{\partial}{\partial x} r_t(x) + \sigma_t(x) \int_0^x \sigma_t(u) du \right\} dt \\ &+ \sigma_t(x) dW_t, \end{aligned}$$

- The foreign forward rate dynamics under Q^d (Heath-Jarrow-Morton-Musiela equation)

$$\begin{aligned} d\tilde{r}_t(x) &= \left\{ \frac{\partial}{\partial x} \tilde{r}_t(x) + \tilde{\sigma}_t(x) \int_0^x \tilde{\sigma}_t(u) du \right. \\ &- \left. \tilde{\sigma}_t(x) \delta_t^\star \right\} dt + \tilde{\sigma}_t(x) dW_t, \end{aligned}$$

- The Q^d-dynamics of the exchange rate (units of domestic currency per unit of foreign currency)

$$dS_t = S_t(r_t(0) - \tilde{r}_t(0))dt + S_t \delta_t dW_t.$$

Assumption

$\sigma(t,x)$, $\tilde{\sigma}(t,x)$ and $\delta(t)$ - adapted proceses:

$$\sigma(t,x) = \sigma(\hat{r}_t, x), \qquad \tilde{\sigma}(t,x) = \tilde{\sigma}(\hat{r}_t, x)$$

$$\delta(t) = \delta(\hat{r}_t)$$

where σ, $\tilde{\sigma}$ and δ denote deterministic mappings

$$\begin{aligned} \sigma &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m \\ \tilde{\sigma} &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}^m \\ \delta &: \mathcal{H} \times \tilde{\mathcal{H}} \times \mathcal{R}_+ \to \mathcal{R}^m, \end{aligned}$$

$$\hat{r}_t = \begin{pmatrix} r_t \\ \tilde{r}_t \\ S_t \end{pmatrix}$$

$\mathcal{H}$ and $\tilde{\mathcal{H}}$ - spaces of the domestic and foreign forward rate curves

We intend to study:

$$\begin{cases} dr_t &= (\mathbf{F}r_t + \sigma(\hat{r}_t)\mathbf{H}\sigma^\star(\hat{r}_t))dt + \sigma(\hat{r}_t)dW_t, \\ \\ d\tilde{r}_t &= (\mathbf{F}\tilde{r}_t + \tilde{\sigma}(\hat{r}_t)\mathbf{H}\tilde{\sigma}^\star(\hat{r}_t) - \tilde{\sigma}(\hat{r}_t)\delta^\star(\hat{r}_t))dt \\ &+ \tilde{\sigma}(\hat{r}_t)dW_t, \\ \\ dS_t &= S_t(\mathbf{B}r_t - \mathbf{B}\tilde{r}_t)dt + S_t\delta(\hat{r}_t)dW_t, \end{cases}$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad \mathbf{B}f(x) = f(0)$$

$$\mathbf{H}f(x) = \int_0^x f(s)ds$$

System of the two SDEs **AND** the exchange rate equation !!!

The main problems are:

- When can the infinite dimensional domestic and foreign forward rate processes be realized by means of a Markovian finite dimensional state space model?

- Given that an FDR exists, how can we construct it?

We know from the previous literature that:

The two-country model admits an FDR **if and only if** the **Lie algebra** generated by vector fields

$\hat{\mu}(\hat{r})$-[$Stratonovich\ correction\ term$] and $\hat{\sigma}_i(\hat{r})$, where

$$\hat{\mu}(\hat{r}) = \begin{pmatrix} \mathbf{F}r + \sigma(\hat{r})\mathbf{H}\sigma^\star(\hat{r}) \\ \\ \mathbf{F}\tilde{r} + \tilde{\sigma}(\hat{r})\mathbf{H}\tilde{\sigma}^\star(\hat{r}) - \tilde{\sigma}(\hat{r})\delta^\star(\hat{r}) \\ \\ \mathbf{B}r - \mathbf{B}\tilde{r} - \frac{1}{2}\|\delta(\hat{r})\|^2 \end{pmatrix}$$

$$\hat{\sigma}_i(\hat{r}) = \begin{pmatrix} \sigma_i(\hat{r}) \\ \\ \tilde{\sigma}_i(\hat{r}) \\ \\ \delta_i(\hat{r}) \end{pmatrix}, \qquad i = 1, \ldots, m.$$

is **finite dimensional** (evaluated pointwise near $\hat{r}^0$).

Existence of an FDR. Results

In the following cases we can derive **necessary and sufficient** conditions for an FDR to exist

- Forward rate volatilities σ and $\tilde{\sigma}$ are deterministic, i.e. of the form

$$\sigma(\hat{r}, x) = \sigma(x), \quad \tilde{\sigma}(\hat{r}, x) = \tilde{\sigma}(x)$$

$$\delta(\hat{r}) = \delta$$

- Forward rate volatilities and the exchange rate volatility are of the form

$$\sigma(\tilde{r}, r, x) = \varphi(r, \tilde{r})\lambda(x) \quad \tilde{\sigma}(\tilde{r}, r, x) = \varphi(r, \tilde{r})\tilde{\lambda}(x),$$

$$\delta(\tilde{r}, r) = \varphi(r, \tilde{r})$$

Constant direction volatility. The length of the fields, φ depends on the entire (domestic and foreign) forward rate curves and is **the same** for both countries.

Results (cont.)

- The two-country model admits an FDR if and only if $\sigma(x)$ and $\tilde{\sigma}(x)$ ($\lambda(x)$ and $\tilde{\lambda}(x)$) are **quasi-exponential**.

  The exchange rate volatility does not play a role!

- I. In the deterministic volatility case the dimension of the relevant Lie algebra

$$dim\,\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + m + \sum_{i=1}^{m} n_i,$$

  n_i is the degree of the **minimal** polynomial $\mathbf{M^i}$:

$$M^i \sigma_i(x) = M^i \tilde{\sigma}_i(x) = 0$$

  II. In the scalar constant direction volatility case the dimension of the relevant Lie algebra is

$$dim\,\{\hat{\mu}, \hat{\sigma}\}_{LA} \leq 3 + 2n$$

Smaller dimension of the state space

There exists an FDR with a **smaller** dimension of the Lie algebra

$$dim\left\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\right\}_{LA} \leq 1 + \sum_{i=1}^{m} n_i.$$

if and only if for each i the exchange rate volatility δ and the forward rate volatilities satisfy

$$\mathbf{B}M_1^i(\mathbf{F})\left(\sigma_i - \tilde{\sigma}_i\right) + M^i(0)\delta_i = 0, \qquad M^i(0) \neq 0,$$

where if M is defined as above and M_1 relates to M by

$$M(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$M_1(x) = x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2 x + a_1$$

The exchange rate volatility plays a role!

A more general case

- The domestic and foreign forward rate volatilities have the form

$$\sigma_i(r, \tilde{r}, S, x) = \varphi_i(r, \tilde{r}, S)\lambda_i(x), \quad i = 1, \dots, m,$$

$$\tilde{\sigma}_i(r, \tilde{r}, S, x) = \tilde{\varphi}_i(r, \tilde{r}, S)\tilde{\lambda}_i(x), \quad i = 1, \dots, m,$$

- The exchange rate volatility takes the form

$$\delta_i(r, \tilde{r}, S), \quad i = 1, \dots, m$$

Different $\varphi_i(r, \tilde{r}, S)$ **and** $\tilde{\varphi}_i(r, \tilde{r}, S)$**!!**

In this more general case we can study only **larger** Lie algebras and thus find only **sufficient** conditions!!

A sufficient condition - $\lambda(x)$ and $\tilde{\lambda}(x)$ are quasi-exponential functions. (φ_i, $\tilde{\varphi}_i$ and δ_i are any arbitrary scalar fields.)

Construction of an FDR.

From the literature we know that if the Lie algebra $\{\hat{\mu}, \hat{\sigma}\}_{LA}$ is spanned by the smooth vector fields

$$\hat{f}_1, \ldots, \hat{f}_d,$$

then , for the initial point $\hat{r}^0$, all forward rate curves produced by the model will belong to the manifold $\hat{\mathcal{G}} \in \hat{\mathcal{H}}$, which can be parametrized as

$$\hat{G}(z_1, \ldots, z_d) = e^{f_d z_d} \ldots e^{f_1 z_1} \hat{r}^0,$$

and where the operator $e^{f_i z_i}$ is defined as a solution to

$$\left\{ \begin{array}{rcl} \dfrac{\partial \hat{r}_t}{\partial t} & = & f(\hat{r}_t), \\ \hat{r}_0 & = & \hat{r}^0. \end{array} \right.$$

- This leads to a construction algorithm.
- Basically compute dynamics of the local coordinates on $\hat{\mathcal{G}}$.

Construction. Results.

- In the simpler cases we can construct **minimal** realizations

- Since in the general constant direction volatility case we can study only enlarged Lie algebras $\Rightarrow$

  we can obtain only **non-minimal** realizations.

Construction. Example.

$$\sigma(x) = e^{-\alpha x} \quad \tilde{\sigma}(x) = Ce^{-2\alpha x} \quad \delta = const$$

$$\begin{aligned} r(z,x) &= G_1(z^0, z_0, z_1, z_2) \\ \tilde{r}(z,x) &= G_2(z^0, z_0, z_1, z_2) \\ S(z) &= G_3(z^0, z_0, z_1, z_2) \end{aligned}$$

and the dynamics of the state space variables are given by

$$\begin{aligned} dZ^0(t) &= dt \\ dZ_0(t) &= dW(t) \\ dZ_1(t) &= [-2\alpha^2 Z_2(t) + Z_0(t)]dt \\ dZ_2(t) &= [-3\alpha Z_2(t) + Z_1(t)]dt. \end{aligned}$$